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A proof of the optimal leapfrogging conjecture

Sam K. Miller and Arthur T. Benjamin

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Consider a collection of checkers that move on the integer lattice by shifting or jumping. It has been shown that a configuration of pieces can translate itself at a speed of at most 1, and this can only be achieved with a collection of 1, 2 or 4 checkers. We prove the conjecture that with any other number of checkers, the fastest obtainable speed is $\frac{2}{3}$.

1. Introduction

Suppose we have some checkers placed in the lower left corner of a Go board, and we wish to move them to the upper right corner in as few moves as possible. There are no opponent pieces present, and the pieces move as they would in the game of Chinese Checkers, where for one move, a piece may either shift one unit in any direction, or repeatedly leapfrog over other pieces.

Let us consider the Go board as a subset of the nonnegative integer lattice $\mathbb{Z}^2$. As an example, suppose we have four pieces placed at the coordinates $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$, and wish to move them to the squares $(9, 9)$, $(10, 9)$, $(9, 10)$, and $(10, 10)$. For the pieces to complete the task in as few moves as possible, the pieces must first be moved into a configuration such that they may jump over each other in an optimal way.

We may intuitively attempt lining the checkers up diagonally in what we will call a *snake configuration*, that is, moving the pieces to coordinates $(0, 0)$, $(1, 1)$, $(2, 2)$, and $(3, 3)$. By repeating the three-move process of shifting the backmost piece to the right $[(0, 0) \to (1, 0)]$, leapfrogging that piece to the front $[(1, 0) \to (3, 4)]$, then shifting it right again $[(3, 4) \to (4, 4)]$, we can reach our destination in $4+4+(3\times 7) = 29$ moves. We say that the snake configuration has a speed of $\frac{2}{3}$ since in 3 moves, it makes a forward progress of 2. (Every piece is shifted in the direction $(1, 1)$.)

However a faster method exists. We first move the pieces into what we call a *serpent configuration*, with the pieces on coordinates $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(2, 1)$.

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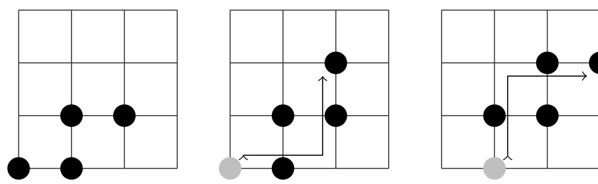

Figure 1. An example of the serpent's two-move trajectory, consisting of two jumps. The placements in the leftmost and rightmost diagrams are translates, and represented by the same configuration, the serpent. These two placements have displacement 2, and require two moves to reach one from the other. Hence, the serpent is a speed of light configuration, i.e., it has speed 1.

Then we repeat the two-move process of leapfrogging the backmost piece to the front $[(0,0) \to (2,2)]$ then leapfrogging the new backmost piece to the front again $[(1,0) \to (3,2)]$, we may reach our destination in $1+1+(2\times 8) = 18$ moves. This is indeed the fastest way of moving the checkers from the bottom left to the upper right. See Figure 1. Note that the serpent configuration has a speed of 1, since it requires only two moves to give it a forward progress of 2.

It was proved in [Benjamin 1990] that in a wide class of optimization problems on lattice structures (such as $\mathbb{Z}^n$), the optimal way to move objects from one location to another far away location, is to spend most of the time repeatedly translating very efficient configurations. It was shown by Auslander, Benjamin, and Wilkerson [Auslander et al. 1993] that the maximum speed of any configuration is 1, there are essentially only three configurations that achieve this speed, and they have at most four pieces. It was conjectured that with five or more pieces, the maximum attainable speed is $\frac{2}{3}$. In this paper, we prove that conjecture.

2. Abstracting the game

Suppose we have p indistinguishable pieces and wish to move them in the positive direction over the integer lattice $\mathbb{Z}^n$. If a piece is located at coordinate $l \in \mathbb{Z}^n$, and some other coordinate $l + e_i$ is not occupied by a piece (for unit vector e_i), then the piece may *shift* there. Alternatively if $l + e_i$ is occupied but $l + 2e_i$ is not, the piece may *hop* over the occupant of $l + e_i$ to land at $l + 2e_i$, where it may remain or continue hopping over other adjacent pieces. One legal *move* consists of either a shift or a *jump*, a sequence of one or more hops by a single piece.

Define a *placement* of size p as a finite subset of $\mathbb{Z}^n$, denoted by $X = \{\vec{x}_1, \ldots, \vec{x}_p\}$. Define the *centroid* of placement X to be

$$c(X) = \frac{1}{p}\sum_{u=1}^{p} \vec{x}_u.$$

For placements X, Y, define their *displacement* as

$$d(X, Y) = \sum_{i=1}^{n} |c_i(X) - c_i(Y)|.$$

For $m \geq 1$, an *m-move trajectory* $X_0, X_1, \ldots, X_m$ is a sequence of placements where X_{u+1} is reachable from X_u in a single legal move. The *speed* of an m-move trajectory from X_0 to X_m is

$$s = \frac{d(X_0, X_m)}{m}.$$

We say that placements X, Y are *translates* if there exists $\vec{a} \in \mathbb{Z}^n$ such that $X + \vec{a} = Y$. We say that translates X and Y are represented by the same *configuration* of pieces, and define the *speed* of a configuration C to be the maximum speed attained by any trajectory between two translates represented by C.

Auslander, Benjamin, and Wilkerson proved in 1993 the following: the maximum speed of any configuration C is 1, and that only three configurations (called *speed of light* configurations) attain this speed in $\mathbb{Z}^n$ for $n \geq 1$ [Auslander et al. 1993]. These configurations are:

- The *atom* $\{x\}$ (if $p = 1$).
- The *frog* $\{x, x + e_i\}$, $1 \leq i \leq n$ (if $p = 2$).
- The *serpent* $\{x, x + e_i, x + e_i + e_j, x + 2e_i + e_j\}$ $1 \leq i \neq j \leq n$ (if $p = 4$ and $n > 1$).

It was conjectured that the maximum attainable speed for any configuration on $p \neq 1, 2, 4$ is $\frac{2}{3}$, which we may observe is attained by the snake configuration with any number of pieces [Auslander et al. 1993]. We will show that in $\mathbb{Z}^2$, aside from these speed of light configurations, the maximum achievable speed is $\frac{2}{3}$.

3. Definitions and properties

Let $m \in \mathbb{Z}$ and placement $X \in \mathbb{Z}^n$. Then *border* l_m is defined by

$$l_m = \{x \in X : \|x\| = m\}.$$

For a placement X, we may define the *tail* (respectively, *head*) of X, by $t(X) = \min_u |l_u| \geq 0$ (respectively, $h(X) = \max_u |l_u| \geq 0$). Define the *width* of a placement X as $w(X) = h(X) - t(X) + 1$. Define the *back border* (respectively, *front border*) of X as $T(X) := l_{t(X)}$ (respectively, $H(X) := l_{h(X)}$). For example, if in the first diagram in Figure 1, the lower left piece is at $(0, 0)$, then its configuration has a tail of 0 and a head of 3.

We now define an underlying configuration which reoccurs in optimal play. A *ladder* of length $k > 0$ is subset of a placement X: $L = \{p_0, p_1, \ldots, p_k\} \subseteq X$ such

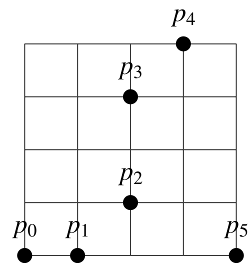

Figure 2. An example of a placement X which contains a ladder. The ladder is $\{p_0, p_1, p_2, p_3, p_4\}$, with p_0 is the base of the ladder, and pieces p_1, p_2, p_3, p_4 are the rungs. Here $\{p_0\} = T(X)$ and $\{p_4\} = H(X)$, so the ladder is a true ladder of X. X has width 8.

that p_0 is able to hop over $p_1, \ldots, p_k$ successively. If $\{p_0\} = T(X)$ and $p_k \in H(X)$, then we say L is a *true ladder* of X. We call the move consisting of p_0 jumping over the rest of the ladder pieces a *climb*, call p_0 the *base* of the ladder, and the other pieces the *rungs*.

Proposition 3.1. *If a configuration X contains a true ladder L, X has even width.*

Proof. Observe that when a piece p hops over another piece p', then p either increases or decreases its border by 2. Therefore since l_0 jumps from $B_{t(X)}$ to $B_{h(X)+1}$, this implies $h(X) + 1 - t(X)$ is even. □

Proposition 3.2. *A placement X with more than one piece can perform a move that simultaneously increases $t(X)$ and $h(X)$ if and only if it has a true ladder.*

Proof. ($\Leftarrow$) Performing a ladder climb increases both $t(X)$ and $h(X)$, as l_0 moves from $T(X)$, leaving that border empty, and jumps in front of $l_k \in H(X)$, thus advancing the front border.

($\Rightarrow$) If a move on X exists that advances the front and back borders forward, since only one piece can change positions, the back border must only have one piece, and it must be the piece which moves. Call this piece p. Since X has more than one piece, $w(X) > 1$, so p must jump from $T(X)$ to in front of $H(X)$. Denote the sequence of pieces hopped over by p by $p_1, p_2, \ldots, p_k$. Since $p_k \in H(X)$, $\{p_0, p_1, \ldots, p_k\}$ is a true ladder. □

We may classify possible moves into seven categories.

- *Ascent*: A move that increases $h(X)$ and $t(X)$. This is necessarily a ladder climb.
- *Front push*: A move that increases $h(X)$ but not $t(X)$.
- *Back push*: A move that increases $t(X)$ but not $h(X)$.
- *Dead move*: A move that changes neither the tail nor head of X.
- *Front retreat*: A move that decreases the head of X.

- *Back retreat*: A move that decreases the tail of X.
- *Reverse ascent*: A move that decreases both the head and the tail of X.

For example, in the placement in Figure 2, p_0 climbing the ladder would be an ascent. p_4 shifting to the right would be a front push. p_0 jumping over p_1 and p_2 only would be a back push. p_5 shifting in any direction would be a dead move.

An ascent is necessarily a ladder climb for nontrivial placements. For a legal m-move trajectory $M = \{X_0, X_1, \dots, X_m\}$, where X_0 is a translate of X_M, define the *moveset* of M as a collection of moves $m(M) = \{x_0 \to x'_0, \dots, x_{m-1} \to x'_{m-1}\}$, where x_i is the location of the piece that moves in X_i, and x'_i is the location of the moved piece in X_{i+1}.

Proposition 3.3. *In any moveset, the total number of front pushes, back pushes, front retreats and back retreats which occur between two ascents must be even.*

Proof. Since ascents can only occur when a configuration has even width, the configurations immediately before and after any ascents must have even width. Therefore, the number of moves between two ascents which change the width parity must be even. The four listed move types are the only move types which change the parity of the width of a configuration, so the result follows. □

For a move trajectory M, let $\mathrm{A}(M)$ represent the number of ascents in $m(M)$, $\mathrm{FP}(M)$ represent the number of front pushes, $\mathrm{BP}(M)$ the number of back pushes, $\mathrm{DM}(M)$ the number of dead moves, $\mathrm{FR}(M)$ the number of front retreats, $\mathrm{BR}(M)$ the number of back retreats, and $\mathrm{RA}(M)$ the number of reverse ascents.

Now, define the *weight* $\omega(M)$ of a trajectory M or its corresponding moveset $m(M)$ as follows:

$$\omega(M) := \mathrm{A}(M) - \left(\tfrac{1}{2}\right) \cdot (\mathrm{FP}(M) + \mathrm{BP}(M)) - 2 \cdot \mathrm{DM}(M) - \left(\tfrac{7}{2}\right) \cdot (\mathrm{FR}(M) + \mathrm{BR}(M)) - 5 \cdot \mathrm{RA}(M).$$

This definition rewards (with higher weight) trajectories that make many efficient moves (e.g., ascents) and penalizes trajectories that use moves that make little forward progress or worse. For example, consider the trajectory where we begin with a snake with five pieces, located at points $\{(0,0), (1,1), (2,2), (3,3), (4,4)\}$, and in three moves (consisting of one back push, one ascent, and one front push), translate it to the points $\{(1,1), (2,2), (3,3), (4,4), (5,5)\}$. Such a trajectory, with speed $\frac{2}{3}$, would have weight $1 - \left(\frac{1}{2}\right)(1+1) = 0$. Additionally, call the coefficient corresponding to each move type the *move weight*. If a sequence of moves are performed, then the weight of the sequence is the sum of all the move weights. If we partition a trajectory $M = M_1 \oplus M_2 \oplus \cdots \oplus M_k$, then $\omega(M) = \omega(M_1) + \cdots + \omega(M_k)$.

Lemma 3.4. *A m-move trajectory M of a configuration C has speed greater than $\frac{2}{3}$ if and only if $\omega(M) > 0$.*

Proof. Since the move types are mutually exclusive, $\mathrm{A}(M)+\mathrm{FP}(M)+\mathrm{BP}(M)+\mathrm{DM}(M)+\mathrm{FR}(M)+\mathrm{BR}(M)+\mathrm{RA}(M)=m$. Additionally, the displacement of M can be characterized by

$$\mathrm{A}(M)-\mathrm{RA}(M)+\left(\tfrac{1}{2}\right)\times(\mathrm{FP}(M)-\mathrm{FR}(M)+\mathrm{BP}(M)-\mathrm{BR}(M)).$$

Therefore the speed of M is

$$\frac{\mathrm{A}(M)-\mathrm{RA}(M)+\mathrm{FP}(M)/2-\mathrm{FR}(M)/2+\mathrm{BP}(M)/2-\mathrm{BR}(M)/2}{\mathrm{A}(M)+\mathrm{FP}(M)+\mathrm{BP}(M)+\mathrm{DM}(M)+\mathrm{FR}(M)+\mathrm{BR}(M)+\mathrm{RA}(M)}.$$

It is straightforward to check

$$\frac{2}{3}<\frac{\mathrm{A}(M)-\mathrm{RA}(M)+\mathrm{FP}(M)/2-\mathrm{FR}(M)/2+\mathrm{BP}(M)/2-\mathrm{BR}(M)/2}{\mathrm{A}(M)+\mathrm{FP}(M)+\mathrm{BP}(M)+\mathrm{DM}(M)+\mathrm{FR}(M)+\mathrm{BR}(M)+\mathrm{RA}(M)} \iff 0<\omega(M). \qquad \square$$

We next introduce an important theorem which, as a corollary, demonstrates that most movesets do not have speed greater than $\frac{2}{3}$.

Theorem 3.5. *For any trajectory M with no two ascents occurring in a row and not both beginning and ending with an ascent, $\omega(M)\le 0$.*

Proof. After an ascent, if a front push, back push, front retreat, or back retreat occurs, this changes the width of the placement, and since prior to the first move the placement had even parity, for another ascent to occur, another one of the listed moves must occur.

Without loss of generality let us assume M begins with an ascent, therefore it cannot end with one. Let us partition the moveset $m(M)$ into separate blocks $B_1, \ldots, B_k$ where each block begins with an ascent and contains no other ascents. Since no two ascents can occur in a row, each block must consist of at least two moves. Additionally since each new block must begin with an ascent, a block must end with a placement with even width.

B_i contains one ascent and at least one other move a. If a is any type of move besides a front or back push, $\omega(B_i)$ is negative. Otherwise, if a is a front push or back push the resulting placement has odd width and another front push, back push, front retreat, or back retreat must occur. This implies $\omega(B_i)\le 0$, with equality holding only when B_i is of the form $\{\mathrm{A}, \mathrm{FP}\,/\,\mathrm{BP}, \mathrm{FP}\,/\,\mathrm{BP}\}$. Since $B_i \le 0$ for all $1\le i\le k$, $\omega(M)\le 0$, as desired. $\square$

Since we know that the maximum speed of a configuration is 1, and we are looking at nonmaximal configurations, every moveset has at least one nonascent move. Therefore, we will assume for the remainder of this paper that the last move in any moveset is not an ascent.

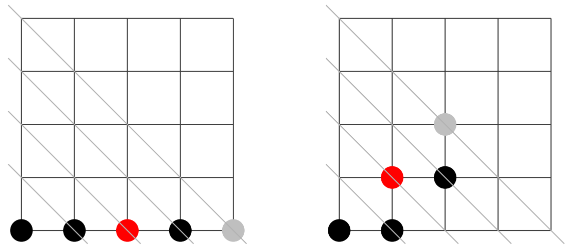

Figure 3. In the case of $p = 4$, we cannot place p_3 without a contradiction.

4. $p = 1, 2, 4$

For a configuration which is not speed of light, observe that it may reach speed arbitrarily close to speed 1 by first shifting into a speed-of-light configuration, repeating its set of moves sufficiently long, then moving back to the original configuration. We will consider only trajectories which do not use this strategy. Rigorously, if a trajectory between translates does not perform any moves corresponding to any "speed-of-light" configuration's optimal moveset, call the corresponding configuration *nonspeed-of-light*.

Theorem 4.1. *A nonspeed-of-light configuration C with* 1, 2, *or* 4 *pieces cannot have speed greater than* $\frac{2}{3}$.

Proof. When $p = 1$, this is immediately true, since there is only one configuration, and it can translate itself with speed 1 or -1.

For $p = 2$, any trajectory containing a jump (which must be an ascent) must contain the frog's optimal trajectory (a jump), therefore any nonspeed-of-light configuration is limited to only shifts. Therefore, the optimal trajectory of C cannot contain any ascents, thus C cannot have speed greater than $\frac{2}{3}$.

For $p = 4$, if C has optimal trajectory without two ascents in a row, C has speed at most $\frac{2}{3}$. Suppose then that C has two ascents in a row, without loss of generality let us assume its trajectory begins with two ascents. C must have even width, and every border must contain a piece due to the parity of the ends of each successive ladder. Since C can perform two ascents in a row, it has width 4. $C = \{p_1, p_2, p_3, p_4\}$ with p_i on l_i. Say p_1 has jump $p_1 :\xrightarrow{p_2} a_1 \xrightarrow{p_4} a_2$ for open locations a_1, a_2. $d(p_1, p_2) = 1$, $d(a_1, p_2) = 1$, and $d(a_1, p_4) = 1$. Similarly, write the next move by p_2 as $p_2 :\xrightarrow{p_3} b_1 \xrightarrow{p_1 = a_2} b_2$. $d(p_2, p_3) = 1$ $d(p_3, b_1) = 1$, and $d(a_2, b_1) = 1$.

Suppose p_1, p_2, and p_4 are collinear (p_1 on x, p_2 on $x + u_i$, p_4 on $x + 3u_i$), $a_2 = x + 4u_i$ and $a_1 = x + 2u_i$. However for p_2 to jump over p_3 and p_1, p_3 must occupy a_1, contradicting the assumption that a_1 is open. Say p_1 starts on x, p_2 on $x + u_i$, then a_1 is $x + 2u_i$, p_4 starts on $x + 2u_i + u_j$, and a_2 is $x + 2u_i + 2u_j$. There are only two locations both adjacent to p_2 and 2 away from a_2, a_1 and $x + u_i + u_j$. However if p_3 is on $x + u_i + u_j$ and we perform the two ascents, we have performed the serpent configuration's trajectory, a contradiction. These cases are visualized in Figure 3. Thus it is impossible for a nonspeed-of-light configuration C to have two ascents in a row, so C must have speed at most $\frac{2}{3}$. □

5. $p = 3$

Theorem 5.1. *No configuration of three pieces C exists with speed greater than $\frac{2}{3}$.*

Proof. To show this, we will demonstrate that no placement X exists for $p = 3$ such that two successive ascents are possible. If X has two or three pieces occupying the same border, then the back or front border has two pieces, rendering successive ascents impossible. Otherwise assume X has pieces occupying all different borders. The only possible way for X to be able to perform a ladder climb is if it has width 4, since a piece jumping over two other pieces can travel distance at most 4. Let us consider the four borders passing through X, without loss of generality say $l_1 - l_4$. l_1 and l_4 must contain one piece each, p_1 and p_3 respectively, implying the last piece, p_2 can either lay on l_2 or l_3. If p_2 lies on l_3, p_1 cannot jump. Otherwise, p_2 lays on l_2. If p_1 can perform an ascent, the pieces now lay on l_2, l_4, and l_5, which implies p_2 cannot jump, as in Figure 4.

Therefore no placement X with $p = 3$ pieces exists such that two consecutive ascents can be performed, as desired. No such configuration of $\mathbb{Z}^n$ exists with speed greater than $\frac{2}{3}$. □

6. $p > 4$ for $\mathbb{Z}^2$

We provide a proof that no configuration of greater than four pieces has a speed greater than $\frac{2}{3}$ in the 2-dimensional case. We are confident that the approach taken here will produce the same sort of result in $\mathbb{Z}^n$, but we shall not pursue that here. Recall that if a moveset has no consecutive ascents, by Theorem 3.5, the moveset has speed at most $\frac{2}{3}$, so our focus will be on movesets that have consecutive ascents.

Lemma 6.1. *For $p > 4$, there does not exist a configuration with a moveset containing four or more consecutive ascents.*

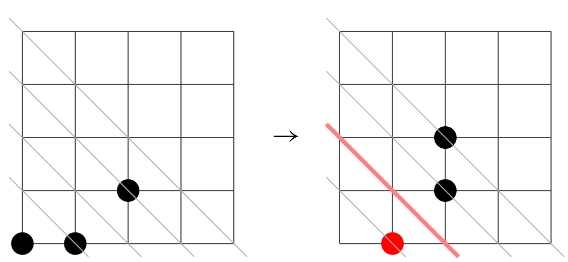

Figure 4. After a jump, p is isolated and cannot jump.

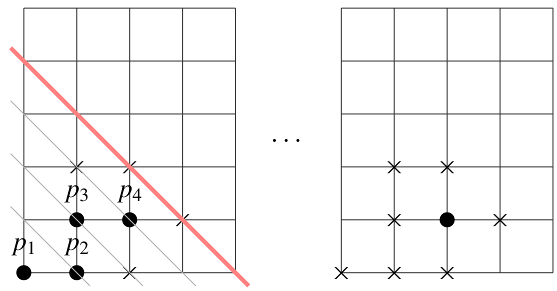

Figure 5. The back borders of C before any moves on the left, and after the first three ascents on the right. An $\times$ indicates a location a piece cannot be located in that board state.

Proof. Suppose for contradiction that there exists a configuration C with moveset containing more than three consecutive ascents. Then there can only be one piece on each of the four backmost borders. Additionally, the width of C prior to moving must be at least 6. It follows that without loss of generality, we can assume the first four pieces which ascend are located at $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(2, 1)$, as in Figure 5.

However, after the first three ascents, the piece at $(2, 1)$ p_4 is not adjacent to any pieces and therefore cannot ascend, a contradiction. $\square$

Define a piece's *measure* by taking its location modulo 2, $\overline{(x, y)}_2$. For example, p_4 in the above diagram has measure $(0, 1)$. Note that when a piece jumps, its measure stays constant. This restricts the number of locations in $\mathbb{Z}^2$ a piece can jump to, given its starting position.

We assume without loss of generality that a moveset M begins with the maximum number of ascents. Note that this implies M ends in a nonascent. Define an *isolating partition* of M as follows. First, partition $m(M)$ sequentially into blocks $A_1, \ldots, A_k$ such that each block begins with two or more consecutive ascents, but does not have consecutive ascents anywhere else and does not end with an ascent. So each new block begins at every occurrence of a sequence of two or more consecutive ascents, and ends with a nonascent. This partition of M is unique. For example, if a configuration had moveset of type {A, A, A, FP, DM, BP, A, A, DM, DM, A, FP, BP}, then the moveset would be partitioned into blocks $A_1 =$ {A, A, A, FP, DM, BP}, $A_2 =$ {A, A, DM, DM, A, FP, BP}.

Let $L(A_i)$ be the number of ascents A_i begins with. In our example, $L(A_1) = 3$ and $L(A_2) = 2$. By Lemma 6.1, $L(A_i) \le 3$. Since A_i ends with a nonascent, $\omega(A_i) < L(A_i)$. We wish to show $\omega(A_i) \le 0$ for all i, since $\omega(M) = \omega\big(\sum A_i\big) = \sum \omega(A_i)$. Hence if $\omega(A_i) \le 0$ for all i, then $\omega(M) \le 0$. So it suffices to only consider blocks rather than entire movesets. Since A_i can only begin with two or three ladder climbs, we only have these two cases to consider.

Call a block A_i for which $\omega(A_i) \leq 0$ *suboptimal*. Recall that:

Lemma 6.2. *If a consecutive sequence of moves $S \subset A_i$ satisfies $\omega(S) + L(A_i) \leq 0$, then necessarily, $\omega(A_i) \leq 0$, and thus A_i is suboptimal.*

Proof. We may assume without loss of generality that S does not begin with an ascent, since if it does, then removing the initial ascents produces another consecutive sequence of moves satisfying the same condition. We may also assume that the move following S is an ascent or the end of the block, since otherwise, we may extend S until the move following it is an ascent.

Now, partition A_i into smaller blocks in the following way: let the first block contain all consecutive initial ascents. Then, until the beginning of S is reached, partition the moves so each block begins with a nonascent and continues until an ascent is reached, and let the block end with an ascent. Each of these blocks will contain exactly one ascent and contain at least one nonascent, since A_i only contains one sequence of consecutive ascents. By assumption, the last move before S begins is an ascent, so this partitioning continues until S is reached. Let S be the next block. Then, partition the remaining moves of A_i in the same way as in the proof of Theorem 3.5, that is, by letting each block begin with an ascent and continuing until another ascent is reached. These blocks will also contain only one ascent.

By the same arguments used in the proof of Theorem 3.5, all blocks B which are not the first block and S satisfy $\omega(B) \leq 0$. Since the only blocks remaining are S and the first one, which has weight $L(A_i)$, by summing the weights of each block, we conclude that $\omega(A_i) \leq 0$. □

Call S a *suboptimal sequence of moves*, and call any consecutive sequence of moves S that is not suboptimal *optimal*. For example, if two ascents were followed by four front pushes, the four front pushes would be a suboptimal sequence of moves, and the block A_i would therefore be suboptimal. Our strategy is to show that any block A_i must contain a suboptimal sequence of moves.

Lemma 6.3. *If a block A_i begins with exactly three ascents, $\omega(A_i) \leq 0$.*

Proof. Observe that if A_i begins with exactly three consecutive ascents, the initial placement is forced to have a serpent configuration at the back, and after the three ascents, the resulting configuration is forced to have a serpent configuration in the front, as demonstrated in Figure 6.

By considering move weights, it suffices to demonstrate that a consecutive sequence of moves in A_i occurs with weight -3, that is, a suboptimal sequence of moves must occur. Since each nonascent has weight at most $-\frac{1}{2}$ if six nonascents occur between two ascents, then the condition is satisfied. Note that if five nonascents occur between two ascents, a sixth must occur by Proposition 3.1. Alternatively,

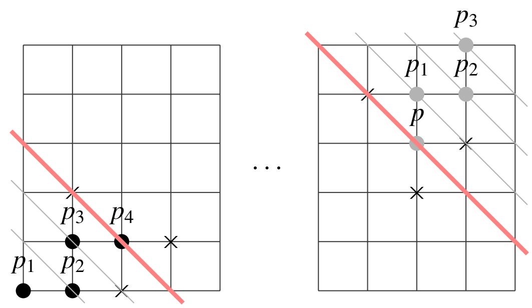

Figure 6. The borders before and after the first three ascents. The red borders denote forced placements. An $\times$ indicates a location a piece cannot be located in that board state.

if two dead moves occur or one dead move and one front/back push occurs, the condition is also satisfied for the same reasoning. Finally, any front retreat, back retreat, or reverse ascent is immediately suboptimal.

Call the border containing the backmost piece prior to moving l_1. We first consider two cases, if l_4 has exactly one piece or if l_4 has two or more pieces.

If l_4 has two or more pieces, first suppose the next ascent occurs from l_4. This implies that one of the pieces on l_4 must front push or dead move from l_4 first. Observe that a ladder climb to the front from l_4 cannot be performed unless p or p_2 moves, since the pieces from l_4 can only hop over p_1 and p_3 by parity. If a piece from l_4 dead moves, then since an ascent can still not occur, another move must be made before an ascent is possible, so since a dead move and one more nonascent has been performed, there is a suboptimal sequence of moves.

Therefore, a possibly optimal sequence of moves must have a piece from l_4 make a front push before the other performs the ascent. A front push consists of a ladder climb from l_4 to beyond the front border, but one of p or p_2 must move before that is possible. If p or p_2 performs a dead move, then after the ladder climb from l_4, a suboptimal sequence of moves will be forced to occur before the next ascent.

If p or p_2 ladder climbs to perform a front push, a ladder climb from l_4 still cannot be performed, so necessarily, a front push must occur by the piece that previously moved. After this, it may be possible for a piece, call it p' from l_4 to front push. The only optimal move afterwards is another front push, and since the remaining piece on l_4 can only hop over p_1 and p_3 in the front borders, for an ascent to be possible, p' must front push. If an ascent is not possible, then since four front pushes have occurred and another nonascent must occur, a suboptimal sequence of moves must happen. Otherwise, suppose an ascent from l_4 happens. The sequence of moves performed is A, A, A, FP, FP, FP, FP, A, and the sequence of moves after the consecutive ascents has weight -1. Now, if a new sequence of moves of weight -2 occurs, the total sequence will be suboptimal.

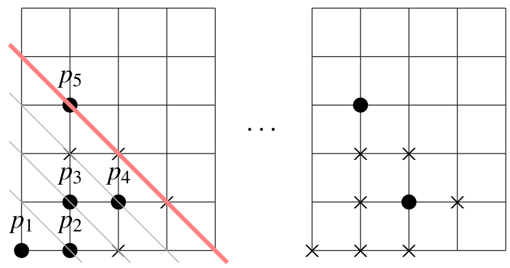

Figure 7. The left diagram is the initial configuration, and the right diagram is the configuration after the three consecutive ascents.

From the initial configuration, Figure 6, it is necessary that the pieces on l_4 were either farther than distance 2 away, or were distance 2 away, but must have hopped over separate pieces. Thus, there are two pieces on l_5. If the next ascent occurs from l_5, the border beyond p_3 is empty and requires a piece there to build a ladder. This border cannot be filled by a back push since l_5 has two pieces, and thus, a dead move must occur before the ascent, forcing a suboptimal sequence of moves. Otherwise, if the next ascent occurs from beyond l_5, since a ladder climb from l_5 to the front is impossible, a dead move must occur when bringing the pieces on l_5 forward, again, forcing suboptimality. We conclude that if l_4 began with two or more pieces, and the first ascent after the three initial ones occurred from l_4, there is a suboptimal sequence of moves.

Otherwise, suppose l_4 contains at least two pieces, and the first ascent after the initial three happens beyond l_4. In this case l_4 must be cleared, but neither piece can front push (a ladder climb to in front of p_3). If one piece dead moves forward and the other front pushes, then a suboptimal sequence of moves occurs. Otherwise, if no dead move occurs, then necessarily, one of p or p_2 front pushes twice to open the ladder, then a piece front pushes from l_4, and another back pushes to clear l_4. However by a previous argument, there must be at least two pieces on l_5, so another move must occur before an ascent can. Since five moves have occurred, suboptimality is forced. Thus, if l_4 began with at least two pieces, a suboptimal sequence of moves must occur.

Now, suppose l_4 has one piece, and let p be the original front border piece. The back of the initial configuration must necessarily must be in the situation in Figure 7.

If the next ascent occurs on l_4, then again, p is forced to dead move, and since p_4 is isolated, another piece must move to be adjacent to it. If these are separate moves, then two dead moves have been performed, a sequence of moves with weight -4. Otherwise, suppose these are the same move, so p_4 ascends right after. Then there is a new serpent configuration at the front, and two pieces on l_5, namely p_5 and the piece p_4 hopped over to ascend (which is necessarily p as in Figure 6), as pictured in Figure 8.

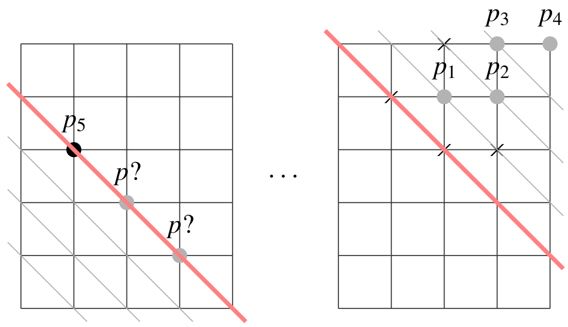

Figure 8. The configuration after p_4 ascends, in the case where l_4 has one piece and the next ascent follows from l_4. There are necessarily two pieces on l_5 and a serpent configuration in the front preventing an ascent from occurring.

Since the previous two moves have weight -1, it is enough to show the next sequence of moves must have weight -2. Thus it suffices to assume no dead move can occur. Note that the parity is correct for an ascent to occur, as the pieces on l_5 can hop over p_2 and p_4, but given the front of the configuration, no ascent can occur. At least two front pushes (or a dead move) must occur in the front borders before one of the pieces on l_5 can climb to the front. However, this is three front pushes, implying a fourth front or back push must occur before an ascent can occur, so there must be a sequence of moves with weight -2 if the next ascent is to occur from l_5.

Otherwise, if the next ascent happens beyond l_5, it is straightforward to see that a sequence of moves must first occur with weight -3, and we conclude that in any case, $\omega(A_i) \leq 0$. $\square$

Lemma 6.4. *If a block A_i begins with exactly two ascents, $\omega(A_i) \leq 0$.*

Proof. By considering move weights, it suffices to demonstrate that a sequence of moves in A_i after the two ascents occurs with weight -2, that is, a suboptimal sequence of moves must occur. In particular, it suffices to show either a dead move occurs or four front or back pushes occur between ascents. However, by Proposition 3.1, it suffices to show only three front or back pushes occur rather than four. First observe that the starting and ending configurations after the two ascents must be as follows at the front and back borders respectively, pictured in Figure 9.

Call the border containing the piece p_1 which is farthest back l_1. We begin by considering two cases, if l_3 has exactly 1 piece, p_3, or if it has multiple pieces p_3 and p_3'. Then, we consider subcases and determine that any trajectory must eventually become suboptimal.

First, suppose l_3 has at least two pieces initially, p_3 and p_3'. Let us consider the next two moves following the ascents. If either is a dead move or worse, the trajectory is suboptimal, so suppose the next two moves are front or back pushes.

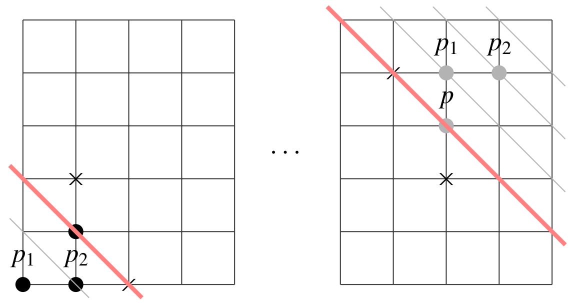

Figure 9. The borders before/after the ones indicated by red in the above diagrams are forced placements. A $\times$ indicates a location a piece cannot be.

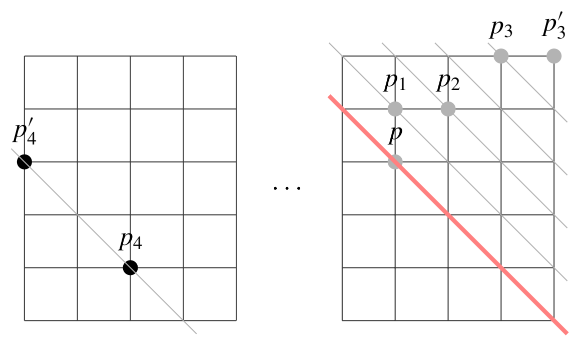

Figure 10. Right: the only possibly optimal trajectory if two pieces started on l_3, five moves in. Left: a corresponding possible placement on l_4 five moves in.

Since l_3 has at least two pieces, the first move must be a front push. If the front push is not performed by p_3 or p'_3, then the next move must also be a front push, but since the width of the configuration is odd after the first front push, p_3 or p'_3 could not have performed the front push. Since l_3 contains two pieces still, the next move is not an ascent, so the sequence of moves is suboptimal.

Otherwise, suppose the first front push was a ladder climb from l_3 to the front. Without loss of generality suppose p_3 makes the climb. If the second move is a back push from l_3, since there is a serpent configuration consisting of p, p_1, p_2, p_3, a ladder climb is not possible afterwards, and thus the trajectory is suboptimal.

Instead, suppose the second move is a front push. Since there is a serpent in the front, the only move which allows for a ladder climb (necessarily by p'_3) is p_3 shifting forward. Then, a ladder climb by p'_3 must finish by hopping to where p_3 was prior to shifting, then hopping over p_3. This implies p_3 and p'_3 originally had the same measure, so they were distance at least 4 away. Therefore, there must be at least two pieces on l_4 which p_3 and p'_3 hopped over, p_4 and p'_4, as pictured in Figure 10.

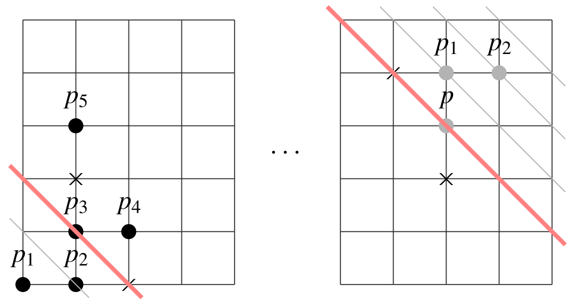

Figure 11. On the left, part of the necessary configuration if l_4 contains one piece, prior to the first two ascents. On the right, the front borders after the first two ascents.

By a parity argument, p_4 and p_4' cannot hop over p, p_2, or p_3, as they are on an even-numbered border. If the next ladder climb occurs from beyond l_4, then p_4, p_4', and p_3' all must move first, which is necessarily a sequence of moves with weight at maximum -2. Otherwise, if the next ladder climb occurs on l_4, the pieces in front of p which can be used in the ladder are p_1 and p_3'. However, a piece needs to move to the border between the ones containing p_2 and p_3, and by parity, this piece cannot come from l_4. Hence, this must be a dead move, and we conclude that if l_3 began with 2 or more pieces, then $\omega(A_i) \leq 0$.

Otherwise, suppose l_3 began with exactly one piece on it. We consider subcases based on where the next ascent in the trajectory occurs from and the number of pieces on the following borders. First, suppose the next ascent comes from l_4 or beyond. If l_4 contains more than one piece, then in order for the trajectory to be optimal, the next two moves must be a back push from l_3, then a front push from l_4. However, this is impossible, since a back push from l_4 must be a ladder climb, but by a parity argument of where the frontmost border is, a ladder climb from l_4 cannot advance the front border.

Otherwise, suppose the next ascent comes from l_4 or beyond and that l_4 contains exactly one piece. The necessary starting configuration is pictured in Figure 11.

Then for an optimal trajectory, the next two moves following the two ascents must be a back push from l_3 and a front push elsewhere (by assumption an ascent is not allowed). Assume the same piece does not perform the two moves, since otherwise, an ascent would be a faster trajectory, which would reduce to the previous lemma. Note that the front push cannot occur from l_4 by parity. Moreover, the back push from l_3 must hop over the sole piece on l_4, because otherwise, after the two moves there would be two pieces on l_4.

Finally, note that the front push must necessarily come from p_2 if an ascent is possible afterwards, since it is clear that a front push from p_1 prevent ascents, and front pushes from any other piece result in a serpent configuration at the front which

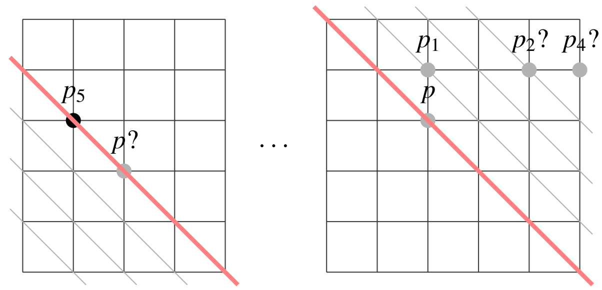

Figure 12. The configuration after p_4 ascends. The question marks denote where piece locations are not forced.

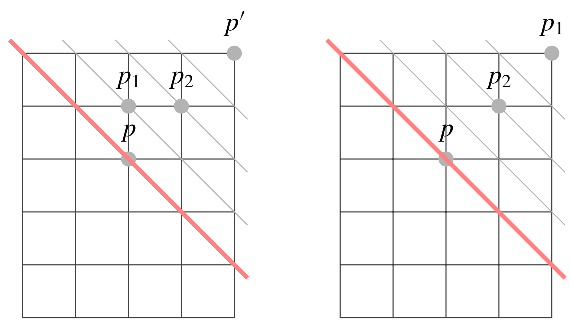

Figure 13. Note that the location of p' on the left or p_1 on the right are not uniquely determined.

prevents an ascent. Hence, after the front and back push, if an ascent is possible, there must be two pieces on l_5: the piece which p_2 hopped over in its ascent, and the piece l_4 is to hop over first in its ascent (this may be p_3). After p_4 ascends, we now are in the configuration pictured in Figure 12.

Now, for the trajectory to stay optimal, the next ascent must occur from l_5, since otherwise it would take two moves to clear l_5 and at least one to alter the front borders so that an ascent is possible. Pieces from l_5 may only hop over p and p_4 in the front borders, so they must be in the next ladder. This implies that in the next two moves before the ascent, a piece must move to the border between p_1 and p_2. However, this must be a dead move or worse, since l_5 contains at least two pieces and thus cannot initiate a back push. Thus, all movesets for which the next ascent comes from l_4 or beyond are suboptimal.

Finally, we consider the case where the next ascent comes from l_3, and l_3 contains only one piece. Since an ascent cannot immediately occur, the next two moves if they are to be optimal must be front pushes. It is quick to see that the only possibilities are either p_1 pushing twice or another unlabeled piece coming from an odd-numbered border pushing twice, as pictured in Figure 13.

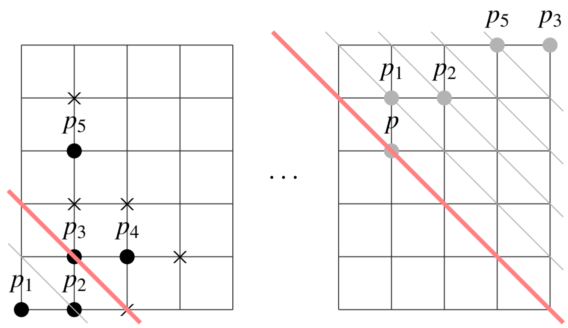

Figure 14. On the left, the necessary setup if l_4 and l_5 only have one piece, and on the right, in the lone scenario where l_5 front pushed before p_3 ascended.

If p_3 cannot ascend, the sequence of moves must be suboptimal, so assume p_3 can ascend, the third ascent so far in the moveset. We now consider cases based on which border the fourth ascent occurs from. If the fourth ascent occurs from l_4 or any even numbered border, these pieces cannot hop over p, p_2, or $p'(p_1)$ in the left-hand (resp. right-hand) cases.

In the left-hand case from Figure 13, before the next ascent, a piece must move to the border between p_2 and p', p_2 must move for p_1 to be free to be hopped over, and p' must move so the frontmost piece can be hopped over. These must necessarily be performed as front or back pushes for optimality, however if these are performed as front or back pushes, this is at least three separate moves, making the sequence suboptimal.

In the right-hand case from Figure 13, pieces must move to the border between p and p_2 and the border between p_2 and p_1, and p_1 must move so the frontmost piece can be hopped over. Again, these must be performed as front or back pushes for optimality, but as front or back pushes, this is at least three separate moves, making the sequence suboptimal.

Finally, suppose the fourth ascent occurs from l_5 or beyond. There is necessarily at least one piece on l_4 which must front push, and p_3 must move to open the ladder it climbed. If there are two pieces on l_4, then there are three moves that must occur which is suboptimal, so suppose there is only one piece on l_4, p_4, which both p_3 and p_1 hopped over. By similar arguments to earlier, p_4 cannot hop forward, it must shift to l_5. Now, if there are two or more pieces on l_5, then suboptimality is forced, as one must move before the ascent, which totals three moves before the ascent. However, it is possible that l_5 contains only one piece, p_4, as p_5 as shown in Figure 14 could have been the piece to front push prior to the ascent of p_3.

In this case, then p_4 can ascend and optimality is still preserved. However, there must be two different pieces on l_6, the piece p_5 hops over when it front pushes, and the piece p_3 hops over after hopping over p_4 when it ascends. By a similar argument as earlier, the next ascent occurring from l_6 forces suboptimality, and it is straightforward to see that at least three unique moves must occur for the next ascent to occur from l_7 or beyond. Thus, if after the initial two ascents, the next ascent occurs from l_3, and l_3 only contained one piece, suboptimality is forced. We have exhausted all cases, and conclude that $\omega(A_i) \leq 0$. □

Theorem 6.5. *If C is a configuration with five or more pieces, then C has speed less than or equal to $\frac{2}{3}$.*

Proof. Consider any m-move trajectory M of C. If M has no consecutive ascents, then Theorem 3.5 implies M has speed at most $\frac{2}{3}$. Otherwise, perform an isolating partition of $M = \sum_{i=1}^{n} A_i$. By the two previous lemmas, $\omega(M) = \omega\left(\sum A_i\right) = \sum \omega(A_i) \leq 0$, as desired. □

Acknowledgments

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sakmille@ucsc.edu *Department of Mathematics, University of California Santa Cruz, Santa Cruz, CA, United States*

benjamin@hmc.edu *Department of Mathematics, Harvey Mudd College, Claremont, CA, United States*