Uniformity, Coherence, and Regularity

Preliminary report

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Introduction In 1970 G. Grätzer [1] showed that a variety of algebras is regular if and only if the free algebra on 3 generators is regular. Recently Kaarli and Pixley proved the following specialized description:

> ([4],Lemma 3.5)*If an arithmetical variety is generated by a finite minimal algebra which is regular, then the variety is regular.* (An algebra is **minimal** if it has no proper subalgebras.)

The purpose of the present note is to present the following companion result for congruence uniformity and coherence.

Theorem *If an arithmetical variety is generated by a finite minimal algebra which is congruence uniform, then the variety is both congruence uniform and congruence coherent.*

From the congruence distributivity of the variety it follows that the subdirectly irreducible members are homomorphic images of a minimal generating algebra. Thus the proof of Lemma 3.5 actually required only that the minimal generating algebra $\mathbf{A}$ satisfy the condition:

> a) for each meet irreducible congruence θ of $\mathbf{A}$, if μ is the unique cover of θ, then no μ-class consists of a single element.

Likewise the proof of the present Theorem will require only that $\mathbf{A}$ satisfy:

> b) for each meet irreducible congruence θ of $\mathbf{A}$, if μ is the unique cover of θ, then all μ-classes are the same size.

Background An algebra is **congruence regular** if no pair of distinct congruence relations share a congruence class; an algebra is **congruence uniform** if for every congruence relation θ, all θ-classes have the same cardinality. A variety is congruence regular (uniform) if each algebra in the variety is congruence regular (uniform).

Because for varieties regularity is equivalent to the condition that no nonzero congruence has a singleton congruence class, every congruence uniform variety is obviously congruence regular, which is reflected by the fact that condition b) above obviously implies a). (Though not relevant for our present concerns it is also easy to see that every finite congruence uniform algebra is also congruence regular, and that for finite congruence uniform algebras, homomorphic images are also congruence uniform.)

An algebra **A** is **congruence coherent** if for each sub-universe B and congruence $\theta \in Con\mathbf{A}$, if B contains some θ-class, then B is the union of θ-classes. Varieties are coherent if the algebras in the variety are coherent. It is known that for varieties, congruence coherence implies regularity.

Recall [2] that a variety V is **affine complete** if all congruence compatible operations of algebras in V are polynomials. The study of affine complete varieties has been inspired by the early observation by G. Grätzer that the variety of Boolean algebras is affine complete and the subsequent discovery that many properties of Boolean algebras are inherited in some more general form by affine complete varieties. While on the one hand no strictly algebraic characterization of affine complete varieties is known, it is important to know that on the other hand *arithmetical* affine complete varieties of finite type are precisely those arithmetical varieties generated by some finite minimal algebra and we tend to use these descriptions interchangeably. [2], Section 4.2.

The proof of our Theorem uses the arithmeticity of the variety to construct a certain compatible function by induction on the congruence lattice of the minimum generating algebra. This is the method described in the proof of Theorem 2.2.6 of [2] and as adapted in the proofs of Theorems 3.2 of [3] and 3.2 of [4].

In addition several of the general properties of affine complete varieties referred to above are used in the proof; in order to make the proof more accessible we gather together some of these as follows:

P1 All affine complete varieties V have the properties:

- residual finiteness,
- congruence distributivity,
- V if **B** is any subalgebra of any algebra $\underset{\sim}{A} \in V$, then $V\mathbf{B}) = V(\mathbf{A})$. An immediate consequence is that no non-trivial algebra in V has a one-element subalgebra. ([2], Section 4.2) (This property is usually called semi-degeneracy.)

- the properties of being *finitely generated*, *locally finite*, and equivalent to a variety of *finite type* (i.e.: only finitely many basic operations) are all equivalent. ([2], Section 4.4)
- if V is of finitely generated then it is generated by a unique *minimal algebra* (i.e.: a finite algebra $\mathbf{A}$ of more than one element and having no proper subalgebras), which is also the largest minimal algebra in the variety. This generator is constructable from any finite generator and is characterized by the property of being *weakly diagonal*, i.e.: every subuniverse of A^2 contains the graph of some automorphism of $\mathbf{A}$.([3], Section 2)

P2 If $\mathbf{A}$ is a weakly diagonal algebra and $t(x, \ldots, z, a)$ and $u(x, \ldots, z, a)$ are polynomials containing only the element $a \in A$ and $t(x, \ldots, z, a) = u(x, \ldots, z, a$ for all $x, \ldots, z \in A$, then for some unary term $s(x)$, $t(x, \ldots, z, s(x)) = u(x, \ldots, z, s(x))$ is a term identity of $\mathbf{A}$.

Thus the polynomial identities of $\mathbf{A}$ determine the equational theory of $\mathbf{A}$. [3]), Section 2.)

A ternary function $u(x, y, z)$ is a **minority function** on a set A if

$$\text{for all } x, y \in A,\ u(x, x, y) = u(x, y, x) = u(y, x, x) = y \quad (min)$$

For example, if p is any Pixley function and m a ternary majority, then

$$u(x, y, z) = m(p(z, x, y), p(z, y, x), p(x, z, y)) \quad (\dagger)$$

is a minority function.

Proof of the Theorem We let $\mathbf{A}$ be a minimal generator of V which satisfies the uniformity hypothesis of condition b) of the **Introduction**. We show that V is both uniform and coherent. We will use the same proof strategy as was used to prove Theorem 3.2 of [3] and Theorem 3.2 of [4] and hence will require our hypothesis for the largst minimal algebra $\mathbf{M}$ in V. But since the subdirectly irreducible members of V are homomorphic images of both $\mathbf{A}$ and $\mathbf{M}$, condition b) transfers to $\mathbf{M}$. Henceforth we shall therefore assume that $\mathbf{A}$ is the largest minimal algebra of V.

We shall first prove, by induction on the height m of $Con(\mathbf{A}/\theta)$, for every $\theta \in Con\mathbf{A}$, the statement S_m:

There exists a system of functions

$$\{u_\theta \mid \theta \in Con\mathbf{A},\ \text{height}(\theta) \leq m\},$$

such that every u_θ is a minority function on A/θ which, for fixed $x/\theta, y/\theta$, as a function of z/θ , is an involution on $\mathbf{A}/\theta$:

$$u_\theta(x/_\theta, y/_\theta, u_\theta(x/_\theta, y/_\theta, z/_\theta)) = z/_\theta. \quad (inv)$$

The system is compatible in the sense that each of the functions induces on each previous level the functions belonging to that system. (Notice that if p is the discriminator, u defined by (†) is an involution.)

To prove S_1 ($\mathbf{A}/\theta$ is simple), we define u_θ by $u_\theta(x/\theta, y/\theta, z/\theta) = z/\theta$ if $x/\theta, y/\theta, z/\theta$ are all different, and as a minority function otherwise. Then u_θ is obviously a minority function which is an involution as a function of z/θ. (If p were a principal function, then since $\mathbf{A}/\theta$ is simple so p is the discriminator, we could take u_θ to be the term (†). Of course there are other involutions, and this is important later in the proof.)

We proceed with the induction step just as in the proof of Theorem 3.2 of [4]. Thus we first suppose that ρ has more than one cover in $Con\mathbf{A}$. Then as before u_ρ is uniquely determined by the functions u_σ with $\rho < \sigma$. Since each of these is a compatible minority function satisfying (inv), the same is true of u_ρ on $\mathbf{A}/\rho$.

Next suppose ρ has only one cover in $Con\mathbf{A}$ so $\mathbf{A}/\rho$ is subdirectly irreducible; let the cover of ρ be μ. To simplify notation, without loss of generality we may take $\rho = 0_A$; so that $a/\rho = a$ for all $a \in A$, $\mathbf{A}$ is subdirectly irreducible, and $u_\rho = u$ when defined.

For $x, y, z \in A$ the system of congruences

$$\begin{aligned} w &\equiv z \ (Cg(x,y)) \\ w &\equiv x \ (Cg(y,z)) \\ w &\equiv y \ (Cg(x,z)) \end{aligned}$$

is pairwise compatible so, by arithmeticity, always has solutions $w \in A$. (The formula (†) provides one solution.) The set of all solutions is $w/Cg(x,y) \cap w/Cg(y,z) \cap w/Cg(x,z)$. If any two of x, y, z are equal, this set contains only one element; otherwise, since **A** is subdirectly irreducible, it contains w/μ. We will choose $u(x,y,z)$ in this set. From this it follows that any such choice will determine $u(x,y,z)$ as a minority function. Also, each element of $u_\mu(x/\mu, y/\mu, z/\mu)$ is obviously a solution of the system of congruences. Hence, since all μ classes are the same size, for each fixed pair x, y we can choose $u(x,y,z) \in u_\mu(x/\mu, y/\mu, z/\mu)$, so that, as a function of z, it is an involution between z/μ and $u_\mu(x/\mu, y/\mu, z/\mu)$. In this way $u(x,y,z)$ is defined for all x, y, z in A and is a compatible minority function satisfying (inv).

Now since **A** is affine complete and u is congruence compatible by construction, it is a polynomial which is a minority function which satisfying (inv). Therefore, since **A** is the largest minimal algebra of V, by **P1** it is weakly diagonal and hence by **P2** there is a minority term, which we also call $u(x,y,z)$, which also satisfies (inv) in V. Hence for any $\mathbf{A} \in V$, since u is a minority function, for each $x, y \in A$ and $\theta \in Con\mathbf{A}$, it induces a bijection between x/θ and y/θ. We conclude that V is congruence uniform.

Finally,for any $\mathbf{A} \in V$ by (inv) we have $u(x,y,x/\theta) = y/\theta$ for any $\theta \in Con\mathbf{A}$ and, since $u(x,y,z)$ is a term, y/θ is contained in any subalgebra containing x/θ and y; thus V is also congruence coherent. •

If **A** is a finite simple algebra in an arithmetical variety, it obviously satisfies all of the conditions described by Theorem 3.2 of [3], Theorems 3.1, 3.2, and Lemma 3.5 of [4] and the Theorem of the present note. Hence we have

Corollary *If* **A** *is a finite, simple, minimal algebra which generates an arithmetical variety V, the Pixley term t can be chosen so that the following conditions are satisfiedf by V and for every algebra* **A** *in V:*

a) $(x,y) \in Cg(u,v) \leftrightarrow t(u,v,x) = t(u,v,y)$, *(Equationaly definable principal congruences),*

b) V is congruence regular,

c) the principal congruences of **A** *form a sublattice of $Con\mathbf{A}$ with*

$$\begin{aligned} Cg(a,b) \vee Cg(c,d) &= Cg(t(a,b,c), t(b,a,d)), \text{ and} \\ Cg(a,b) \wedge Cg(c,d) &= Cg(m(a,b,c), m(a,b,d)), \end{aligned}$$

where m is the majoority term $m(x,y,z) = t(x, t(x,y,z), z)$.

d) V is both congruence uniform and congruence coherent.

For example, from the Corollary it follows at once that the variety of Boolean algebras, or the variety generated by any primal algebra has all of the properties $a), \ldots, d)$.

Comments: 1. In contrast to the proofs of both Theorem 3.2 of [3] and Theorem 3.2 of [4], this theorem does not require the construction of a "special" Pixley function, and, in fact the function does not even have to be principal.

2. The method of proof of this theorem has been used with variations ([3],[4]) to obtain conditions under which an arithmetical affine complete variety has any of the conditions $a), \ldots, d)$. A question: are there further properties of arithmetical affine complete varieties yet to be obtained using this method?

References

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