Infinitesimals from Archimedes to Galileo

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To my old friend Ellis Cumberbatch

This essay is the result first, of my learning some years ago about Archimedes ingenious use of infinitesimals and then of my consequent interest in the history of these mysterious entities and of their place in the history of science, and more recently, of my deeper appreiation of the crucial role played by them in two key discoveries of Archimedes and Galileo. As a result of my admittedly somewhat cursory study I am led to think that these discoveries are indicative of the greatest achievements of the human intellect. They are respectively, Archimedes' determination of the area of a segment of a parabola (the quadrature of the parabola), and Galileo's implicit discovery of the formula $s = \frac{1}{2}aT^2$ for the distance traveled by a body under constant acceleration a during elapsed time T. These discoveries are not important just for themselves, and as such are rather narrow and specialized, but more because they opened the door to a vast new array of possibilities and led directly to rise of science in the seventeenth century. In particular Galileo's discovery, for the first time ever as far as I know, gave a quantitative description of physical change, a quest which had stumped the Greeks and, though there were earlier rumblings, was an opening shot in the scientific revolution. In each case their discoveries amounted to inventing special cases of integral calculus. Also, in each of these cases infinitesimals were used and I don't see how they might have made their discoveries without them. This, in spite of the resistance to their use, particularly in Galileo's time but also, according to what we know, with uncertainty by Archimedes himself.

I have written this primarily as an exercise for my own benefit; if there would be any intended reader, I guess it could be anyone, e.g.: a student, with a slight knowledge of calculus.

Archimedes (287-211 BCE) was the greatest of the Greek mathematicians and as far as I can tell, the only one to make full blown use of infinitesimals in the way modern mathematicians would. Greek mathematics as well as mathematics overall gradually fell into decline after his death. It was not until Galileo (1564 -1642), when infinitesimals were again used to truly spectacular advantage from his time on. Galileo died the year Newton was born and thus the torch was passed and after a gap of more than 1800 years modern science was born or perhaps one should say reborn, and infinitesimals again played a vital role. Oddly, as we shall see later, Galileo's discovery was essentially also made in the 14th century by the French philosopher Bishop Nicole Oresme, so one might ask why didn't the scientific revolution ocurr 300 years earlier than it did?

First, what are infinitesimals (sometimes also called indivisibles)? Infinitesimals are like numbers both real and complex; since a complex infinitesimal z can be written as $z = x + iy$, for x and y real, we consider real infinitesimals, which can be either positive or negative (0 is not an infinitesimal), so what are the positive ones? They are conceived to be "numbers" greater than 0 and less than any positive real number. Now positive real numbers can be identified with decimal numbers so numbers of the form $\frac{1}{10^n}$, n a positive integer, grow smaller, but are never equal to 0, and can be made as close to 0 as we choose as n grows larger. So any positive infinitesimal must be less than $\frac{1}{10^n}$ no matter how large n is. It is as though there is a gap, containing all positive infinitesimals between 0 and the positive real numbers. Thus infinitesimals, within the realm of numbers as we know them are purely fictitious. (But then, of course, the real numbers themselves, conceived in their entirety, are also somewhat fictitious; but that's another story.) A great achievement of 19th century mathematics was the establishment of a rigorous foundation of mathematical analysis without infinitesimals and this became the standard approach, though we still teach our students to use infinitesimals as a heuristic device, thinking of the differential $f(x)dx$ as a strip of height $f(x)$ and (infinitesimal) width dx, knowing that we can justify ourselves by reverting to the modern epsilons and deltas if needed. In the 20th century through the development of a rich branch of mathematical logic known as model theory, the existence of infinitesimals has been established as an enlargement of the system of real numbers together with enough of the properties of the real numbers to justify their use in several areas of mathematics and in particular, mathematical analysis has been successfully enlarged to "Non-standard Analysis" to incorporate the completely rigorous use of infinitesimals. Thus freshman calculus can be presented via infinitesimals just as easily as via the traditional $\epsilon - \delta$ approach, though this is seldom done, so strong is the latter tradition.

Archimedes Method Now let us discuss Archimedes' quadrature of the parabola. First we paraphrase the following from the Wikipedia article on Archimedes Method [1]:

"The Method of Mechanical Theorems, also referred to as The Method, is considered one of the major surviving works of Archimedes. The Method takes the form of a letter from Archimedes to Eratosthenes, the chief librarian at the Library of Alexandria, and contains the first attested use of infinitesimals.The work was originally thought to be lost, but in 1906 was rediscovered in the celebrated Archimedes Palimpsest. The palimpsest includes Archimedes' account of the 'mechanical method', so-called because it relies on the law of the lever, which was first demonstrated by Archimedes, and of the center of mass (or centroid), which he had found for many special shapes. See [2] for a fascinating account of the history of the palimpsest.

Archimedes did not admit the use of infinitesimals as part of rigorous mathematics, and therefore did not publish his method in the formal treatises that contain the results. In these treatises, he proves the same theorems by exhaustion, finding rigorous upper and lower bounds which both converge to the answer required. Nevertheless, the mechanical method was what he used to discover the relations for which he later gave rigorous proofs."

We want to get at the essence of the work so we present it in modern terms, using a coordinate system, (invented by Descartes, who was a contemporary of Galileo, though the two probably were unacquainted), using the simplest equation for a parabola, $y = x^2$, and determine the area of the region P of the plane bounded by the graph of this equation, the x-axis, and the vertical line $x = 1$. These concepts, whose discovery/invention were also great achievements in their own right on a par with Archimedes, are well known to us from our early education and their use will simplify the exposition of Archimedes' crucial idea without encumbering it with collateral details. For example, where Archimedes used triangles to approximate the area P, our use of rectangles greatly simplifies the argument without losing it's great central idea.

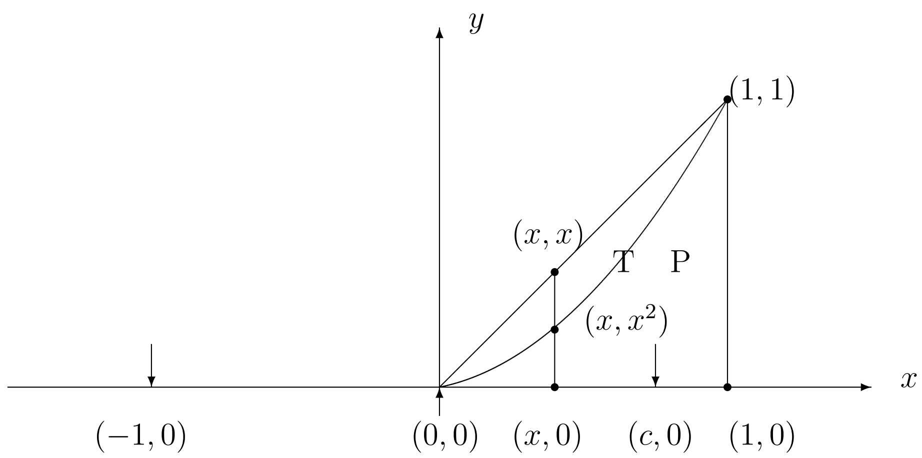

Figure 1

Referring to Figure1, we compare the region P with the region T bounded by the circumscribed right triangle with vertices $(0,0),(1,0),(1,1)$. Thus the graph of $y = x^2$ lies below the hypotenuse (the segment joining $(0,0)$ and $(1,1)$) (and thus P has area less than the area 1/2 of T). Also think of the regions P and T as consisting of lamina of uniform area density so their masses and areas can be identified and their centers of mass become geometric concepts and are usually referred to as centroids. Archimedes created this concept and used it often but we are not quite sure that his precise definition was exactly the same as ours since none of his surviving works explain it. After we complete the discussion of quadrature, we shall return to Archimedes' work on centroids. Meanwhile we assume our usual definition coincides with his.

Consider vertical line segments from $(x,0)$ to the points (x,x^2), and (x,x) on the graph of the parabola and the hypotenuse respectively. Following Archimedes' idea we think of these lines as thin strips of infinitesimal width and heights x^2 and x respectively. Taking this infinitesimal width as our unit of length, these infinitesimal strips then have area (and mass) of x^2 and x respectively. We then denote the total mass obtained by sweeping strips of mass x^2 through P from $x = 0$ to $x = 1$, by $A(P) = \Sigma x^2$. Further, imagine a fulcrum located at the origin $(0,0)$ and a pan balance centered at the fulcrum and with left pan at $(-1,0)$ and right pan at at some point $(c,0)$ on the positive x-axis. Imagine the mass $A(P)$ of P placed at the point $(-1,0)$, the left pan, so it will have moment counter-clockwise about the fulcrum of magnitude $A(P) \cdot 1 = A(P)$.

For the triangle, recall that it was proved in elementary geometry that the medians of a triangle, i.e.: the lines joining the vertices to the mid-points of their opposite sides, intersect in a point which is located at 2/3 of the distance from each vertex to the opposite side. Simple experiments, almost certainly performed by Archimedes, confirm that this point is the centroid of the triangle. (We shall describe how he further justified this later when we discuss centroids.)

Observe that for a point (x,x) on the hypotenuse of the triangle, the moment of a strip of height x and infinitesimal width and distance x from the fulcrum has magnitude $x \cdot x = x^2$ so that Σx^2 can also be taken to represent the magnitude of the total moment of the triangle clockwise about the fulcrum. But this moment also equals $A(T) \cdot c$ where $A(T)$ is the area of the triangle and c is the x- coordinate of the centroid of the triangle. Moreover the centroid of the triangle is located at a point whose x-coordinate is 2/3 and the area is $1/2$ so this total moment $A(T) \cdot c = 1/2 \cdot 2/3 = 1/3$. Since the moments of the triangle and parabola both equal Σx^2, the scales balance and we we have $A(P) = A(T) \cdot c = 1/3$. Eureka!

Archimedes corrected As remarked eralier, Archimedes apparently did not trust this ingenious "Method" as a more than a heuristic device and provided a rigorous proof as well using approximation by inscribed triangles. An account of his proof may be found in [2].

In the following we will take advantage of the benefit of coordinates, and use approximating rectangles instead of the triangles used by Archimedes. His proof that $A(P) = 1/3$ is then essentially the following, which can be found in any elementary calculus text. For each positive integer n and integers $k = 0, \ldots, n$, we introduce on the interval $[0, 1]$, $n + 1$ mesh points $x_k = k/n$. These divide $[0, 1]$ into n sub-intervals $[x_{k-1}, x_k], k = 1, ..., n$, each of width $1/n$. With bases $[x_{k-1}, x_k]$ inscribe inner rectangles of height x_{k-1}^2 and outer rectangles of height x_k^2. Denoting the sums of all of the inner and outer rectangles by R_n^I and R_n^O respectively, and observing that for all n, $R_n^I < A(P) < R_n^O$ he shows (or would have if he were not using triangles) —we skip the technical details— that if the area is assumed to be a number a , and if $a < 1/3$ then for some n there is an inner sum $a < R_n^I < A(P)$, and if $a > 1/3$ for some n there is an outer sum $A(P) < R_n^O < a$, so $A(P)$ can be neither greater than nor less than $1/3$. Therefore, by the law of trichotomy, $A(P) = 1/3$.

Except for the much more subtle question of whether the concept of area is actually well-defined for curved regions such as P, an issue of some delicacy which would wait till the late 19th century for resolution, Archimedes is two millennia ahead of his time. He does acknowledge the practical benefits of his Method: it is often easy of use and it discovers the area, while the modern approach seems to requires one to know the area in advance. In a subsequent application of the Method he discovered that the volume of a sphere is $2/3$ the volume of the circumscribed cylinder. Archimedes regarded this discovery as his greatest accomplishment and supposedly asked that the diagram used in making it be inscribed on his gravestone.

Final remark on the origins of infinitesimals. Some authors have attributed the discovery of infinitesimals to several earlier mathematicians. Notable examples are the Greeks mathematician, Eudoxus and Democritus in the 4th century BCE and who died about half a century before the birth of Archimedes. Eudoxus is important in the history of mathematics, most notably for his theory of proportion and for the so-called Method of Exhaustion, whereby areas of plane figures or solids such as the area of a circle or volume of a sphere, are approximated by inscribing or circumscribing polygons of increasingly many sides converging on the given plane figure or solid. Democritus, also using Exhaustion, used what is now known as Cavelieri's Principle (of which more later), to find the volume of a cone. It is important that the Method of Exhaustion considers infinitesimals, if at all, only as "limiting quantities" in the context of convergence proofs. But the concept and utility of infinitesimals, so it seems to me, lies not in their approximation properties but rather in the idea that plane figures or solids actually consist of the union of infinitely many infinitesimal area or volume elements which can somehow be treated with standard computational methods to obtain areas or volumes, with no concept of convergence involved. It seems to me that this point of view is critical and that this idea of infinitesimals is truly a radical departure from previous practice, and as far as I can tell, Archimedes was the originator of this novel idea.

It is also important to emphasize that Archimedes' "Method" is a "Mechanical Method": it treats infinitesimal areas and moments as real in the sense of occurring in nature and regards them as amenable to the ordinary laws of physics. Thus Archimedes' genius is not bound by the severe restrictions of Euclid's rather sterile Platonic school of thought; he is far more imaginative, and to great effect!

Archimedes centroids How did Archimedes come upon such a radical idea as infinitesimals? For the answer we turn to his work on centroids. Archimedes sets forth certain basic properties of centroids in his treatise *On the equilibrium of planes*. For justification of these properties he unfortunately refers to his lost treatise *On levers*, so we can only conjecture how he justified them.

Before describing these properties we recall the basic concept of center of mass from a modern point of view. We will follow the delightful paper by T. M Apostol and M. A.Mnatsakanian, *Finding centroids the easy way*, [3].

The laws of classical Newtonian mechanics are normally stated for "point particles" as given by their position, velocity, and acceleration. Since naturally occurring objects are composite and are made from smaller objects, atoms and molecules, so to apply the laws of physics to such composite objects we replace the objects by single points—centers of mass— where all of the mass is concentrated. (Note that these "point particles", or "point masses", are infinitesimals by another name.) If the object consists of finitely many point masses $m_1, \ldots, m_n$ located in a plane or in space at positions described by vectors $\mathbf{c}_1, \ldots, \mathbf{c}_n$, then the center of mass of the object is defined by the vector

$$\mathbf{c} = \frac{m_1\mathbf{c}_1 + \cdots + m_n\mathbf{c}_n}{m_1 + \cdots + m_n}.$$

In case all of the masses are the same, then the individual masses can be factored from the numerator and canceled from the numerator and denominator, leaving

$$\mathbf{c} = \frac{\mathbf{c}_1 + \cdots + \mathbf{c}_n}{n},$$

which a purely geometric property called the *centroid.* If the object consists of all of the points of some region of a line , plane, or space, this formula is replaced by the ratio of two integrals, the total geometric moment divided by the total length, area, or volume, and of course this is what Archimedes is anticipating. Henceforth we restrict our attention to plane regions and refer to such regions as "lamina". A lamina is "symmetric" if there is a point, called the "center of symmetry", with the property that a line through this point intersects the lamina in equal segments in each direction from the point. A lamina has an "axis of symmetry" if there is a line with the property that perpendiculars to the line intersect the lamina in equal segments in each direction from the line.

Archimedes recognized these ideas and formulated certain properties of centroids which we list below. In particular what we call "Archimedes Lemma" resembles a proposition explicitly presented in his writing.

Basic properties of centroids:

- The centroid of a lamina having a center of symmetry is at the center of symmetry.
- The centroid of a lamina having axis of symmetry lies on the the axis of symmetry.
- Archimedes Lemma. If a lamina A consists of the union of lamina B and C then the centroid of A lies on the line segment joining the centroids of B and C.

These properties are probably intuitively clear and are commonly justified by thought experiments or by actual experiments (e.g.: hanging mobiles made from cardboard.) Their application enables us to determine centroids of complex non-symmetric lamina. Again following T. M Apostol and M.

A.Mnatsakanian we illustrate this with the following example of an L-shaped lamina (Figure 2).

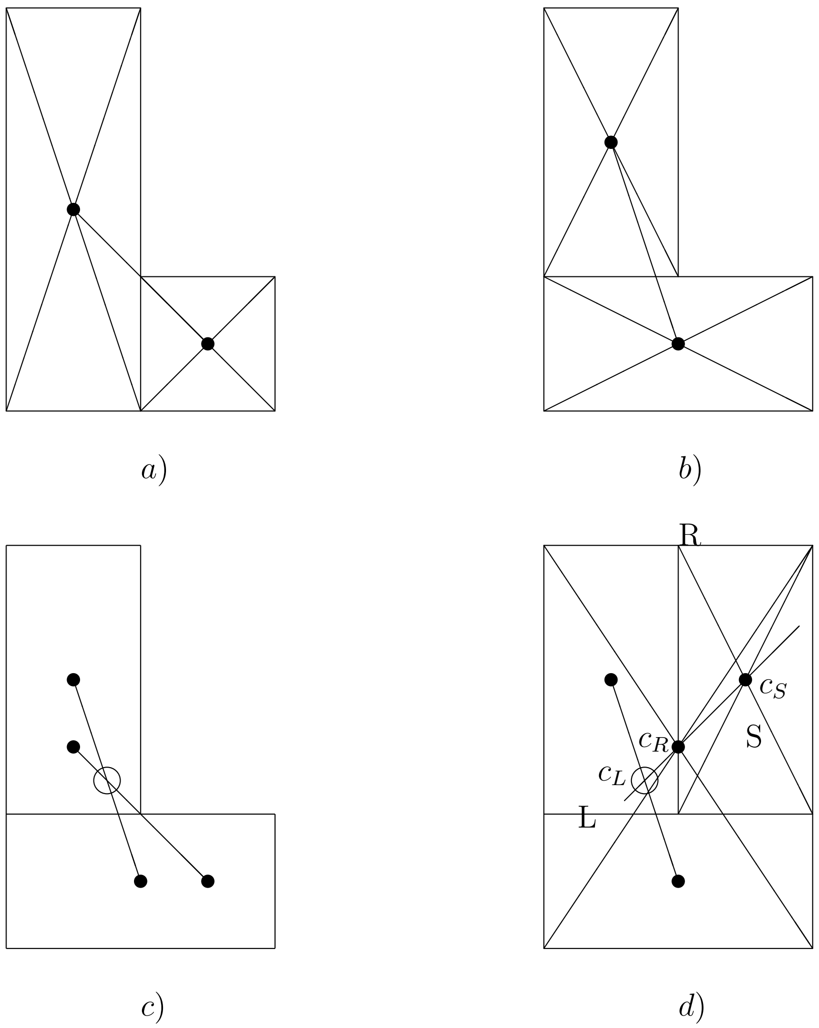

Figure 2

Paraphrasing [3], Figures 2 a) and b) show the L-shaped lamina L decomposed in two d different ways into rectangular pieces. In each case the centroid of L lies on the line segment joining the centroids of the two pieces. Consequently the centroid of L is at the intersection of these segments, as shown in Figure 2 c).

As a practical matter the intersection of the segments of Figures 2 a) and b) may be difficult to determine since they may be close to parallel. Figure 2 d) shows another way to determine the centroid of L: Enlarge L by adjoining the rectangle S to form the large rectangle R. By Archimedes Lemma the centroid c_R of R must lie on the line segment joining the centroids c_L and c_S of L and S respectively; but this line is determined by any pair of c_L, c_S, and c_R, and hence c_L lies on the line determined by c_S and c_R. Consequently c_L is at the intersection of this line with either of the segments of Figure 2 a) or b).

Similarly, ingeniously applying these properties of centroids, along with some others concerned with rotating or translating a lamina, Archimedes was able to determine centroids of a large number on non-symmetric lamina, for example, the center of mass of an arbitrary triangular lamina.

So back to our primary concern, which is: how did Archimedes justify these basic properties of centroids?

First we should recognize that for Archimedes the relation between the physical and mathematical worlds was probably different from ours and was probably rather blurred.. Plato's influence and in particular his theory of "ideas" and "forms" permeated Euclid's Elements and had to be familiar to Archimedes. But we are told that Plato rejected infinitesimals and we know that Archimedes famously blended mathematics and physics. So while Euclidean points had location but no extension, lines had extension but no width, and planes had extension in two dimension but no thickness, all of this was in the world of the Platonic "forms" and Archimedes somehow seems to have grasped the idea that certain lines, "infinitesimal" lines, could have extension but no "measurable" width, meaning that no real number could be assigned to their width.

The reason he came to this must have been in his justification of the principles of centroids above. The experimental evidence for lamina having a center or axis of symmetry is so compelling and, for example, in the case of a center of symmetry it is also so compelling to imagine each pair of equal segments extending in each direction from this center as being rectangles of infinitesimal width and centroids at the intersection of their diagonals, i.e.: at half their length, and thus exerting equal but opposite torques. Likewise for lamina having an axis of symmetry, the experimental evidence is so simple and compelling that it cries out for an equally simple mathematical justification; in this case each pair of equal segments perpendicular to the axis is conceived of as a pair of rectangles of infinitesimal width as before.

But this argument is essentially circular: all we have done is reduce the relevant cases to the special case of a symmetrical lamina consisting of a rectangle (of infinitesimal width.)

I think a possible way out of this dilemma is to recognize the role of atomism and the influence of its most cited founder, Democritus (460-370 BCE), and the influence of Epicurus (341-270 BCE) with whom Archimedes was roughly contemporary. Archimedes is considered to be inspired by Plato and Aristotle who were ardently anti-atomists, specifically through his familiarity with Euclid's work which in its strictly logical hierarchical structure is certainly antithetical to atomism. But, remarkably, Democritus is the only philosopher mentioned by Archimedes in his work while Plato and Aristotle are not mentioned at all! All of this, the product of recent scholarship, together with the fact that Archimedes, in his published work, seems to have kept the source of his ideas hidden (as mentioned earlier, in his published proof of the area of the segment of the parabola, how did he know in advance that the area was 1/3?) leads us to conclude that Archimedes was a kind of "closet" atomist all along and that infinitesimals were justified for him, to the extent they could be, by being associated with atoms. In this way of thinking a point mass can be thought of as a single atom, a rectangle of infinitesimal width can be thought of as consisting of a row of atoms arranged linearly, a lamina of infinitesimal thickness as a lamina of atoms. Though it cannot be known for sure, I think it quite possible that this is roughly the way Archimedes thought. If this is so, that Archimedes regarded infinitesimals as "real" and outside the real number system, then in a way he was anticipating non-standard analysis. Of course, given that this would leave the problem of the compatibility of "real" physical atoms and the abstraction of the real number system (as understood at that time) , we can better appreciate his unwillingness to accept his proofs using the Method as fully justified.

Cavalieri An important contemporary of Galileo (1564-1642) was Cavalieri (1598-1647). As he was thirty years older, Galileo was at first something of a mentor to Cavalieri and even helped secure for him a position at the University of Bologna, and over the years Cavalieri maintained systematic scientific relations with Galileo. He wrote two important books, the first of which, *Geometrica*, which has been described as "a likely winner if there were a prize for obscurity", was actually widely studied, while the second, with a longer name, was much less popular, but contains what is known as

Cavalieri's Principle, which we state as follows:

> Suppose two solids are located in space between two horizontal parallel planes so that each is tangent to each of the solids. Suppose further that each plane moving parallel to and between the original two planes intersects each of the solids in plane figures of equal area. Then the two solids have equal volumes.

This principle, which is usually taught today in beginning geometry courses, is quite powerful. For example, suppose the two solids are a right circular cone of height equal to the distance between the planes, while the second is a right pyramid of the same height with square base, each placed with base in the lower plane and vertex in the upper, and such that each of the movable planes intersects the cone in a disc of area which is equal to the area of the square which it intersects with the pyramid. (If the radius of base of the cone is R then the edge of the base of the pyramid should be $\sqrt{\pi}R$.) Then Cavalieri's Principle asserts that the two solids have the same volume. Since the volume of the pyramid is 1/3 the volume of the right square cylinder of the same base and height (easily proven in Euclid), the same must be true of the cone, which is not so easy without Cavalieri's Principle. By considering the corresponding intersections of the movable plane to be lamina of infinitesimal thickness, and since those of the cone are in 1-1 correspondence with those of the pyramid, then if one accepts the idea that each of the solids is the union of these lamina, then Cavalieri's Principle is immediate. Moreover, the important fact for us is that Galileo could hardly be ignorant of this principle and its significance.

Galileo In the *Dialogs Concerning Two New Sciences*, after a protracted discussion of the nature of acceleration Galileo decides that since he is familiar with constant velocity, the simplest assumption to make regarding acceleration in the case of freely falling bodies is that it is also constant. After discussing how experimenting with inclined planes would confirm this assumption in the case of a freely falling body, he then says, in the words of his mouthpiece Salviati (quotation from [4].):

" Let us then, for the present, take this (constant acceleration) as a postulate, the absolute truth of which will be established when we find that the inferences from it correspond to and agree perfectly with experiment. The author having assumed this single principle passes next to the propositions which he clearly demonstrates; the first of these is as follows: " (Theorem I, Proposition I, below)

(Comment: Galileo takes constant acceleration as a postulate, then proves $s = \frac{1}{2}gt^2$ as a purely mathematical consequence, and then proposes an experiment to "demonstrate the absolute truth of" what? not the formula $s = \frac{1}{2}gt^2$ which has been proven for *any* constant acceleration, but rather, the constant acceleration postulate for free fall. I suspect many high school physics classes present the formula as only about free fall and, more significant, that the formula was established by experiment. As I recall, my HS physics class made this claim. My college physics course, having a calculus co-requisite, did much better, first establishing constant g for free fall by experiment, and then quickly integrating gt to obtain the formula. Galileo does it the other way around, and sort of invents a case of integral on the way.)

"Theorem I, Proposition I The time in which any space is traversed by a body starting from rest and uniformly accelerated is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed and the speed just before acceleration began.

(Now he presents the proof) (See Figure 3, which appears as Figure 47 in Galileo's Dialogs [4].) In the emphasized text he clearly is thinking of summing infinitesimals.)

Let us represent by the line AB the time in which the space CD is traversed by a body which starts from rest at C and is uniformly accelerated; let the final and highest value of the speed gained during the interval AB be represented by the line EB drawn at right angles to AB; draw the line AE, then all lines drawn from equidistant points onAB and parallel to BE will represent the increasing values of the speed, beginning with the instant A. Let the point F bisect the line EB; draw FG parallel to BA, and GA parallel to FB, thus forming a parallelogram (rectangle) AGFB which will be equal in area to the triangle AEB, since the side GF bisects the side AE at the point I; for if the parallel lines in the triangle AEB are extended to GI, then *the sum of all the parallels contained in the quadrilateral is equal to the sum of those contained in the triangle* (my emphasis) AEB; for those in the triangle IEF are equal to those contained in the triangle GIA,while those included in the trapezium AIFB are common.

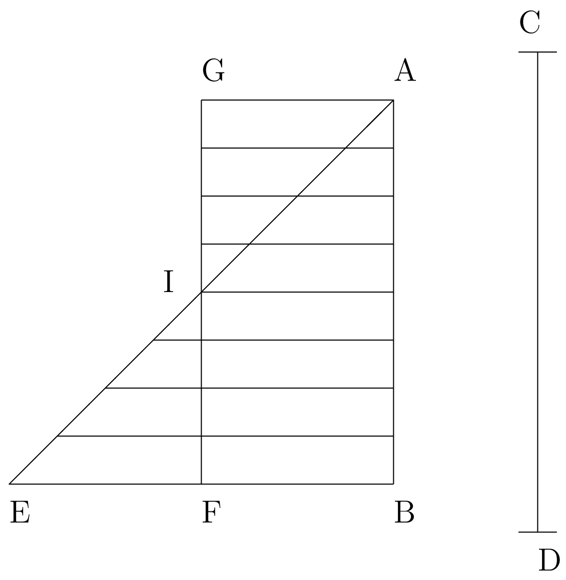

Figure 3

Since each and every instant of time in the time-interval AB has its corresponding point on the line AB,from which points parallels drawn in and limited by the triangle AEB represent the increasing values of the growing velocity, and since parallels contained within the rectangle represent the values of a speed which is not increasing, but constant, it appears, in like manner, that the momenta [momenta] assumed by the moving body may also be represented, in the case of the accelerated motion, by the increasing parallels of the triangle AEB, and, in the case of the uniform motion, by the parallels of the rectangle GB. For, what the momenta may lack in the first part of the accelerated motion (the deficiency of the momenta being represented by the parallels of the triangle AGI) is made up by the momenta represented by the parallels of the triangle IEF.Fig. 47 Hence it is clear that equal spaces will be traversed in equal times by two bodies, one of which, starting from rest, moves with a uniform acceleration, while the momentum of the other, moving with uniform speed, is one-half its maximum momentum under accelerated motion. q.e.d." (End of quotation)

What the theorem states is: if the constant acceleration assumed is g, then distance traveled from time $t = 0$ to $t = T$ is

$$\int_0^T v(t)\,dt = \int_0^T gt\,dt = \frac{1}{2}gt^2|_0^T = \frac{1}{2}gT^2. \quad (\dagger)$$

Relating these equations to the proof above, if the velocity $v(t)$ at time t is constant over the interval from 0 to T, then the "final and highest value of the speed gained during the interval is AB" is " represented by the line EB $\cdots$...", so $\frac{1}{2}gT \cdot T = \frac{1}{2}gT^2$ equals the total distance traveled at half the the final speed and is just the area of the parallelogram AGFB, and the area of this parallelogram equals the area of the triangle AEB.

By comparing the line segments parallel to EB with those parallel to FB Galileo asserts that "since parallels contained within the rectangle represent values of a speed which is not increasing, but constant, it appears, in like manner, that the momenta assumed by the moving body may also be represented, in the case of the accelerated motion, by the increasing parallels of the triangle AEB, and, in the case of uniform motion, by the parallels of the rectangle GB."

From earlier discussion Galileo also understands that for constant acceleration g, the velocity at the instant t is $v(t) = gt$ and that this also somehow represents the infinitesimal distance traveled at time t. Thus he is asserting that the distance traveled equals the integrals in (†). The second equality of (†) follows from the equality of the areas of the rectangle and triangle, and of course amounts to a case of the Fundamental Theorem of Calculus, appearing once again after 1800 years.

In his definitive biography of Galileo, J. L. Heilbron writes (pp. 128) "By 1610 Galileo possessed virtually everything he later made public in *Two new sciences* (1638). He, Sarpi, and at least one or two others then knew that in free fall bodies increase their distance from their starting point in proportion to the square of the time elapsed" etc., But Heilbron then finds little certainty about more information and writes that "To obtain an answer in this muddle of uncertain dates, perplexed attributions, and reciprocal influences without claiming greater certainty than our information allows, a Galilean dialogue might suit." He then presents such a dialogue with participants Alexander, "Galileo's alter ego", and Galileo himself. During the course of this dialogue, as Galileo is explaining his "square of the time" result, Alexander (pp137) says "I do not see what significance to assign to the concept of 'the sum of all the velocities.' He is apparently is referring to the phrase "sum of all the parallels" used several times in the proof quoted above from *Two new sciences*. Galileo responds

"The sum of the degrees or moments of velocity, that is , of all the lines that can be drawn parallel to —. I blush to say that I took the concept and the analysis from a medieval technique called the 'latitude of forms.' That was not very clever of me since the scholastic philosophers were interested in making a picture, not in describing the accidents of, motion. But I had known the technique for a long time and seized upon it to answer Sarpii's demand for a principle."

Assuming the correctness of Heilbron's "Galilean dialogue" it seem clear that Galileo couldn't care less about how "the latitude of forms" is justified and certainly issues concerning infinitesimals were lost on him. In this he is like many modern scientists who uncritically grab any convenient result from from another discipline to justify a claim in their own.

Oresme A quest for the meaning of "latitude of forms" led me to to Oresme. From the Stanford Encyclopedia of Philosophy SEP article on the French philosopher Bishop Nicole Oresme (1323-1382): 2.6 Mathematics. We quote from this as follows:

(Begin SEP quotation) Oresmes main contributions to mathematics are contained in his Questiones super geometriam Euclidis and his Tractatus de configurationibus qualitatum et motuum. In these works Oresme conceived of the idea of using rectangular coordinates (latitudo and longitudo) and the resulting geometric figures (configurationes) to distinguish between uniform and nonuniform distributions of various quantities, such as the change of velocity in relation to time or the distribution of the intensities of a quality in relation to the extension of the subject. In the discussion of motions the base line (longitudo) is the time, while the perpendiculars raised on the base line (latitudines) represent the velocity from instant to instant in the motion. Thus uniform acceleration is represented by a right triangle. Oresme even extended his definition to include three-dimensional figures (Clagett 1974, 226228). Thus, he helped lay the foundation that later led to the discovery of analytic geometry by Rene Descartes (15961650).

Furthermore, Oresme used his figures to give the first proof of the "Merton theorem", discovered at Oxford in the 1330s: the distance traveled in any given period by a body moving under uniform acceleration is the same as if the body moved at a uniform speed equal to its speed at the midpoint of the period (Clagett 1974, 225226; Smorynski 2017, 216222). Some scholars believe that Oresmes graphical representation of velocities was of great influence in the further development of kinematics, affecting in particular the work of Galileo (15641642). (End of SEP quotation)

Some think Galileo has been too much paraised and Oresme not enough. Certainly Oresme, in his proof and using essentially the same diagram as Figure3 , makes implicit use of infinitesimals. Galileo seems to just uncritically accept Oresme's proof.

Conclusion From what we have seen above it seems fair to note that while Archimedes used physics, in particular his knowledge of levers and centroids, to discover new results in geometry (areas and volumes), Galileo on the other hand used geometry to discover new results in physics (in particular a basic law of motion). Certainly the latter, a great first step in the development of applied mathematics so important to the scientific revolution, has been more significant than the former. It seems that Archimedes was more interested in mathematics while Galileo was not really interested in mathematics and, in particular, seems primarily interested in justifying the constant acceleration hypothesis and his discovery of the $1/2gt^2$ formula as providing a means for experimental verification. Indeed, recall that he says

" Let us then, for the present, take this (constant acceleration) as a postulate, the absolute truth of which will be established when we find that the inferences from it correspond to and agree perfectly with experiment."

Thus while I think that the discovery of the formula is paramount, Galileo seems more interested in justifying the constant acceleration hypothesis.

Another thing: he speaks of establishing the "absolute truth' of the hypothesis'. This seems strange: he seems to claim that since constant acceleration implies the formula which is true experimentally, then constant acceleration must be true, an obvious logical fallacy. I think he means that the experiment will provide "strong" evidence for constant acceleration. Is he just speaking hyperbolically?

My conjecture about Archimedes' identification of infinitesimals with atoms is perhaps wildly far fetched, and while we will never know for sure what Archimedes thought, I would like to know if my conjecture is consistent with what is known. Unfortunately my background in mathematics and a slender knowledge of the history of philosophy poorly equip me to make an evaluation. Nonetheless Archimedes work on centroids makes the conjecture tempting.

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