Affine Completeness and Compact 1

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János Kollár [5] showed that for a variety V the following are equivalent: (i) No non-trivial algebra $\mathbf{A} \in V$ has a one-element subalgebra, (ii) for all $\mathbf{A} \in V$, $1_A \in Con\mathbf{A}$ is compact, i.e.: is finitely generated or, briefly, $\mathbf{A}$ has compact 1.

Any variety of lattices with nullary 0 and 1 and the variety of rings with nullary 1 are examples. Both obviously satisfy both conditions of Kollár's theorem.

The variety of Boolean algebras, because of complementation, has both of these properties and more. In particular for every element x of a Boolean algebra $\mathbf{B}$, $1_{\mathbf{B}} = Cg(x, x')$, so that $1_{\mathbf{B}}$ is both *principal* and *term defined* by the same terms x and x' for all members of the variety.

This latter example leads to the folowing tentative

Definition A variety V has *uniform term defined compact* 1 if there is an integer n and terms $u_i, v_i, i = 1, \ldots, n$, such that for all $\mathbf{A} \in V$

$$1_{\mathbf{A}} = \bigvee_{i=1,\ldots,n} Cg^A(u_i, v_i)$$

The purpose of the present note is to present the **Theorem** below which asserts that any arithmetical affine complete variety of finite type has uniform term defined compact 1 and if the variety is also regular then 1_A is both principal and term defined for all $\mathbf{A} \in V$.

Varieties with the property (i) are called *semi-degenerate* — terminology due to Lausch and Nöbouer [6] (which is somewhat unfortunate since on the one hand a variety is called *degenerate* if it consists solely of the one element algebra, while on the other hand Boolean algebras and rings with nullary 1 seem in no way close to being degenerate in any ordinary sense of the word. In addition the presence of nullaries, which may seem artificial in these cases, is not necessary. This was shown by B. Csákány [1] who shows that (i) is actually a Malcev condition.)

For background, recall [2] that a variety V is *affine complete* if all congruence compatible operations of algebras in V are polynomials. The study of affine complete varieties has been inspired by the early observation by G. Grätzer that the variety of Boolean algebras is affine complete and the subsequent discovery that many properties of Boolean algebras are inherited in some more general form by affine complete varieties. Our Theorem is an easy consequence of several of these general properties and in order to make the proof easily accessible we gather together those we require:

P1 All affine complete varieties V have the properties:

- residual finiteness,
- congruence distributivity,
- V if $\mathbf{B}$ is any subalgebra of any algebra $\underset{\sim}{A} \in V$, then $V\mathbf{B}) = V(\mathbf{A})$. The semi-degeneracy of V is an immediate consequence. ([2], Section 4.2)
- the properties of being *finitely generated*, *locally finite*, and of *finite type* (only finitely many basic operations) are all equivalent. ([2], Section 4.4)
- if V is of finitely generated then it is generated by a unique *minimal algebra* (i.e.: a finite algebra $\mathbf{A}$ of more than one element and having no proper subalgebras), which is also the largest minimal algebra in the variety. This generator is constructable from any finite generator and is characterized by the property of being *weakly diagonal*, i.e.: every subuniverse of A^2 contains the graph of some automorphism of $\mathbf{A}$. ([3], Section 2)

P2 If $\mathbf{A}$ is a weakly diagonal algebra and $t(x, \ldots, z, a)$ and $u(x, \ldots, z, a)$ are polynomials containing only the element $a \in A$ and $t(x, \ldots, z, a) = u(x, \ldots, z, a)$ for all $x, \ldots, z \in A$, then for some unary term $s(x)$, $t(x, \ldots, z, s(x)) = u(x, \ldots, z, s(x))$ is a term identity of $\mathbf{A}$.

Thus the polynomial identities of $\mathbf{A}$ determine the equational theory of $\mathbf{A}$. ([3], Section 2.)

P3 If V is an arithmetical affine complete variety of finite type a term $p(x, y, z)$ of V may be chosen which not only satisfies the three identities characterizing arithmeticity,

$$p(x, x, z) = p(z, y, z) = p(z, y, y) = z,$$

but also so that the variety is a *principal arithmetical variety*, i.e.: for all $\mathbf{A}$ in V,

$$(z, w) \in Cg^{\mathbf{A}}(x, y) \Leftrightarrow p(x, y, z) = p(x, y, w).$$

In consequence of this, if θ is a finitely generated congruence, say

$$\theta = Cg^{\mathbf{A}}(a_1, b_1) \vee \cdots \vee Cg^{\mathbf{A}}(a_n, b_n),$$

then $(x, y) \in \theta \Leftrightarrow$

$$p(a_1, b_1, p(a_2, b_2, \ldots, p(a_n, b_n, x) \cdots)) = \\ p(a_1, b_1, p(a_2, b_2, \ldots, p(a_n, b_n, y) \cdots)).$$

([4], Theorem 1.1, Theorem 2.2.)

P4 If V is an arithmetical affine complete variety of finite type which is also regular then the term $p(x, y, z)$ may be chosen so that in addition to satisfying **P3**, for all algebras in V the join of principal congruences is given by the formula

$$Cg(a, b) \vee Cg(c, d) = Cg(p(a, b, c), p(b, a, d)).$$

Thus all finitely generated congruences are principal. ([4], Theorem 3.2.)

Now we can easily assemble the proof of our Theorem.

Theorem *If V is an arithmetical affine complete variety of finite type then a principal term $p(x, y, z)$ for V may be chosen so that for some integer n,*
(a) 1_A is compact for all $\mathbf{A} \in V$ and is the join of n term defined principal congruences. i.e.: 1_A is uniform term defined.
(b) If V is, in addition, regular, then $p(x, y, z)$ may be chosen so that in addition to (a) there are unary terms $u(x)$ and $s(x)$ of V such that for every $\mathbf{A} \in V$, and $x \in A$, $1_A = Cg(u(x), s(u(x)))$, i.e. 1_A is principal and is term defined.

In Boolean algebras we may take $u(x) = x$ and $s(x) = x'$.

Proof From **P1** we may take $\mathbf{A}$ to be the largest minimal algebra in V.

(a) Since $\mathbf{A}$ is finite, for some integer n and $a_i.b_i \in A$, $1_A = Cg(a_1, b_1) \vee \cdots \vee Cg(a_n, b_n)$; from **P3**, since $\mathbf{A}$ is minimal, for $a = a_1$ there are unary terms h^i, k^i such that for all x, y

$$p(p(a, k^1(a), p(h^2(a), k^2(a), \ldots, p(h^n(a), k^n(a), x) \cdots) = \\ p(p(a, k^1(a), p(h^2(a), k^2(a), \ldots, p(h^n(a), k^n(a), y) \cdots).$$

By **P2** the same identity holds with some unary term $u(x)$ replacing each occurrence of a. It follows from **P3** again that for all $\mathbf{A} \in V$ and $x \in A$,

$$1_A = Cg(u(x), k^1(u(x))) \vee \cdots \vee Cg(h^n(u(x)), k^n(u(x))),$$

so that $1_{\mathbf{A}}$ is term defined.

(b) Since $\mathbf{A}$ is regular and finite, by **P4** $1_A = Cg(a, b)$ for some $a, b \in A$. Since $\mathbf{A}$ is minimal, by **P2**, for some unary term $s(x)$, $b = s(a)$, so $Cg(a, s(a)) = 1_A$. Since p is principal, by **P3**, $p(a, s(a), z) = p(a, s(a), w)$ for all $z, w \in A$; again by **P2**, for some term $u(x)$, $p(u(x), s(u(x)), z) = p(u(x), s(u(x)), w)$ is an identity of A and hence of V. It follows by **P3** again that for every $x \in A, \mathbf{A} \in V$, $1_A = Cg(u(x), s(u(x)))$. •

Example. The algebra $D3 = \langle\{0, 1\}, xy \vee xz \vee yz, x'\rangle$ with xy and $x \vee y$ equal min and max respectively, is a quasi-primal algebra with the discriminator $t(x, y, z) = xy' \vee xz \vee y'z$, hence the variety $V = V(D3)$ is a principal arithmetical variety with $(z, w) \in Cg^{\mathbf{A}}(x, y) \Leftrightarrow t(x, y, z) = t(x, y, w)$ for all algebras in V. But $t(x, x', z) = t(x, x', w)$ is an identity of $D3$ and hence of V. So $1_A = Cg^{\mathbf{A}}(x, x')$ for all $A \in V$.

An alternative proof of (b): Let V be a regular arithmetical affine complete variety of finite type. Let $F = F(x, y, z)$ be the free algebra of the variety with free generators x, y, z, and let $S = F(x)$ be the subalgebra generated by x. F is finite by **P4**. Let $\theta = Cg^F(S \times S)$. Then $\theta|_S = S \times S$, so $S/\theta|_S$ consists of only one element. But $S/\theta|_S$ is a subalgebra of F/θ. Then, since no proper algebra in the variety can have a 1 element subalgebra, by **P1**, F/θ must have just one element; therefore $\theta = 1_F$. Since everything is finite θ is compact and hence is principal, by **P4**, so for some unary terms $u(x), v(x) \in S, 1_F = Cg^F(u(x), v(x))$. Therefore, for all $z, w \in F$, we have $(z, w) \in Cg^F(u(x), v(x))$, so by **P3** $p(x, y, z)$ may be chosen so that $p((u(x), v(x), z) = p(u(x), v(x), w)$ is an identity of the variety. It follows that for every algebra A of the variety, $1_A = Cg^A(u(x), v(x))$. •

References

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