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# Consensus-halving via theorems of Borsuk-Ulam and Tucker

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## Abstract

In this paper we show how theorems of Borsuk-Ulam and Tucker can be used to construct a *consensus-halving*: a division of an object into two portions so that each of  $n$  people believes the portions are equal. Moreover, the division takes at most  $n$  cuts, which is best possible. This extends prior work using methods from combinatorial topology to solve fair division problems. Several applications of consensus-halving are discussed.

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## 1. Introduction

The study of fair division problems is concerned with finding ways to divide an object among several parties according to some notion of fairness. The cake-cutting problem of Steinhaus (1948) is perhaps the best known example. Aside from the division of goods, other fair-division problems address the division of burdens (e.g., the chore-division problem (Gardner, 1978; Peterson and Su, 2002)) and the division of mixtures of goods and burdens (e.g., the *rent-partitioning* problem (Brams and Kilgour, 2001; Haake et al., 2002; Su, 1999): how to split rent so that housemates are satisfied by different rooms).

Recently, ideas from combinatorial topology have provided new and constructive

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methods for obtaining solutions to fair-division problems. Su (1999) discusses a cake-cutting procedure of Simmons that can be extended to obtain envy-free solutions for chore division and rent-partitioning using variants of a result known as Sperner's lemma, which is the combinatorial equivalent of the Brouwer fixed point theorem of topology.

In this paper, we demonstrate how a result known as Tucker's lemma, which is the combinatorial equivalent of the Borsuk-Ulam theorem of topology, can be used to solve a different kind of fair-division problem: is it possible to divide a mixture into two portions so that each of  $n$  people believes both portions are the same size (*a consensus-halving*)? Moreover, a constructive proof of Tucker's lemma yields an effective algorithm for locating an approximate solution that uses a minimal number of cuts.

As an application, an approximate consensus-halving procedure could allow two families to split a piece of land into two regions such that every member of both families believes the land is nearly equally divided. Using a minimal number of cuts would also be desirable in this setting. We discuss potential applications to the Law of the Sea Treaty (Brams and Taylor, 1996) and the *necklace-splitting* problem of Alon (1987). Another application solves a *team-splitting* problem: given a territory and a pair each of zoologists, botanists, and archaeologists, is it possible to divide the territory into two portions in such a way that members of any given pair will prefer to explore different portions? Thus the group could be split in an *envy-free* fashion—into two teams with one member of each specialty among them. We explain how a consensus-halving method can be adapted for this purpose near the end of this article.

## 2. Remarks on procedures and algorithms

As we shall be interested in procedures of various kinds, we make a few associated definitions. We use the term *algorithm* to refer to a procedure that proceeds in discrete time steps and terminates after a finite number of steps. (Thus the so-called 'moving-knife' procedures (Brams et al., 1997) are not algorithms because they require continuous evaluation of the cake.) For problems that have a finite number of solution candidates, one algorithm that can be used to find a solution is a brute-force *exhaustive search*, which involves sequentially testing each of the finite number of possibilities. Any algorithm that locates a solution more systematically than an exhaustive search will be called *effective*. We shall be concerned with effective algorithms in this paper.

## 3. Consensus-halving

To be precise, we assume that any *object*  $A$  to be divided is a measurable bounded set in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , infinitely divisible, and that each player  $i$  has a bounded continuous measure  $\mu_i$  on (measurable) subsets of  $A$  which describes the (positive or negative) value that she assigns to that subset. Thus the object we consider can be a mixture of desirable and undesirable parts (the players may in fact disagree on which parts are desirable and undesirable). By a *continuous* measure, we mean absolutely continuous with respect to

Lebesgue measure, so that continuity of the measures (with respect to Lebesgue measure) forbids the existence of ‘point masses’—zero volume subsets with non-zero worth.

Although we model player preferences with measures, we remark that none of our proofs will require the  $\mu_i$  to be additive over subsets. (The  $\mu_i$  may as well be continuous set functions defined on the Borel  $\sigma$ -algebra and satisfying all the properties of measures except countable additivity.) Thus we *do not* need to require that player valuations over subsets of  $A$  be additively separable.

We establish the following theorem.

**Theorem 1.** (Consensus-Halving) *Consider an object  $A$ , and  $n$  people whose preferences on  $A$  are modeled by continuous measures  $\{\mu_1, \dots, \mu_n\}$ . Using at most  $n$  cuts by parallel planes,  $A$  may be divided into two portions  $A_+$  and  $A_-$  such that each of  $n$  people thinks that  $A_+$  and  $A_-$  are exactly equal, i.e.,  $\mu_i(A_+) = \mu_i(A_-)$  for all  $i \in \{1, 2, \dots, n\}$ . This number of cuts by parallel planes is best possible; there is a collection of player measures which requires  $n$  cuts. Moreover, an effective algorithm exists for locating an  $\varepsilon$ -approximate solution, i.e., two portions  $A_+, A_-$  such that  $|\mu_i(A_+) - \mu_i(A_-)| \leq \varepsilon$  for all  $i \in \{1, 2, \dots, n\}$ .*

Non-constructive versions of Theorem 1 have already been obtained; for instance, see Goldberg and West (1985) and Alon and West (1986). The latter uses the Borsuk-Ulam theorem but in a fashion that requires additivity of the measures  $\mu_i$ . Alon (1987) proves a generalization that produces  $k$  equal portions according to  $n$  probability measures; it yields our result when  $k = 2$ , but it is also non-constructive and based on a topological result of Bárány et al. (1981). Another approach to produce the existence of the sets  $A_+$  and  $A_-$  is to use Lyapunov’s theorem (see Barbanel (1996)); however, it is even less constructive because it does not even say how many cuts are required or what the sets  $A_+$  and  $A_-$  might look like.

By contrast our result is constructive in the sense that the proof yields a method for converging on a solution. If only an approximate solution is desired, then there is an effective algorithm for locating it, using a combinatorial result known as Tucker’s lemma. Our proof does not require additivity nor positivity of the measures, nor must the measures satisfy  $\mu_i(A) = 1$  (but they should be bounded for the conclusion to make sense). A constructive proof of Tucker’s lemma yields a simplicial algorithm that guarantees an  $\varepsilon$ -approximate solution, so that each person believes the two portions are within size  $\varepsilon$  of each other.

In some sense approximate solutions are the best one can hope for; Robertson and Webb (1988, p. 104) have shown that there is no finite algorithm that will produce an *exact* equal division. If one allows continuous procedures such as ‘moving-knife’ solutions (Brams et al., 1997), then a procedure due to Austin (1982) will achieve an exact consensus-halving for  $n = 2$ . However, there is no known generalization of Austin’s procedure for  $n \geq 3$ .

We make a couple of remarks about our  $\varepsilon$ -approximate solution. First, note that a brute-force search algorithm is always possible for locating an  $\varepsilon$ -approximate solution. Because we know (by this theorem or Theorem 1.2 of Alon (1987)) that a solution

exists, and since the set of possible divisions by  $n$  parallel planes is a compact set  $D$ , only a finite number of divisions need be checked to find one that is within  $\varepsilon$  of a consensus-halving. However, while our approach would use those same divisions, regarded as vertices in an associated triangulation of  $D$  (see Section 4), our approach is more systematic than a brute-force search because it only checks the vertices near a ‘path’ in the triangulation, which for reasonable player preferences is generally very straight and does not wind throughout all of  $D$ .

Second, we emphasize that the number of cuts involved is minimal. Brams and Taylor (1996, pp. 131–132) give an inductive procedure for an approximate consensus-halving, and Robertson and Webb (1988, p. 128) propose a procedure for approximate division into ratios; however, both these procedures involve a large number of cuts which grows as  $\varepsilon$  decreases. By contrast, our approach never uses more than  $n$  cuts.

#### 4. Tucker’s combinatorial lemma and the Borsuk-Ulam theorem

Recall that an  $n$ -simplex in  $\mathbb{R}^n$  is the convex hull of  $n + 1$  affinely independent points (vertices) in  $\mathbb{R}^n$ . A  $k$ -face of an  $n$ -simplex is the  $k$ -simplex spanned by any subset of  $k + 1$  vertices. A triangulation of a set  $X$  is a collection of (distinct)  $n$ -simplices whose union is  $X$ , with the property that any two of them intersect in a face common to both, or not at all.

Represent the  $n$ -ball  $B^n$  by the set of all points  $\mathbf{x} = (x_i) \in \mathbb{R}^n$  such that  $|x_1| + \cdots + |x_n| \leq 1$ . The boundary of this ‘octahedral’ ball is the set of all points in  $\mathbb{R}^n$  satisfying  $|x_1| + \cdots + |x_n| = 1$  and may be thought of as an  $(n - 1)$ -sphere  $S^{n-1}$ . A centrally symmetric triangulation of  $S^{n-1}$  is one such that if  $\sigma$  is any simplex of the triangulation, then  $-\sigma$  also is.

The following combinatorial theorem of Tucker (1945) was proved for the case  $n = 2$ . The proof for general  $n$  may be found in Lefschetz (1949).

**Tucker’s Lemma.** *Let  $T$  be a centrally symmetric triangulation of  $S^n$  whose vertices are assigned labels from  $\{\pm 1, \pm 2, \dots, \pm n\}$  such that labels of antipodal vertices sum to zero, i.e., the labelling function  $l$  satisfies  $l(-\mathbf{x}) = -l(\mathbf{x})$  for any vertex  $\mathbf{x}$ . Then there exist adjacent vertices in the triangulation whose labels sum to zero.*

This result is often stated for a triangulation of a ball, but for our purposes later, we have cast it for a triangulation of a sphere (obtained by gluing two  $n$ -balls along their boundaries) (see Fig. 1).

Tucker’s lemma is equivalent (see Freund and Todd, 1981) to the following famous theorem from topology:

**The Borsuk-Ulam Theorem.** *For any continuous function  $f: S^n \rightarrow \mathbb{R}^n$ , there exist antipodal points  $\mathbf{x}, -\mathbf{x} \in S^n$  such that  $f(\mathbf{x}) = f(-\mathbf{x})$ .*

The equivalence is valuable because Tucker’s lemma has a constructive proof, while the Borsuk-Ulam theorem can be used to prove fair division theorems. For instance, the

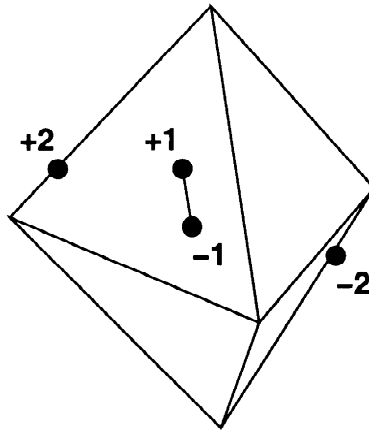


Fig. 1. An ‘octahedral’ sphere  $S^2$  (with triangulation not shown). The conditions of Tucker’s lemma (e.g., see the antipodal  $+2, -2$  labels) implies the existence of an edge in the triangulation whose labels sum to zero (e.g., the  $+1, -1$  labels).

Ham Sandwich Theorem, which says that there exists a hyperplane that perfectly bisects  $n$  sets of positive measure in  $\mathbb{R}^n$ , is well-known to be a consequence of the Borsuk-Ulam theorem (Rotman, 1988, p. 413) and therefore Tucker’s lemma can be used to find such a hyperplane. However, as a fair-division theorem, the Ham Sandwich Theorem is of little practical value when the dimension of the sets is greater than 3, and even in dimension 3, it is unneeded if one allows several cuts.

We seek somewhat more practical applications of the Borsuk-Ulam theorem and Tucker’s lemma, such as the consensus-halving result of Theorem 1.

**Proof of Theorem 1.** For ease of expression, we refer to the object  $A$  to be divided as ‘cake’ even though players may find certain subsets undesirable.

Place  $A$  in a coordinate system aligned with the cardinal directions of the compass. Assume without loss of generality that the (east/west) width of  $A$  is one unit. Suppose further that  $A$  is to be divided by vertical, parallel north–south planes.

Each point  $(x_1, \dots, x_{n+1})$  of  $S^n$  corresponds to a set of cuts of the cake (called a *cut-set*) obtained by making north/south cuts so that (from west to east) the pieces have widths of  $|x_1|, |x_2|, \dots, |x_{n+1}|$ . Use the respective signs of  $x_1, x_2, \dots, x_{n+1}$  to determine which portion of the division gets the corresponding piece: collect all the pieces for which  $x_i$  is positive, lump them together, and call this the portion  $A_+$ . The other pieces will be lumped together to create portion  $A_-$  (see Fig. 2).

The existence of a division such that  $A_+$  and  $A_-$  are deemed exactly equal by all players follows easily from the Borsuk-Ulam theorem; consider the function  $f: S^n \rightarrow \mathbb{R}^n$  such that the  $i$ -th coordinate function  $f_i(\mathbf{x}) = \mu_i(A_+)$ , player  $i$ ’s measure of the ‘value’ of  $A_+$ . This is a continuous function of  $\mathbf{x}$  (because of the continuity assumption on the measures), hence by the Borsuk-Ulam theorem there exists a point  $\mathbf{x}$  such that  $f(\mathbf{x}) = f(-\mathbf{x})$ . But since antipodal points on  $S^{n-1}$  correspond to the same division with the roles of  $A_+$  and  $A_-$  interchanged, the Borsuk-Ulam point  $\mathbf{x}$  corresponds to a set of (at

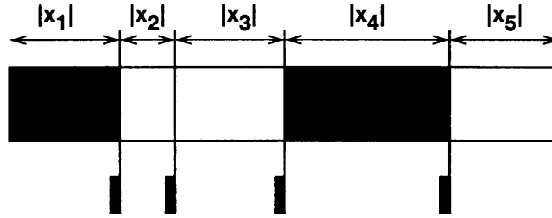


Fig. 2. A cut-set represented by  $(-0.2, +0.1, +0.2, -0.3, +0.2)$ . The portion  $A_+$  is the union of the white pieces;  $A_-$  is the union of the shaded pieces.

most)  $n$  cuts (and fewer if the components of  $A_i$  are adjacent) such that  $\mu_i(A_+) = \mu_i(A_-)$  for all  $i$ , i.e., the pieces are deemed equal by all players. One may see that  $n$  cuts are also necessary in the case in which  $A$  is a line segment and the player measures have support in  $n$  disjoint subintervals of  $A$ . This shows the existence of a solution to the consensus halving problem.

To construct an approximate solution (to any pre-specified error tolerance  $\epsilon$ ), use Tucker’s lemma. Recall that every point in  $S^n$  corresponds to a cut-set. Given  $\epsilon > 0$ , choose a triangulation of  $S^n$  with mesh size so small that, in the cut-sets corresponding to any two adjacent vertices, the portions  $A_+$  (and also  $A_-$ ) differ by no more than  $\epsilon$  in any of the player measures.

We now assign to every vertex a label in the set  $\{+1, -1, \dots, +n, -n\}$  which consists of a number and a sign. The *number* assigned to a vertex will be the number of the player who believes the difference between  $A_+$  and  $A_-$  is greatest for the cut-set corresponding to that vertex. (If there are players equally distressed about the difference, choose the smallest-numbered player.) The *sign* assigned to a vertex will signify the portion that the ‘most distressed’ player *prefers* in the cut-set corresponding to that vertex: if piece  $A_+$  (resp.  $A_-$ ) is preferred, the sign assigned is  $+$  (resp.  $-$ ). (In case that player prefers both pieces equally, choose the portion containing the west edge of the cake.)

Note that this gives an anti-symmetric labelling  $l$  in which  $l(-\mathbf{x}) = -l(\mathbf{x})$  at every vertex, because when the roles of  $A_+$  and  $A_-$  are reversed, the same player is most distressed but her preference is reversed. (Thus moving to the antipodal vertex leaves the label number the same but flips the sign.)

Applying Tucker’s lemma, there exists a pair of adjacent vertices in the triangulation with the same label number but opposite signs. Either of these vertices corresponds to a cut-set that is an *approximate* consensus-halving, since at these two nearby cut-sets, the maximally distressed player prefers different portions. For this player, both portions are within  $\epsilon$  of each other in value, and since this player’s distress was maximal, no other player will dispute this assessment by more than  $\epsilon$ .

These adjacent vertices may be found effectively by using the algorithm of Freund and Todd (1981), or more recent methods found in Yang (1999). These are *simplicial algorithms* that follow paths of simplices in the triangulation; we do not review them here for lack of space. However, we do emphasize the important feature of such algorithms is the fact that they use vertex labels to determine a path that finds the desired

adjacent vertices. This is far more efficient than a brute-force search of the vertices. In our setting, each vertex corresponds to a cut-set, so the vertex labels can be determined on the fly by moving from vertex to vertex and interactively polling the players for their preferences at the cut-sets along the path. (T. Prescott has noted that one can reduce the length of the path produced by the Freund-Todd algorithm by modifying the above labelling rule: assign a vertex the number of the most-distressed player (as before), but let the sign of the vertex signify the portion which that player *least* prefers.)  $\square$

We remark that the proof of Theorem 1 can be modified to address preference measures on a measurable set of any dimension as long as it can be mapped onto a bounded real interval such that the image measures (of the players' measures) are absolutely continuous. In this case, the inverse images of the cut sets of the interval yield the cut sets of the object.

## 5. Remarks on implementation and efficiency

In an actual implementation, the algorithm for consensus-halving can be coded so that a computer could proceed through the algorithm and interactively ask players at each step which portion they would prefer and their perceived difference in size between the portions. (See Su (1999) for a similar fair division procedure based on Sperner's lemma.) Convergence to a solution can be enhanced by existent homotopy algorithms in which  $\varepsilon$  need not be specified in advance and generally decreases with the run time. See Todd (1976) or Yang (1999) for a survey of such methods applied to fixed point problems.

An additional feature of our approach is that since the algorithm is interactive, players do not have to reveal their a priori preferences (which may in general be very hard to describe). Moreover, during the procedure they do not need to reveal their preferences over all possible cut-sets, but only for cut-sets near a path followed by the simplicial algorithm. On the other hand, if players can fully describe their preferences beforehand, the algorithm can be run from the initial data alone. This may be possible with sufficiently nice preferences, or if the measures are describing some objective data, such as in the 'necklace-splitting' problem below.

We remark that issues such as strategic manipulability of the algorithm and Pareto-efficiency of the outcome are not meaningful in the consensus-halving problem if players are not assigned to one of the portions that result. For instance, a team of  $n$  people might wish to divide a project into two portions so that they can equally divide their work over 2 days. In this case all players desire the same goal: to get agreement by all  $n$  people that two portions of the project are nearly equal in size.

On the other hand, efficiency and manipulability are important issues when each player seeks *different* goals, such as maximizing different pieces. For instance, if feuding families wish to divide some territory between the two families, then a consensus-halving among all involved may not be as desirable as a division in which each person believes that the portion his family received was *bigger* than the other portion. In that case, a consensus-halving would not be Pareto-efficient.

What can be said about the efficiency of a consensus-halving? Some heuristic evidence suggests that for large numbers of players, any division is likely to be efficient. An interesting model of O'Neill, discussed by Roth (1985), shows that in a negotiation of  $n$  players among  $m$  settlements, if the rank-orderings of the settlements by players are assumed equally likely, the likelihood of a Pareto-optimal settlement approaches 1 as the number of players grows. In fact, for as few as 10 players and 1000 settlements, the likelihood of a Pareto-optimal settlement is more than 76%; for 15 players/1000 settlements, more than 98%. Roth notes that “the larger the number of negotiators with independent interests, the harder it is to propose a change from one settlement to another that all regard as an improvement”. In the consensus-halving problem, any cake division is a settlement. Thus if we consider divisions of the cake into 10 strips of width  $1/10$ , there are  $2^{10} = 1024$  possible settlements. The O'Neill data suggests that for 10 players, about  $3/4$  of these settlements are efficient with respect to all 10-strip divisions.

As for strategic play in applications where players are assigned to one of the portions, we can discourage such actions by specifying that the assignment be made only after the halving has already been decided. Thus, no player would have any assurance that she would be assigned to a piece that she tried to fatten up, and strategic play might backfire. On the other hand, if she states her true intentions, she will be guaranteed an approximate consensus-halving. This provides a strong incentive to be truthful.

## 6. The law of the sea treaty and necklace splitting

We remarked earlier that a consensus-halving procedure could help families to split a piece of land into two regions in such a way that every member of both families believes the land is nearly equally divided.

An analogous situation arises in the 1994 Convention of the Law of the Sea (Brams and Taylor, 1996, p. 10), which uses a divide-and-choose procedure to protect the interests of developing countries when an industrialized nation wants to mine a portion of the seabed in international waters. An agency representing the developing countries chooses one of the two halves to reserve it for future mining by less-developed nations. If a consensus-halving procedure were used instead of divide-and-choose, it would yield a division into two portions such that every nation agreed both portions were almost equally valuable.

Our algorithm also provides an effective solution to the discrete problem of ‘splitting necklaces’ (Alon, 1987). Imagine a necklace of jewels of  $n$  different colors but an even number of identical jewels of each color. (The position of each jewel is fixed relative to the other jewels.) Using a minimal number of cuts, we desire a division of the necklace into two portions such that for any color, both portions have the same number of jewels of that color. (We imagine the necklace laid out along a straight line, with cuts made perpendicular to this line.) Theorem 1 can be applied in this context by assigning each player a jewel color, and replacing each player’s subjective measure by a precise count of the number of jewels of her assigned color in each portion. (These measures are absolutely continuous with respect to Lebesgue measure, as required by Theorem 1.) In this case, since the measures are completely known from initial data, a simplicial



algorithm can be adapted to compute an exact solution: if  $\varepsilon$  is chosen to be one jewel, then the simplicial algorithm will find adjacent vertices possessing opposite labels, and a little thought reveals that one of these vertices must represent cuts that divide all the jewel colors in half. (Otherwise the same player could not be ‘most distressed’ at both vertices and still change preferences.)

## 7. Team-splitting

Each consensus-halving result corresponds to a related *envy-free* division problem for twice the number of people, by averaging measures. For instance, our consensus-halving result can be used to address the following ‘team-splitting’ problem.

Suppose among the  $2n$  explorers on an expedition there are two of each specialty: two zoologists, two botanists, two archaeologists etc. They want to know the fairest way to split both their team and their territory. In other words, they want to split into two teams in such a way that each specialty is represented on each team, and such that each team member is satisfied that she is on the team with the best half of the territory to explore.

**Theorem 2.** (Team-Splitting) *Given a territory and such a collection of  $2n$  explorers, there exists a way to divide the territory and the people into two teams of  $n$  explorers (one of each type) such that each explorer is satisfied with his/her territory.*

This result assumes there are no coalitions (sets of people who desire to be on the same team) and that the players have continuous (though not necessarily additive) measures over the territory.

**Proof.** The territory is the object  $A$  that will be divided by consensus-halving. Consider the  $i$ -th pair of scientists by specialty, with measures  $\lambda_i$  and  $\lambda'_i$ . Let  $\mu_i = \lambda_i + \lambda'_i$ . These form a collection of  $n$  measures with which to apply consensus-halving, obtaining two portions  $A_+$  and  $A_-$  for which

$$\lambda_i(A_+) + \lambda'_i(A_+) = \lambda_i(A_-) + \lambda'_i(A_-)$$

for all  $i$ . If  $\lambda_i(A_+) = \lambda_i(A_-)$  and  $\lambda'_i(A_+) = \lambda'_i(A_-)$  then both scientists of the  $i$ -th pair are indifferent between the portions; we can flip a coin to make assignments. Otherwise we conclude that one member of the  $i$ -th pair believes  $A_+$  is more valuable than  $A_-$ , and the other believes the opposite. In this case assign each scientist of the  $i$ -th pair the portion of the territory that she prefers.  $\square$

From the consensus-halving theorem, we see that the team-splitting solution is even reasonably practical—it would never involve more than  $n$  straight cuts through the territory.

## 8. Open problems

We close with some open problems.

- (1) *Consensus-splitting* in an arbitrary ratio. Suppose we desired a division of cake into two portions so that each of  $n$  people agreed the split was some other ratio, say two-to-one? Under what conditions can this be achieved effectively using a minimal number of cuts?
- (2) *Consensus-1/k-division*. Is there an effective algorithm for obtaining a division into  $k$  portions such that each of  $n$  people believes all  $k$  portions are equal in size? Such a method could be used, for instance, to divide an estate among  $k$  children such that each of  $n$  people (parents, children, and others) agreed that all children received equal portions.
- (3) A *generalized Tucker's lemma*. It seems quite likely that the above problem could be addressed by proving some generalization of Tucker's lemma. What is the appropriate combinatorial generalization, and is there a constructive proof?

We pose the following conjecture, a generalization of Tucker's lemma which is motivated by our cake-cutting interpretation. It would solve the consensus-1/k-division question if true. Let

$$S_k^n = \{(z_0, \dots, z_n) : z_i \in \mathbb{C}, \sum |z_i| = 1, z_i^k = |z_i|^k\}$$

This is a kind of a 'branched sphere' embedded in  $\mathbb{C}^{n+1}$ ; it is a manifold that branches in  $k$  directions at every seam, and when  $k = 2$  it is just the usual sphere  $S^n$ . Each point in  $S_k^n$  can be associated to a division of cake. Think of  $|z_i|$  as widths of pieces of cake; the  $z_i^k = |z_i|^k$  condition guarantees that the  $z_i$  are real scalar multiples of a  $k$ -th root of unity, which is the 'sign' of the piece. These signs associate pieces of the cake into  $k$  portions. (The signs are  $+1, -1$  when  $k = 2$ , i.e., consensus-halving). There is a natural action of the symmetric group  $S_k$  on the  $k$  signs; it induces an action of  $S_k^n$  which permutes the assignment of the pieces to portions, i.e., permuting the signs will permute the signs of the portions accordingly.

**Conjecture.** Suppose that  $S_k^{n(k-1)}$  is triangulated symmetrically, i.e., if  $\sigma$  is a face of the triangulation and  $s \in S_k$ , then  $s(\sigma)$  is also a face of the triangulation. Let  $V$  be the vertices of the triangulation, and let  $l:V \rightarrow L$  be a labelling function such that the labels in  $L = \{\omega^j m : \omega = e^{2\pi i/k}, 1 \leq j \leq k, 1 \leq m \leq n\}$  consist of a number  $m$  and a sign  $\omega^j$ . Suppose that  $l$  satisfies a 'consistency condition', such that if  $s \in S_k$  then  $l(s(v)) = s(l(v))$ . Then there must exist  $k$  adjacent vertices in the triangulation with the same label number but different signs.

A constructive proof of this conjecture would give an effective algorithm for approximate consensus-1/k-division as well as associated discrete versions (Alon, 1987). This conjecture appears to be related to recent work of G. Ziegler on colorings of Kneser hypergraphs (Ziegler, 2002).

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