Fair Division Procedures: Why use Mathematics?

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> Good sense is, of all things among men, the most equally distributed; for everyone thinks himself so abundantly provided with it, that those even who are the most difficult to satisfy in everything else, do not usually desire a larger measure of the quality than they already possess.
>
> —René Descartes, *Discourse on Method*, 1637

Introduction

Unlike "good sense", there are many things in the world that (by necessity or desire) must be distributed among people or parties, but for which agreements on how to make that division are not easy to construct. This is a seed for conflict and a starting point of a negotiation problem. As Descartes alluded to, the fundamental problem is how to "satisfy" the parties according to a criterion such as "equal distribution" or some notion of fairness. Difficulties arise because people often view the shares in any proposed distribution differently. And "good sense" is often not enough to help the parties resolve their conflict!

Mathematics may have something to offer in the quest to develop methods for conflict resolution. Problems of *fair division* —how to divide an object fairly— is one area in which mathematics has not only been fruitful in modeling the basic structure of the conflict and locating solutions, but also has suggested useful methods and procedures for parties to reach agreement.

Our goal in this paper is to indicate for a non-technical audience how mathematics can be used to support the development of procedures for fair division problems. We wish to advance an approach to conflict resolution, which is guided by the use of theoretical models, as we consider it to be complementary to "traditional" ones. To demonstrate this, consider the Exercise below. Although it is a simple division problem, it provides a concrete context in which to illustrate our points (without being distracted by the technical mathematics that a more complicated example would require).

> **Exercise:** Tony and Chris wish to share a cake. They have differing opinions of the worth of various portions of cake; for instance, Tony prefers portions with chocolate and Chris prefers portions with cherries. They would like to determine a "fair" division of the cake. As a mediator, can you suggest a method for helping them solve their problem? □

Using mathematics to model (and resolve) conflicts follows a general pattern. First, the *structure* of the conflict is described in mathematical terms. Since a class of seemingly different division problems may have a similar structure, a mathematical model can be used to analyze a whole class of such conflicts. The model clarifies what the *possible* outcomes in the conflict look like.

Second, one formulates desirable *properties* that a solution should satisfy. Here, the word *solution* does not only refer to a particular outcome in a particular problem, but it means a "rule" of which outcome to pick for any problem in the class of conflicts described by the model.

Third, the work of the mathematician is to show that one can really find a solution that satisfies the desired properties. The work does not end there, since knowing the existence of a solution does not necessarily mean knowing how to reach it; there is often a gap between theoretical existence results and a practical, applicable solution. A very important task, then, is to convert theoretical results into a step-by-step manual to a solution. Each step should be easy to comprehend, acceptable to the parties, and should bring the parties one step further to a solution. We call such a manual a *procedure*. A *procedural approach* to resolve a conflict is a process that runs in stages to produce a solution, with a theoretical foundation that shows why each step in the procedure works. The solution obtained fulfills all desired properties that were built into this process right at the beginning.

In this paper, we demonstrate by our running example (the Exercise) how to take a conflict and use mathematics to produce procedures for resolving the conflict. The reader may compare her solution to the Exercise with our approaches, and judge whether he or she sees advantages of using formal methods for developing procedures for a solution.

This paper is organized as follows: In Section 1 we start with the description of a conflict by means of a mathematical model. In the remainder of this section we discuss fairness concepts (envy-freeness and equitability) and their "formal" counterparts. In Section 2 we discuss the step from theory to applicability. The question we address is: how can we achieve a solution by a procedure? Section 3 addresses the need, use and advantages of using mathematics in conflict resolution processes. Section 4 gives historical and technical as well as non-technical references for further reading.

1 Mathematical Descriptions

1.1 Modeling a Conflict

Mathematics offers a wide range of tools for analyzing a fair division problem. Such an analysis begins with a mathematical *model*, whose function is to describe the structure of the problem and the set of possible outcomes unambiguously. Then, in order to determine solutions for the problem, we have to know (and model) the persons' *preferences* over the possible outcomes, which may vary from person to person (and indeed often do— it is embodied in the nature of a conflict that some of the possible outcomes are liked by some but not by others).

We show how to do this in the context of our Exercise, which we now examine in a more detailed fashion.

Example 1.1 Suppose Tony and Chris wish to divide a cake among themselves. We use the word "cake" only for convenience; the object to be divided could be anything (e.g., land, money, time). Certainly, the cake might have a complicated form (in the Figure, for instance, it consists of a number of geometric lumps), but as a first step in modeling we make a simplifying assumption that does not change the fundamental nature of the problem: we assume that the cake has a rectangular form and is 10 cm long. The mathematical description

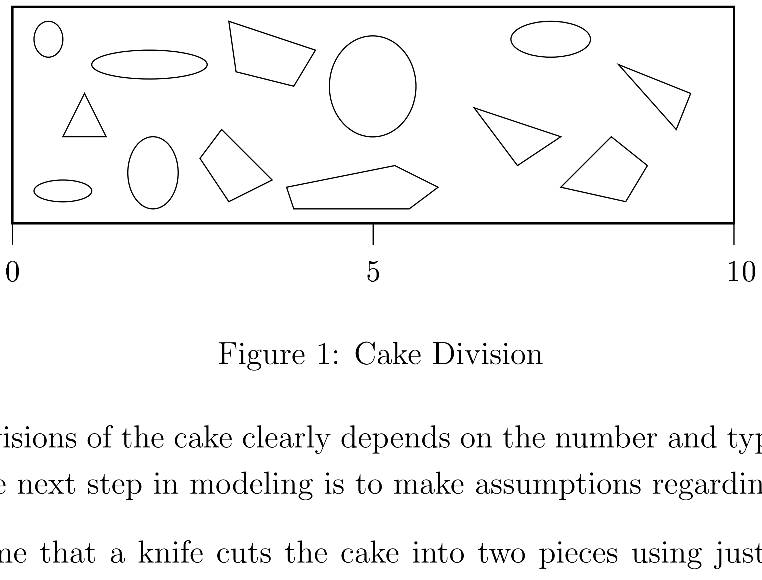

Figure 1: Cake Division

of the possible divisions of the cake clearly depends on the number and type of cuts that are allowed; hence the next step in modeling is to make assumptions regarding this.

Suppose we assume that a knife cuts the cake into two pieces using just one vertical cut. Then each piece is specified by (a) a cut location, and (b) the side of the cut on which the piece lies, either "left" or "right". For instance, (x, left) denotes the piece to the left of a cut

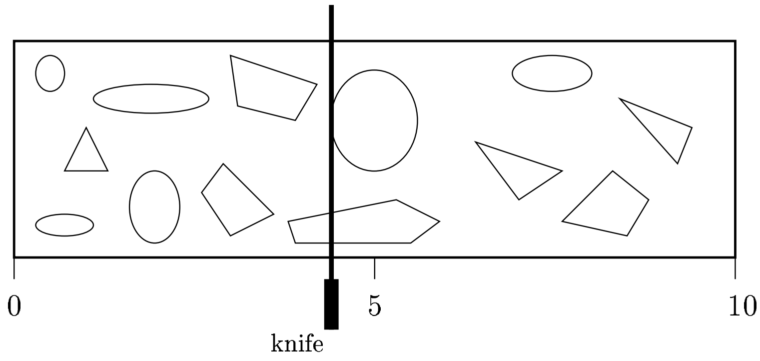

Figure 2: Cake Division: One vertical cut

at x. Hence each possible division is determined by (a) a cut location, and (b) an assignment of who gets which piece. (Mathematically, this is a pair (x, π), where x is a number between 0 and 10, and π is a function that takes C or T to either "left" or "right". We write π^C for the piece that Chris gets, and π^T for the remaining piece, the one that Tony gets.) This describes the structure of the problem.

Now we describe the preferences. It is likely that Chris and Tony have different preferences over such divisions of the cake, and the size of a piece of cake may not be their main criterion for deciding which piece they want. For example, if Chris prefers cherries (which are more frequent on the left side of the cake), and Tony likes chocolate (which is distributed evenly across the cake), and neither care about the mass of the cake itself, then cutting the cake exactly in the middle and assigning Chris the left hand side might have the following effect: Chris might think that he has more than 50% of what the total cake is worth to him. Tony might think that the two pieces are equal (if both contain the same amount of chocolate) and might be indifferent between getting the left or the right piece (which means he evaluates his piece at 50%).

To elicit a complete description of a person's preferences, we must know how that person would evaluate each possible division of cake. As a "scale" for evaluation, we could let each person assign a total of 100 points to the whole cake and distribute this number among the two pieces. Thus preferences could be described by the graphs in Figure 3.

The solid lines in Figure 3 correspond to those divisions of the cake in which Chris gets the left piece and Tony gets the right piece, whereas the dashed lines represent the opposite assignment. Figure 3 tells us how Chris and Tony will evaluate their piece of cake as a function of the position of the cut. □

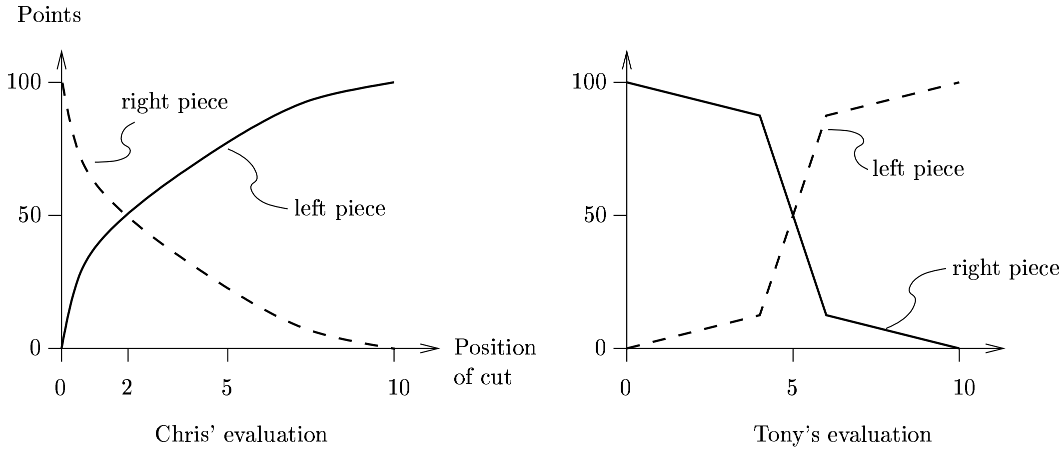

Figure 3: Preferences over divisions of the cake

This example reveals a few issues concerning the mathematical description of the two-person cake-cutting problem.

1. What is the appropriate method to cut the cake?

2. What is the best way to express preferences?

3. How practical is the process of specifying preferences?

We address each of these issues in turn:

1) We assumed that the cake is divided by a single vertical cut, but in practice, one could think of many other versions of cutting (e.g. rotating the knife, cutting more than once, or making no cuts). All these alternatives could just as well be formalized by slight modifications. In doing so, one must bear in mind some external considerations: what is practicable (e.g., can we rotate the knife?), what will allow for the existence of a solution (e.g., making no cuts will not allow for an envy-free division), what is efficient (e.g., is more than one cut necessary?), and what is procedural (e.g., can the solution be obtained by some practical method)?

2) The scale used to express preferences should be chosen with respect to the desired outcome. In the example, each person was to divide 100 points. With this kind of preference announcement, it is not possible to compare how one person evaluates the cake with respect to how the other one does. If we were interested in dividing the cake such that each person gets the "same total benefit" (suppose this could be measured) then we would have to give a small piece to the person who loves cake and a large piece to the person who does not really

care about cake, and the scale chosen above would not be the appropriate one. On the other hand, if we were interested in an envy-free division of the cake (i.e. neither person wants to trade their assigned pieces), then the point assessment is helpful. In assigning 100 points, it is only important how each person evaluates the two pieces, but it is not relevant how he evaluates them in comparison to the other person. Thus the method by which preferences are described is strongly connected with the kind of solution of concern.

3) In the example, each person must evaluate both pieces for a continuum of possible cuts. This may not be practical. Thus there is a trade-off between practicability and accuracy when describing preferences. For practical reasons, it may be sufficient to ask Chris and Tony how they assess the pieces for cuts at 0cm, 1cm, 2cm, ... , 10cm. Both would then only have to announce 22 numbers (2 pieces x 11 cuts) in order to describe their preferences, which is much less demanding than before. In this scenario, one could either estimate Chris and Tony's values of pieces at other cut locations, or simply limit the set of possible cuts to these 11 positions. Both versions have their drawbacks— estimation runs the risk that the assumed preferences may not be the correct ones, and limitation runs the risk of not reaching an agreement at all. In the mathematical description of fair division problems one always has to balance practicability with the accuracy of the data.

All these questions show that one should carefully decide how to describe the problem and how to express personal preferences, before one can even speak about how to resolve the conflict.

1.2 Fairness Concepts

In the previous subsection we were concerned with the descriptions of *all* possible outcomes of the conflict. Now we account for the fact that the desired outcome will be assessed by its fairness properties. For instance, in our Exercise, Chris and Tony may desire an *envy-free* outcome, i.e., a division of cake so that neither envies the other. Alternatively, they may want an *equitable* division, i.e., one in which Chris' assessment of his piece (in points) is the same as Tony's assessment of his piece. Thus the set of desirable outcomes will depend on the notion of fairness chosen.

Envy-freeness

If Chris and Tony are interested in an envy-free division of the cake, then we might model each of their preferences by a *utility function* that assigns to each piece of cake a number that gives a measure of its "worth" to that person. Let u^C and u^T denote Chris and Tony's utility functions. Then, for instance, $u^T(x, \text{left})$ is a number that describes Tony's utility for the piece to the left of x. Then Chris does not envy Tony if the inequality $u^C(x, \pi^C) \geq u^C(x, \pi^T)$ holds true. Analogously, Tony is not envious if $u^T(x, \pi^T) \geq u^T(x, \pi^C)$ is satisfied.

Example 1.2 (Example 1.1 cont.)
A closer look at the preferences in Figure 3 reveals the following: if the cut were made at $x = 2$, Chris would be indifferent between the left and the right piece. For $x > 2$ he prefers the left piece and for $x < 2$ he would choose the right one. Hence, Chris' condition

$$u^C(x, \text{left}) \geq u^C(x, \text{right})$$

is satisfied if and only if $x \geq 2$. Analogously, Tony's no-envy condition $u^T(x, \text{left}) \geq u^T(x, \text{right})$ is satisfied if and only if $x \geq 5$. To summarize, we can say:

- For $x \leq 2$ both prefer the right piece over the left one.
- For $x \geq 5$ both prefer the left piece over the right one.
- For $2 \leq x \leq 5$ Chris prefers the left piece and Tony prefers the right one.

This means that the set of all solutions that do not create envy are those where π^C is "left", π^T is "right" and x is between 2 and 5. This is a complete description of the envy-free solutions for this problem. □

Note that in the example a cut at $x = 2$ and assigning the left piece to Chris, leaves Chris indifferent, i.e. he thinks he has got 50% of what the whole cake is worth to him. Tony, receiving the right piece thinks he got more than 90% of what the cake is worth to him. Although this is an envy-free solution, it may be a division that is undesirable with respect to other fairness criteria, such as equitability.

Equitability

The analysis in example 1.2 has to be modified if we change the fairness criterion. Let us assume now that Tony and Chris both consider a solution as "fair" if each one feels he got just as much in his evaluation as the other did in his. We say a solution is *equitable* if the equation $u^C(x, \pi^C) = u^T(x, \pi^T)$ is satisfied. That is, Chris assigns the same number of points to his piece (obtained by the cut at x) as Tony does to his piece.

Example 1.3 (Example 1.1 cont.)
To solve the above equation in our Exercise, Figure 4 rearranges the utility functions. In the left diagram we have Chris' and Tony's utility functions for the case that Chris gets the left and Tony the right piece. The right diagram shows utilities for the opposite assignment.

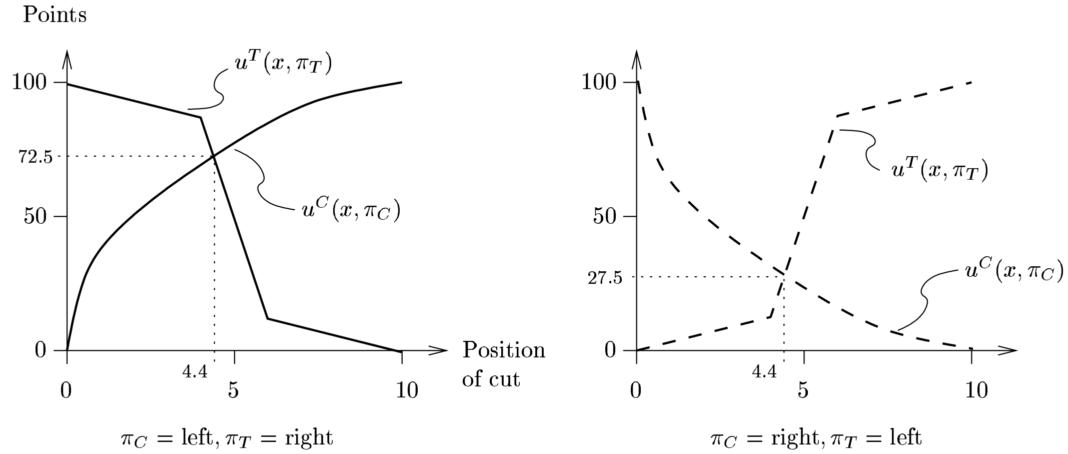

Figure 4: Piecewise preferences over divisions of the cake

The two utility functions on the left diagram intersect for $x = 4.4$, i.e. $u^C(4.4, \text{left}) = u^T(4.4, \text{right}) = 72.5$. With a cut at position 4.4 and Chris obtaining the left piece, both persons assign 72.5 points to their shares. Switching the piece assignment results in the right diagram of Figure 4, where at the intersection of graphs we have $u^C(4.4, \text{right}) = u^T(4.4, \text{left}) = 27.5$ (since the assessments of left and right pieces for any person must add to 100). Thus there are two different outcomes that are equitable (although it is easy to see that one of the two solutions is strictly preferred by both). □

In this example, we see that the concepts of envy-freeness and equitability do not exclude each other. For example, the outcome $(x = 4.4, \pi^C = \text{left}, \pi^T = \text{right})$ also satisfies the envy-freeness conditions. Also, observe the differences between the two concepts— whereas each envy-freeness condition involves only a single person's preference, the equation that determines equitability uses both utility functions at the same time.

Next, we consider a different way of presenting the persons' preferences. We have described the set of possible outcomes of the cake cutting problem. To each such outcome we can assign a pair of numbers consisting of Chris' and Tony's assessment (in points). These pairs can be depicted in a two-dimensional diagram as it is done in Figure 5.

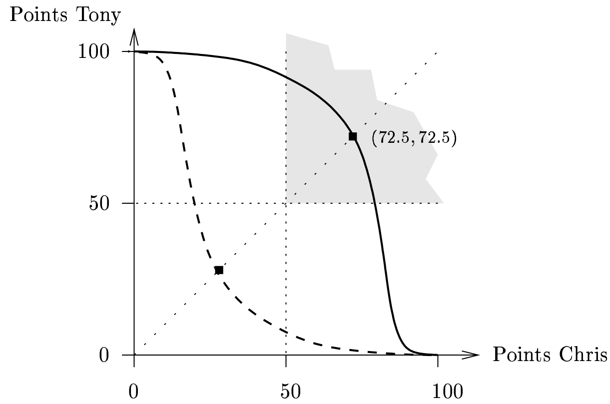

Figure 5: assessments of possible outcomes

The solid line represents all possible pairs of points in which the left piece goes to Chris and the right piece is for Tony, whereas the dashed line corresponds to the opposite assignment. How can we identify equitable solutions from this picture? Since the diagonal (45 degree dotted line) contains all pairs with equal numbers of points, we simply locate the intersection of the diagonal with the two curves. Here we obtain the two pairs $(27.5, 27.5)$ and $(72.5, 72.5)$ (cf. Example 1.3).

Since there are only two persons involved, each person does not envy the other if he thinks his piece is at least 50 points worth (since then the value of the other piece is at most 50 points). So the pairs that correspond to envy-free outcomes are located on the solid line in the shaded region of Figure 5.

1.3 Solutions to the Bargaining Problem

Figure 5 presents preferences in an aggregated form; from the set of possible allocations (pairs) of points, individual preferences cannot be extracted in general. In particular, the Figure does not exhibit the underlying outcomes but only the results in terms of utility (here measured in points).

When one is interested in finding envy-free solutions, this "utility possibility set" is in general not the appropriate description for analysis, because one has to compare the pieces and not the points of the piece assigned. In the previous example we could display envy-free solutions only because there are just two persons involved and utilities were normalized to 100 points. Hence one's valuation of one piece also reveals one's valuation of the other piece. This is no longer possible as soon as there are more than two persons or without normalization constraints.

However, utility possibility sets are a very useful tool to analyze fairness criteria that are based on allocations of utility. For example, an equitable division corresponds to a point in the utility possibility set, in which each agent obtains the same fraction of his maximal possible utility. In the (axiomatic) theory of bargaining, utility possibility sets are used to describe the conflict. This form of representing a conflict was introduced in an early work of Noble-Prize winner John Nash (see Nash (1950)). A *bargaining problem* arises anytime one has a utility possibility set U together with a "disagreement" allocation d in U that represents the utility allocation that prevails when no agreement can be reached. (The source of utilities is a secondary question.) A *bargaining solution* is a "rule" that assigns to each bargaining problem a specific allocation of utilities (a point in U).

The axiomatic approach searches for bargaining solutions that satisfy certain desirable properties *(axioms)*, such as:

Pareto Optimality: The bargaining solution should always select a point for which no other point in U "dominates" it, i.e., there is no other utility allocation in U which leaves everyone with at least as much utility as before and does better for some of the players.

Individual Rationality: The solution should assign to each person at least as much utility as the disagreement point d does. That means everyone should have an incentive to participate in the bargaining process.

Symmetry: The order of the persons does not matter for the solution. This means that the solution only depends on players' preferences and is not influenced by how a player is named.

Scale Invariance: The solution should not depend on the scale that is chosen to measure utilities.

Individual Monotonicity (IM): Suppose there are two bargaining problems (U, d) and (U', d), such that the set of outcomes U' of one contains all the outcomes U of the other, and such that player 1's maximum outcome is the same in both, then the bargaining solution should give player 2 at least as much in U' as in U.

The last two axioms call for an interpretation. The Scale Invariance property roughly says that the solution should not depend on how many points (in total) are assigned to some outcome but on how many points are assigned relative to the maximal number of points.

So, if Chris doubles all points that he assigns (resulting in a maximal number of 200 points), then a solution that is scale invariant selects the same outcome as before.

Figure 6 illustrates the basic idea behind the IM property. In (U', d) there are more possibilities to allocate utility than in (U, d). Facing the "bigger" bargaining problem, Tony could argue that for each fixed utility level (number of points) for Chris, he can obtain more utility in U' than in U. Therefore the solution φ should give him more in U'. In this spirit the situation U' is more advantageous for Tony than U is. Chris is indifferent between U' and U, because he could get the same maximal utility (90 points) in either case. For this reason the IM property only requires that the advantaged player should get more. Applied to the cake

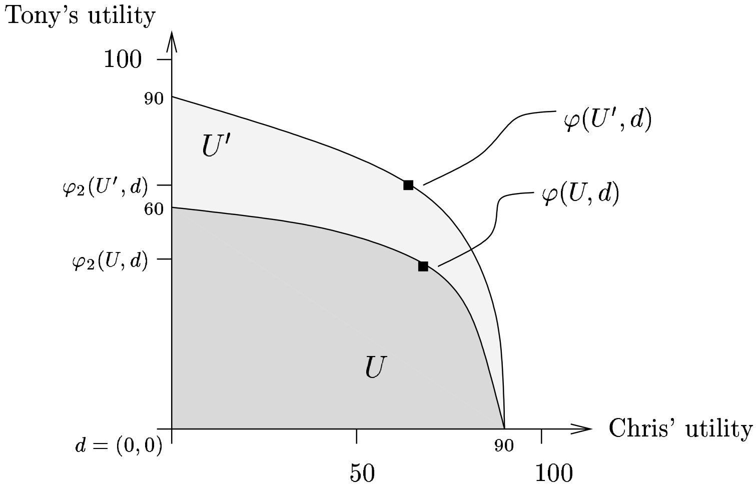

Figure 6: The IM property

cutting problem, an enlargement of the set U to U' could be achieved as follows: suppose Tony loves nuts and Chris is indifferent to them. Now the cake is refined with additional nuts. Tony is delighted, because the cake (and every piece of it) is now worth more to him, while Chris does not realize a change. Comparing the solutions for the two versions of the cake, Tony should be at least as well off in the situation with nuts as without.

In the 1970's, Ehud Kalai and Meir Smorodinsky introduced a bargaining solution[1] for (a certain class of) bargaining problems for two persons that satisfies all these five properties. The idea for this solution goes back to Raiffa (1953). Roughly speaking, the Kalai-Smorodinsky (KS) bargaining solution can be constructed as follows: One connects the disagreement point with the point that represents each player obtaining his maximal utility. The latter point is (due to the nature of a conflict) in general not located in U. The KS solution is now

[1]see Kalai & Smorodinsky (1975)

the upper right most point on this connection line that is still in the utility possibility set Figure 7 illustrates the construction for a typical cake cutting problem from Section 1.

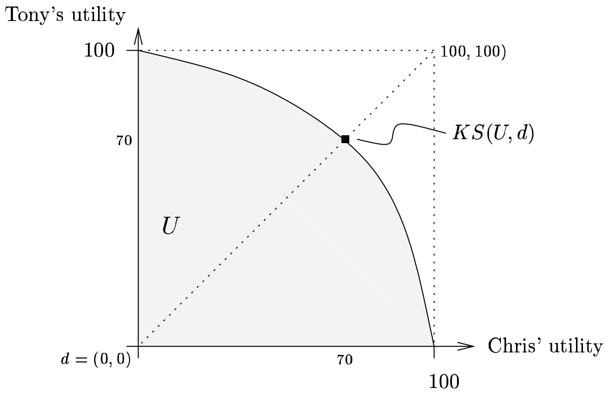

Figure 7: The The Kalai Smorodinsky solution

It can be shown that the above discussed properties are sufficient to uniquely characterize the KS bargaining solution. This means that any bargaining solution with these properties must be the KS bargaining solution. This way of determining solution concepts by their properties is frequently referred to as *axiomatization*.

There is also a description of the fairness concept behind the KS solution that comes from its construction. In the KS solution point, Chris and Tony both get the same fraction of their maximal possible utility (In Figure 7 this fraction is 0.7). In the cake cutting scenario, this means that both get an equal share of what they think the whole cake is worth. This is an equality in terms of utility. Note that this is exactly the same argumentation that we used in Section 1 for equitable solutions. Indeed, the KS solution is the bargaining theoretic counterpart to equitability.[2]

In summary, there are three approaches to the KS solution: 1) the construction method (connection line), 2) the axiomatic approach (KS solution is characterized by certain properties) and 3) the fairness approach (equal fractions of maximal utility).

Besides the KS solution there are many other bargaining solutions that can be approached the same way, e.g., the Nash solution (cf. Nash (1950) and Nash (1953)) or the Perles-Maschler solution (cf. Perles & Maschler (1981)).

[2]Raith & Welzel (1998) showed this analogy by analyzing the *Adjusted Winner* procedure that was introduced in Brams & Taylor (1996) to perform equitable divisions of a set of objects.

2 From Solutions to Procedures

The approaches that we have described above not only model the structure of a fair division problem but also determine solutions. However, there is one decisive question left open: *How do we arrive at a solution?*

For the players, the acceptance of a solution is strongly connected to the transparency of how and why it is determined. Theoretical results can answer the “why”, but the “how” is widely ignored. In Examples 1.2 and 1.3 envy-free and equitable solutions were constructed by modeling and analyzing the preferences. But the calculation of the final solution can be carried out without involving Chris and Tony. Thus, there is a danger that the conflicting parties may not accept the solution because they do not understand the method by which it was determined.

Procedural approaches are exactly concerned with these practical difficulties. Without leaving the ground of a formal (mathematical) description, a *procedure* wants to give “step by step instructions” to the solution. We believe that such a guide should exhibit the following features:[3] Each step should be

1. *intuitive*, i.e. it should be easy to comprehend.

2. *plausible*, i.e. should be simple to argue.

3. *manageable*, i.e. must be straight forward to compute.

Moreover, a procedure should lead to a solution after finitely many steps. In this section we examine three procedures for our Exercise: the well-known *Divide and Choose* procedure, Austin’s *Moving knives* procedure, and a *polling* procedure. In these examples we demonstrate how mathematics is used to reveal fairness properties of the outcomes.

Example 2.1 (Divide and Choose)
Probably the most famous procedure that most people know from the division of chocolate bars in their childhood is *Divide and Choose*.[4] The rules are very simple to understand. Chris (names could be interchanged) takes the knife and cuts the cake into two pieces. After that, Tony may choose among the two pieces. Fine! But what property does the outcome satisfy?
Let us restate the rules combined with suggestions. Chris cuts the cake into what he considers

[3] see also Haake, Raith & Su (2002)
[4] see also Brams & Taylor (1996)

to be equal pieces. Then Tony chooses the piece that he likes most. Now it is clear that the outcome should be envy free. Chris does not envy Tony because he considers both pieces as equal, and Tony had the choice. With respect to envy-freeness the solution is fair. □

However, the procedure has its drawbacks with respect to other notions of fairness, such as equitability, since the outcome generally favors the chooser (since the cutter gets exactly 50% of what the cake is worth to him and the chooser usually gets more). So the final outcome is almost certainly not equitable. (And, if the cutter has information about the chooser's preferences, the procedure can be exploited by the cutter to favor the cutter: he can achieve more than 50% by cutting the cake so that the other player considers the pieces almost equal but leaving the piece the cutter likes a little less attractive to the chooser).

Example 2.2 (Moving knife)
Austin's procedure (Austin (1982)) is a method for dividing a cake into two pieces between two persons so that each person believes it is divided exactly in half. Let Chris hold two knives over the cake, with one at the left edge, such that the portion of cake between them is what he believes to be exactly half. If Tony agrees that it is exactly half, the procedure is finished. Otherwise, let Chris move the knives across the cake from left to right, keeping the portion between them exactly half (in Chris' estimation), until Tony agrees it is exactly half. (There must be such a point because when the rightmost knife reaches the right edge, the leftmost knife must be where the rightmost knife began, hence Tony must by that point have changed preferences.) At this point cuts are made and the cake outside the knives are lumped together, yielding two pieces which both agree are exactly equal. □

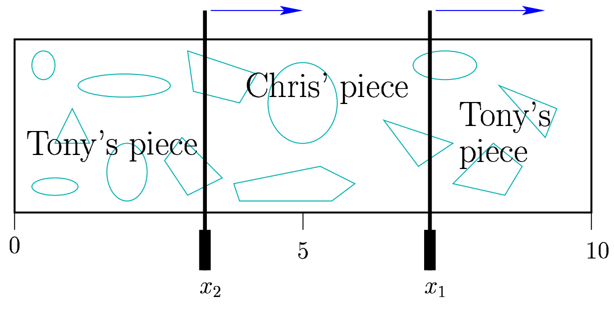

Figure 8: The Moving Knife Procedure

The basic difference between the two procedures is the following: the *Divide and Choose* procedure generates envy-free outcomes, the *Moving Knives* provide an equitable division of the cake. Clearly, Chris and Tony get exactly half of the cake, provided they follow the rules of the procedure and represent their preferences honestly. But there is a drawback with Austin's procedure. As we have seen in Section 1, there are very well equitable divisions so that both get more than 50% in their assessments. Austin's procedure gives both exactly 50%. So the property of *Pareto efficiency* (see Subsection 1.3) is not fulfilled. In general there are divisions that are equitable but ensure both a higher utility.

We may check the fairness properties of both procedures by returning to the formal model developed in Section 1. For this, consider once again Figure 3. *Divide and Choose* tells Chris (as cutter) to select the division of the cake so that he thinks both pieces are equally valuable (i.e. at $x = 2$). In the left diagram, the position of the cut is given by the intersection of Chris' utility functions (since the sum of his points for both pieces is always 100). With a very mild formal condition on preferences (called continuity), one can always guarantee existence of such an intersection point. Therefore the procedure can be carried out as stated, and as argued above, the outcome is envy-free. The mathematical description also reveals that the procedure benefits the chooser.

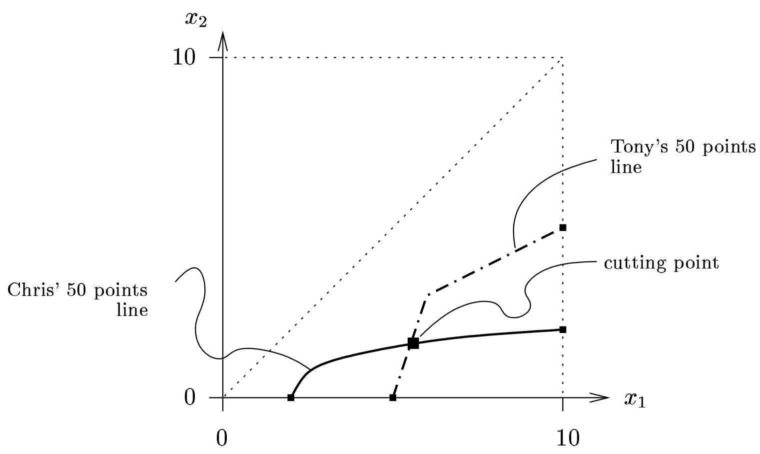

Figure 9: The Moving Knife Preferences

For the *Moving Knife* procedure, consider Figures 8 and 9. Figure 8 simply clarifies who gets which piece. Figure 9 displays Chris and Tony's preferences in a slightly subtle way. Any point in the x_1-x_2 diagram reflects the position of first and second knife. Because the x_1-knife is always on the right of the x_2-knife we only have to consider points below the diagonal (here $x_1 \geq x_2$ holds). The solid line now represents all sets of cuts for which Chris is indifferent between getting the interior or the outer piece. Hence, with all points on that line, Chris obtains a 50% share of the cake. Tony's 50% line is the dashed one. His line starts to the right of Chris' line, because at the beginning, Tony strictly prefers the right (outer) over the left (interior) piece. Tony's line must (and does) terminate above Chris' line, because assignments just have changed compared to the starting situation. Then (this is what math tells us in general) there must be an intersection point of the two lines, i.e. a set of cuts in which both receive exactly a 50% share. Roughly speaking, the procedure finds this point by "moving along" the solid line and stopping at the intersection point.

Example 2.3 (A Polling Procedure)
Finally, we describe a third procedure that illustrates some of the earlier considerations regarding practicability. Suppose, that because of practical reasons (e.g., time or limited resources) we can only ask for the preferences of the players at a finite set of cuts 0cm, 1cm, 2cm, ... , 10cm. Can we find an envy-free division?

This polling procedure begins by successively moving the knife, from left to right, to the positions 0cm, 1cm, 2cm, ... , 10cm, pausing to ask each player at each cut "Which piece would you prefer if the cake were cut here?" Players just answer either L or R at each cut position (they do not need to assess their utilities of pieces). Based on these answers we can suggest a division that is *approximately envy-free.*

First note that if there were a cut for which Chris and Tony answered differently, then we could cut there and assign them the pieces that they requested. But this may not always be possible (e.g., if Chris and Tony were to answer the same way at all 11 cuts). However, we can always find a 1cm segment of cake where at opposite endpoints Chris answered one way and Tony answered the other way. Then cut the cake in the middle of this segment and assign Chris and Tony the side of the cut they preferred at the endpoints. This solution may not be envy-free, but it is envy-free to within the utility specified by one segment of cake.

Take for instance the problem illustrated in Figure 3. If we ask Chris and Tony, which piece they preferred at the 1cm mark then both respond "right piece". At 2cm Chris is indifferent, whereas Tony sticks to "right". We can finally cut the cake at the 3cm mark, as Chris and Tony prefer different pieces. □

How do we know there is such a segment? Again, we appeal to mathematical argument—for the cut at 0cm, both players will prefer piece R, and for the cut at 10cm, both prefer L. We may assume that they answer the same way at all cut locations (else we would have an envy-free division). So as we proceed along the 10 segments from 0cm to 10cm, there must be a segment on whose endpoints their preferences will switch (from L to R).

At first, a procedure to obtain an *approximate* envy-free division seems unnecessary— after all, Divide-and-Choose is a procedure for exact envy-free division and it is simple enough to implement. However, as the number of players increase for the cake-cutting problem, the difficulty of obtaining a practical procedure for exact envy-free division rises quickly (Brams & Taylor 1995), making approximate procedures more attractive. Our two-person polling procedure illustrates the basic idea of the approach advocated by Su (1999) for approximate envy-free solutions of n-person fair division problems.

3 Why do we (have to) use Mathematics?

To summarize, what lessons do we learn about the relationship between mathematics and procedures for solving fair division problems?

Even if we do not "see" the mathematics in the final description of a procedure, it is an indispensable analytic tool. As we observed, it is used to guarantee that the procedure does work and produces a well-defined and desirable outcome. Second, and this is probably more important, mathematical methods reveal the fairness that the outcome of a procedure should satisfy. If all agents follow the rules of the procedure, then the outcome is guaranteed to satisfy some specific fairness criterion like, e.g.,envy-freeness or equitability. Third, mathematical insights also reveal the limitations of a given procedure (e.g., in our discussion on Divide-and-Choose, the proposed solution is envy-free but still favors one party using a different fairness criterion). Therefore, theoretical models not only help to characterize and find fair outcomes, but they are also necessary for a critical assessment of conflict resolution procedures.

Also, the mathematics of setting up a model for a division problem requires preferences to be elicited and the structure of allowable agreements to be made precise. This process can be beneficial to players, as it can help them to pin down what their preferences really are and equip them to assess their desires critically. (People who have not done this critical assessment often have a harder time making decisions.) Moreover, it will help them think broadly about what divisions are possible.

Finally, the axiomatic notion of a "solution" being a rule (that hopefully can also be supported by a *procedure*), which solves a whole class of conflicts, is a powerful idea. It frees us from focusing only on just the one problem at hand, and allows one to pick fairness criteria (axioms) *before* getting caught up in a particular struggle. A solution based on axioms will seem less arbitrary to players than an ad-hoc, situation-specific method developed only for the one conflict at hand.

4 Concluding Remarks

Although fair division problems have been around since antiquity, Steinhaus (1948) was perhaps the first to pose the question as a serious academic endeavor when he asked: *how to cut a cake fairly?* Since then, there has been a lot of work on this classical problem as well as related problems: *burden division* (dividing an undesirable cake), dividing mixtures of goods and burdens, allocations of indivisible objects, division into unequal shares, and explorations of all these types of divisions according to various notions of fairness. Two notable surveys of the fair division literature are the books by Brams & Taylor (1996) and Robertson & Webb (1998), the latter being somewhat more mathematical in nature.

And, just as traced in our paper, there is often a gap between the development of work that focuses on existence of solutions, versus work that develops usable procedures. For instance, the existence of cake-cutting solutions can be traced back to the 1940's but the first general n-person procedure for cake-cutting due to Brams-Taylor was not developed until the 1990's. Even that is not a usable procedure in practice, so there remains much work to be done.

In addition to the debate about the importance of various fairness criteria, there is also (and should be) healthy debate about the aspects of procedures that make them acceptable to participants. For instance, how do various procedures compare on how intuitive, or plausible, or manageable they are? The examples we gave for 2-person cake-cutting are all very easy to follow and would rate well in all three aspects, but n-person counterparts for this and other fair division problems can be a bit more complex.

Even if some of these procedures are lacking in the "manageability" arena, one exciting recent development is the possibility of using computers as "negotiation support tools" to help give a mediator a hand in managing a complex procedure whose foundation is mathematical. The *Fair Division Calculator* (Su 2000) is one example that was recently incorporated into ARTUS, an online negotiation platform (Raith & Wilker 1998). This is a promising first step to give automated procedural support to mediators in a live negotiation.

In this article, we hope that we have given the reader a sense of how mathematics can be used to support the development of procedures for fair division problems, not just ones that are mathematically-supported but ones that can actually lead to practicable uses.

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