Topology Through Inquiry

Instructors' Resource Manual

by

Michael Starbird
Francis Su

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CHAPTER 1

Educational Philosophy

Topology Through Inquiry is designed to be used with an instructional technique variously called Inquiry Based Learning (IBL) or guided discovery or some other phrase that suggests active investigation on the part of the learner. These styles of instruction give students a totally different experience compared to a standard lecture course. We will explain the method in more detail later, but the essence is this: At the beginning of the semester, students are handed the *Topology Through Inquiry* book, which contains a motivated treatment of the ideas of topology, along with a list of theorem statements as well as exercises without proofs. The standing homework assignment is for each student to prove designated theorems or settle the next designated exercises in the book, thus in a sense writing his or her own textbook. In class, the instructor asks students to present their own proofs to the rest of the class. The students have devised an argument for why the assertion is true and present the proof. Other students try to understand the proof. The instructor encourages members of the class to ask questions or re-phrase parts of the proof. Each student is responsible for determining whether the proof is correct. The correctness is not based on an authority figure (the instructor) asserting it is correct, but instead on the ability of the individual students to follow the argument.

Here is the effect of this experience: Students learn to think independently. They learn to depend on their own minds to determine right from wrong. They develop the central, important ideas of topology on their own. From that experience, they learn that they can personally create important ideas. They learn to turn to themselves when they are confronted with a question, whether it's a mathematical question or a question in other parts of life. They develop an attitude of personal reliance and a sense that they can think effectively about difficult problems. They are less likely to accept assertions on the basis of authority alone if they do not personally understand why the assertions are true.

Notice that the effect on students goes far beyond mathematics and far beyond the classroom. Education can shape minds. This type of course systematically develops habits of thought that change students for the better. It often opens their eyes to a new way of looking at their whole life experience. Almost all of education teaches students that they are powerless pawns whose job is to learn some fraction of what some authorities know. But when students daily think through questions that seem hard and create new ideas on their own and determine correctness based on their own ability to discern the soundness of arguments, they frequently break free from the bonds of passive learning and simply explode with an exuberance of independent learning.

A guided discovery or IBL course can liberate students to learn and think on their own. The effect of this type of course can strike very deep. We are not just talking about teaching topology or theorem-proving well. That is important, but it is not as important as teaching students to be better, more independent thinkers every day, in every endeavor of their lives. Not every class or mathematics class needs to emphasize personal discovery, but that experience can be an important piece of education that can affect how students learn, work, and live. A single course is a small fraction of a student's education, but a tiny habit of independent analysis repeated over a lifetime can make the difference of a lifetime. If people like to think and like to explore new ideas, the education they acquire themselves over their lifetime will be vastly greater than any amount of material covered in a semester's class. Attitudes far outlast facts. Knowledge comes and goes, but hatred lasts forever. Perhaps the joys of thought and discovery last forever too.

1. Watching

. We live in a world where watching is a large part of the norm of daily existence. Watching smart phones, watching sporting events, watching the news. Time spent watching consumes a large part of our days. Schools and colleges also tend to emphasize watching. Classes are frequently times for students to watch someone else do something. Apparently, good students are those who watch well. These habits of passivity are hard to break. Our educational system systematically rewards attentive passivity. A high proportion of the experience that our students receive through their education encourages them to look for authorities and to view learning as the acquisition of facts. Yet our society needs leaders who think independently, people who can cope with novelty and find ways to solve problems that have not been clearly posed or answered.

2. Keeping up

. Knowledge is being developed or discovered at a faster rate than ever before. In the technical fields especially, the rate of increase in knowledge is exploding. Each year hundreds of thousands of articles are published in research journals in mathematics, biology, chemistry, and computer science. How do we as educators face the daunting task of keeping up?

The answer is, we cannot keep up. Keeping up is a feeble dream and has been for decades. Now it is all the more important for us to empower our students to be able to deal with situations and knowledge that they do not know and that no one knows. The accelerating production of knowledge means we must abandon the hope of keeping up with new knowledge and must instead take seriously the job of teaching students the skills of navigating in a sea of change. Change is the world for which we are attempting to prepare students. Some knowledge and skills are certainly fundamental and will always be useful, but many specific skills that were previously the bedrock foundation of an educated person must now be looked upon with skepticism. We must try to teach life lessons that students will keep and use in settings we do not foresee. *Topology Through Inquiry* can contribute to creating in our students the skills and values that will empower them to cope effectively with a fast-changing world.

CHAPTER 2

Role in the Mathematical Education of Students

1. Prerequisites

Topology Through Inquiry is designed to foster mathematical maturity by teaching students a suite of attitudes and skills. Students learn to prove theorems on their own and to assess the correctness of proofs. They learn many techniques of proving theorems. They gain experience in presenting their ideas orally and in writing. They learn fundamental ideas of topology. This course is designed to help students adopt a mathematical perspective that features active participation in developing ideas that are new to them and in developing proofs of mathematical assertions.

This course could theoretically be presented to students who have had little or no proof-oriented mathematics previously. No specific mathematical content is required as a prerequisite. However, in practice we have found that the abstraction of topology often presents an obstacle to students who are new to proof. So generally, students will be more likely to succeed with this book if they have had one or more proof courses before, perhaps including an introduction to analysis.

2. Introduction to topology and abstract mathematics

The text provides a solid introduction to topology. The breadth of included topics make it suitable for any undergraduate or first year graduate topology course. In addition, there are many other topics that are not generally part of standard courses and will reward the engaged reader with some beautiful delights.

3. Advanced Transitions-to-Proof

The text can be successfully used in courses meeting many educational goals. The concept of transition-to-proof really embraces a continuum of mathematical maturity. One step is learning specific techniques such as inductions or proof by contradiction. However, learning to prove theorems on ones own also involves additional experience with facing the unknown and effectively struggling to seek insight. We strongly believe that topology is an excellent subject for learning the art and practice of proving theorems independently and for learning the ways of mathematical thought.

4. Future teachers

Future teachers can gain special benefits from this course. They can use a deeper understanding of topology to enrich their future presentations of mathematics to their own students. They will gain experience in presenting mathematics and will be able to use and convey to their students their increased skills in logical thinking and proofs. Also, they will have a classroom experience of a style of pedagogy that they can draw from as they develop their own methods of teaching.

5. Quasi-research in the classroom and beyond

The nature of the Inquiry Based Learning presentation makes the text eminently suitable for giving undergraduates rich research-like experiences. The book can be used in a classroom setting and for independent studies, honors projects, inter-session courses, or other quasi-research experiences. In fact, we believe an inquiry based learning experience can be just as, if not more, valuable than a “true” undergraduate research experience. Individual research projects for undergraduates present many challenges for faculty members and for institutions. Frequently, finding and directing an individual research project is challenging, time-consuming, and the resulting experiences for the students are variable. Sometimes they are the life-changing, exciting, intellectually satisfying experience we seek; and sometimes the project turns out to have some unexpected defect, such as not yielding sufficiently many results or a sufficient variety of challenges to be of great interest. These problems are inherent in research, but present special obstacles as undergraduate experiences.

In almost all cases, finding and overseeing an individual research project is a time-consuming effort for the faculty member. Most colleges and universities do not have the faculty resources to provide individual undergraduate research experiences for more than a small percentage of their undergraduates. An alternative is to offer inquiry based classes that systematically provide a whole class of students with a quasi-research experience by asking students to tackle questions that are new to them, while not being new to the world. In many ways this experience is often superior to individual research projects because the careful control of the challenge questions gives students a predictable range of research experiences. Students can develop attitudes of self-confidence and personal reliance. They can develop intellectual stamina and the sense that they can personally undertake to solve unknown problems that arise in their academic and personal worlds.

CHAPTER 3

Practical Issues

1. Sample Syllabus

What follows is a first day handout for an introductory topology course using *Topology Through Inquiry*. It gives the students some sense of what the class will be like, but they won't really understand what is expected of them until a couple of weeks have passed.

Topology

Instructor: Emmy Noether
Office Hours: By appointment

Rules of the game: The class will be conducted using a method of instruction called Inquiry Based Learning or Guided Discovery. This method fosters creativity and independent thinking. It is also fun. Your book contains lists of exercises and theorem statements without proofs. You will do designated exercises and prove designated theorems on your own and present your results to the class. These presentations are a major part of the course.

The ideal model to follow is to settle the questions and prove the theorems independently and write-up your solutions before the answers are presented in class. You may not consult books, the Internet, other people, or other sources. [Note: Some instructors encourage collaboration among the students in the class. With this approach, it is useful to tell students that before collaborating they should make a serious independent attempt at settling the questions or theorems before discussing them with their fellow students. And the students should be instructed not to tell one another the complete solutions, but just to discuss ideas and give hints. Collaboration is an excellent thing to foster and both strategies (not allowing collaboration or encouraging collaboration) can work well.]

Each day, I will select students to present their solutions in class. Your standing assignment is to be prepared to present your results. When you are presenting your proofs or solutions, strive to make your explanations clear and organized. When you are observing a presentation, it is your responsibility to follow the logic of the solution and verify that it is correct for yourself. You may be asked during class to re-explain an argument that you just heard. If you cannot follow the reasoning, it is your duty to ask a question of the student(s) presenting. If you are truly stuck on a question or proof outside of class, do not hesitate to ask the instructor for help. You should be working far enough ahead of the classroom presentations so that there is time for this consultation.

Homework: Your standing homework assignment is to write up solutions to the designated exercises and theorems in your book *Topology Through Inquiry* before they are presented in class. You will submit your solutions on a daily basis. Please make every effort to keep your solutions neat and clear. You should also keep a notebook containing all your notes and solutions; this notebook will serve as your personal textbook for this course and will help you study for exams. Writing up the proofs and solutions is an excellent way for you to learn the mathematics.

Exams: There will be two hour-exams during the semester. Each will be announced at least one week in advance. The final examination will be comprehensive.

Grade: Homework and presentations: 30%
Hour exams: 20% each
Final: 30%

CHAPTER 4

An Introduction to Inquiry Based Learning Instruction

1. First Day and Early Days

On the first day of class, you might give a little speech about the method and why you are using it. You might mention that the fun of mathematics is doing it for yourself and "doing mathematics" means to prove theorems, make conjectures, and settle questions on your own. Many mathematicians remember the first time they proved things on their own as a special moment in their own histories. Emphasize that you know the students are not expected to prove every theorem. Emphasize that everyone will make mistakes and that mistakes are valuable sources of knowledge. Allay their fears.

At this point start to establish habits for presentations. If you want them to write the statement before the proof, have them do that. Ask them to write complete sentences including the connective words. You might ask students to go to the board in groups of 3 or 4 and write a joint proof of the designated theorem. You might then choose a group with a good proof. After the proof is presented, ask for questions. Ask individuals in the class questions such as, "Why did the presenter write the second sentence?" Answer, "Because that explains what the definition of the symbol is." Etc. Then praise the students' work specifically, that is, say, "This proof is just right because the first sentence stated the assumption of the theorem and every sentence either used a definition to restate the previous statement or used previous statements to deduce their truth." Then go on to the next theorem, again in groups.

Your goals in the first several days are to get the students comfortable with the basic idea that they can prove theorems on their own. You want to establish good form, that is, full sentences, neat presentations, and not leaving out steps.

During these first few days, your students will make some important mistakes. Ask other students if they agree with the proof. Ask them to state whether and, if so, why various statements are true. Sometimes ask them about statements that are true and sometimes ask about those that are not true. Avoid saying that a proof is correct or not correct. But if it is not correct, when you ask a student to explain why some sentence is true, they will be unable to do so. At that point you can say one of several things. You might ask the class to get together in small groups to prove a statement whose proof was omitted. Or you might ask the groups to see how to fix the proof. You might ask the presenters to expand on an idea. If it is seriously flawed, you might ask the presenters to correct the proof and return at the next class period. Or you might ask someone else to do it. Always thank the presenters for helping the students to understand things. Sometimes after a correct proof, we will ask a student whom we know to have an incorrect proof to show the wrong proof. We might say from the start that the upcoming proof is wrong, but we want the class to see it because it shows an important idea. This strategy gives students the idea that wrong proofs can be valuable steps toward understanding and they should not feel embarrassed or crushed by making a mistake.

2. A Typical Day

During a typical day, the instructor might have six or so groups of three or four students (if your class is that large) to go to the board where each group has a designated theorem or exercise to consider. Ask each group to discuss and write down a proof of the designated theorem. Ideally, every student is standing up, working in a group. Often the groups work for perhaps 20 minutes while the instructor walks around from group to group listening and sometimes giving advice. After they have all written up their proofs, ask the first group to present their proof. The instructor stands near the back of the class, watching the presentation and watching the other students. After the group finishes the presentation, the instructor conducts the discussion. Sometimes, you might ask, "Are there any questions?" Other times you might say, "Maria, would you please explain the second sentence of the proof?" And then, "Juana, do you agree with Maria or the presenters and why?"

The instructor's goal is to engage as many students as possible during the class time. Avoid letting the best students dominate the discussion. Generally, don't let the strongest students speak at all during a discussion of an easier theorem. Select a student to present a theorem whom you guess to be about in the middle of those students who are likely to have done that particular theorem correctly. You don't always want a perfect presentation and you don't always want a total loss. Some instructors just choose presenters randomly, but after you get to know the students, it is probably better to select students who are going to give the best experience from the presentation. A perfect presentation is not ideal, and neither is a complete loss.

After the discussion of a presentation is over and the students feel the proof is correct, then you might well make this kind of statement, "One really good aspect of this presentation was this: Did you notice that they wrote down why the open set U containing p contained a point q of A? They stated that p was a limit point of A and that the definition of limit point states that every open set around a limit point must contain a point of the set other than the limit point itself. Then they had a specific point to work with."

If the presenters get stuck, or an error is discovered, you might ask the class to get together in groups of 3 or so and work on an example that might help. Or you might ask students to think of an easier statement that might help. Or you might suggest a hint, basically a lemma or a special case. You might ask them to work in groups or you might give a hint and let them work on that for homework.

At the end of the class session, you might say, "We'll start next time with Theorem 1.7 and should be able to present 1.7, 1.9, 1.10, 1.14, 1.16, 1.22, and 1.25." This gives students some sense of what theorems they should feel responsible for for the next session.

3. Tips

Try variation. A "typical day" is rarely typical, and it is important to vary your inquiry based teaching techniques to find those that work best with your specific class and for you. There are several aspects of the course where choices you make influence the overall effectiveness.

Will you assign problems to individuals or groups in advance or will you ask for volunteers or will you simply assign individuals or groups to exercises or theorems? Assigning theorems in advance speeds the pace of the class, but if Joe knows Sara has been assigned Theorem 2.1, he may be less inclined to work on Theorem 2.1 himself. Typically a mixture of these methods works best.

Will you allow your students to take their written work to the board for their presentation or should they present "notes-free"? It may seem like a small issue, but the answer to this question may make quite a difference in how much time the students spend really learning their proofs. Sometimes we have students use their notes in the early part of the semester while encouraging more "notes-free" presentations near the end of the semester.

Will you present any proofs? There are some hard theorems in the text and it is best to decide ahead of time what strategies you will use when your class inevitably hits a wall with a particular theorem. One solution is to present some of the most difficult proofs yourself. Another is to present a proof to a student outside of class and have the student present it to the class. Another solution is to watch the class stew and sweat as you anxiously await their proof.

The pace of the course will vary with the material, as well as from class to class. If you find that things are proceeding too slowly there are several things you might try. You could only have a subset of the assigned proofs of theorems or exercises presented during class, leaving the others for written homework only. Assigning problems to individuals or groups in advance rather than taking volunteers (or cold calling) tends to pick up the pace. Having multiple students up at the board(s) in groups writing up their proofs at the same time, then going through them one by one is faster than having students present while writing as they present. Using a document camera also decreases the time for actually writing words on the board.

Experienced students. One good way to avoid having experienced students dominate the class or get bored is to give some hard theorems for students to work on if they get ahead. For example, have them prove the Alexander Subbasis Theorem. Then ask them each day whether they have made progress as they walk in or leave and discuss ideas with them. Just a few seconds can keep the more advanced students occupied. And when they succeed with such a challenge theorem then they can present the result. Of course, the more successful students are also a good source for doing the regular harder theorems and for moving on if things start to bog down. That is, if things seem to be going too slowly, just ask the students who can be counted on to give excellent presentations to make suggestions or give presentations. Using those students appropriately can help you regulate the pace of the course. You don't want to go faster than the speed at which most of the students can successfully prove most of the theorems, but you don't want to go too slowly.

Less experienced students. One good way to help shy or less experienced students is to ask them outside of class to come to your office and present some theorem that is coming up. After helping them to understand it thoroughly so you and they know that they can do it and present it well, then during class assign that student to the appropriate group and say in the usual way, "Now we will do Theorem 1.24. Let's see, Oscar would you present this one." In other words, make it seem that you have selected that student in your normal way. When a struggling student has presented something, of course, praise them. Also, help them with questions. In other words, if they are asked a question that they start to get flustered with, then you can help by re-phrasing the question in a helpful way or say, "By the way, in your proof you did . . . , would that answer this question?"

Student attitudes. Generally students are quite appropriate in their comments, that is, they do not tend to be mean or critical. There is a lot of honesty in this format of teaching. That is, students display their weaknesses regularly. It is not in the cultural norm to show weakness or mistakes in public, but you can get the class to accept this reality as simply a good part of what happens by actively praising the value of making mistakes. Learning happens when we push against misconceptions that we are prone to make when we study something new. This experience can be one of the most positive parts of the course, because students can realize that pretending to know more than they do is ultimately not the most productive way to learn or be.

Audience. At first, the students will direct their presentations to the instructor. Don't let them do this. Tell them directly that their fellow students are the audience and then turn your eyes toward the other students to encourage the presenter to do so also. The main thing is do not be a source of confirmation of the correctness of the proof. The presenters must understand that they are convincing the other students, not the instructor.

Students like this experience. Students will enjoy this class if they get a repeated sense of personal accomplishment. The joy of mathematics really is in doing it—seeing patterns and making conjectures, devising convincing proofs that a statement is true, and understanding an argument and having the confidence to know when you know it is true. These experiences are what this course can offer the students throughout the class.

Resources on Inquiry Based Learning Instruction

There are many resources available for IBL instruction. Attending a conference or workshop on IBL is very helpful.

Here is a great place to start to look for resources about IBL and to connect with a community of supportive practitioners is the Academy of Inquiry Based Learning (AIBL) website: http://www.inquirybasedlearning.org/ . From there you can find many resources.

Another source for resources is the Mathematics Learning by Inquiry (MLI) website: http://www.mathlearningbyinquiry.org/ .

We encourage you also to make human connections. There are many members of the IBL community who will welcome your inquiries and connections. If you attend an IBL conference or workshop, you will find an encouraging and welcoming community of dedicated educators.

CHAPTER 5

Possible Paths for Various Potential Courses

Topology Through Inquiry contains far too many exercises and theorems for any semester long or even year-long course. Every instructor will have to decide for yourself which theorems and exercises to assign.

The Preface of *Topology Through Inquiry* contains some sample collections of theorems for different potential courses. For specific sample syllabi, check out the *Topology Through Inquiry* webpage, maintained on Francis Su's website.

If you are an instructor using our book, **we encourage you to submit your sample syllabi for inclusion on this webpage, for the benefit of other instructors.**

In every case, we encourage you to modify these collections to suit you and your students. A lot depends on how mathematically mature, motivated, and knowledgable your students are. In some cases, you will need to provide more scaffolding or hints in order for your students to be able to prove the theorems. In all cases, there are plenty of exercises and theorems to keep even the keenest students happily occupied.