Topology Through Inquiry

Michael Starbird and Francis Edward Su

January 18, 2023

Contents

Part I

Point-Set Topology

Chapter 1

Cardinality: To Infinity and Beyond

Exercise 1.1. *For sets $A_1, A_2 \subset X$ show that*

$$X - (A_1 \cup A_2) = (X - A_1) \cap (X - A_2).$$

Theorem 1.2. *(DeMorgan's Laws) Let X be a set, and let $\{A_k\}_{k=1}^N$ be a finite collection of sets such that $A_k \subset X$ for each $k = 1, 2, \ldots, N$. Then*

$$X - \left(\bigcup_{k=1}^{N} A_k \right) = \bigcap_{k=1}^{N} (X - A_k)$$

and

$$X - \left(\bigcap_{k=1}^{N} A_k \right) = \bigcup_{k=1}^{N} (X - A_k).$$

Exercise 1.3. *For a function $f : X \to Y$, and sets $A, B \subset Y$, show that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.*

Exercise 1.4. *If $f : X \to Y$ is injective and $y \in Y$, then $f^{-1}(y)$ contains at most one point.*

Exercise 1.5. *If $f : X \to Y$ is surjective and $y \in Y$, then $f^{-1}(y)$ contains at least one point.*

Theorem 1.6. *Let $2\mathbb{N}$ denote the even positive integers $\{2, 4, 6, \ldots\}$. Then $2\mathbb{N}$ has the same cardinality as $\mathbb{N}$, that is, $|2\mathbb{N}| = |\mathbb{N}|$.*

Theorem 1.7. *The set $\mathbb{Z}$ has the same cardinality as $\mathbb{N}$, that is, $|\mathbb{Z}| = |\mathbb{N}|$.*

Theorem 1.8. *Every subset of $\mathbb{N}$ is either finite or has the same cardinality as $\mathbb{N}$.*

Theorem 1.9. *Every infinite set has a countably infinite subset.*

Theorem 1.10. *A set is infinite if and only if there is an injection from the set into a proper subset of itself.*

Theorem 1.11. *The union of two countable sets is countable.*

Theorem 1.12. *The union of countably many countable sets is countable.*

Theorem 1.13. *The set $\mathbb{Q}$ is countable.*

Theorem 1.14. *The set of all finite subsets of a countable set is countable.*

Exercise 1.15. *Suppose a submarine is moving in the plane along a straight line at a constant speed such that at each hour, the submarine is at a lattice point, that is, a point whose two coordinates are both integers. Suppose at each hour you can explode one depth charge at a lattice point that will hit the submarine if it is there. You do not know the submarine's direction, speed, or its current position. Prove that you can explode one depth charge each hour in such a way that you will be guaranteed to eventually hit the submarine.*

Theorem 1.16 (Cantor's Theorem)**.** *The cardinality of the set of natural numbers is not the same as the cardinality of the set of real numbers. That is, the set of real numbers is uncountable.*

Exercise 1.17. *Suppose $A = \{a, b, c\}$. Explicitly write out 2^A, the power set of A.*

Theorem 1.18. *If a set A is finite, then the power set of A has cardinality $2^{|A|}$, that is, $|2^A| = 2^{|A|}$.*

Theorem 1.19. *For any set A, there is an injection from A into 2^A.*

Theorem 1.20. *For a set A, let P be the set of all functions from A to the two point set $\{0, 1\}$. Then $|P| = |2^A|$.*

Theorem 1.21. *There is a one-to-one correspondence between $2^{\mathbb{N}}$ and the set of all infinite sequences of 0's and 1's.*

Theorem 1.22 (Cantor's Power Set Theorem)**.** *There is no surjection from a set A onto 2^A. Thus for any set A, the cardinality of A is not the same as the cardinality of its power set. In other words, $|A| \neq |2^A|$.*

Exercise 1.23. *Consider $A = [0, 1]$ and $B = [0, 1)$ and injections $f(x) = x/3$ from A to B and $g(x) = x$ from B to A. Construct a bijection h from A to B such that on some points of A, $h(x) = f(x)$, and for the other points of A, $h(x) = g^{-1}(x)$.*

Exercise 1.24. *Consider $A = [0, 1]$ and $B = [0, 1)$ and injections $f(x) = x/3$ from A to B and $g(x) = x/2$ from B to A. Construct a bijection h from A to B such that on some points of A, $h(x) = f(x)$, and for the other points of A, $h(x) = g^{-1}(x)$.*

Theorem 1.25 (Schroeder-Bernstein)**.** *If A and B are sets such that there exist injections f from A into B and g from B into A, then $|A| = |B|$.*

Theorem 1.26 (Schroeder-Bernstein). *If A and B are sets such that there exist a surjective function $f : A \to B$ and a surjective function $g : B \to A$, then $|A| = |B|$.*

Theorem 1.27. $|\mathbb{R}| = |(0,1)| = |[0,1]|$.

Theorem 1.28. *Let $[0,1] \times [0,1]$ denote the Cartesian product of two closed unit intervals. Then*

$$|[0,1] \times [0,1]| = |[0,1]|.$$

Theorem 1.29. *The set of all functions $f : \mathbb{R} \to \mathbb{R}$ has the same cardinality as $2^{\mathbb{R}}$.*

Theorem 1.30. $|\mathbb{R}| = |2^{\mathbb{N}}|$.

Theorem 1.31. *There are infinitely many different infinite cardinalities.*

Continuum Hypothesis. *There is no uncountable set whose cardinality is greater than the cardinality of $\mathbb{N}$ yet less than the cardinality of $\mathbb{R}$.*

Exercise 1.32. *Given a set X, consider the poset P of all subsets of X partially ordered by inclusion. Show that X is the unique maximal element of P, and show that the empty set is the unique least element of P.*

Exercise 1.33. *Construct an example of a poset with several maximal elements and several least elements.*

Exercise 1.34. *Show that $\mathbb{R}$ with the $\leq$ relation is totally ordered but not well-ordered.*

Zorn's Lemma. *Let X be a partially ordered set in which each totally ordered subset has an upper bound in X. Then X has a maximal element.*

Axiom of Choice. *Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a set of non-empty sets. Then there is a function $f : \lambda \to \bigcup_{\alpha \in \lambda} A_\alpha$ such that for each α in λ, $f(\alpha)$ is an element of A_α.*

Well-Ordering Principle. *Every set can be well-ordered. That is, every set can be put in one-to-one correspondence with a well-ordered set.*

Theorem 1.35. *Zorn's Lemma, the Axiom of Choice, and the Well-Ordering Principle are equivalent.*

Theorem 1.36.

1. *If α is an ordinal number, then any element of α is also an ordinal.*

2. *$S(\alpha) := \alpha \cup \{\alpha\}$, the* successor *of α, is also an ordinal.*

3. *The union of any set of ordinals is an ordinal.*

4. *The ordinal numbers are naturally ordered by inclusion.*

5. *The intersection of any set of ordinals is an ordinal contained in that set of ordinals and is the least element in the set. Hence, any set of ordinal numbers has a least element. Hence, ordinals are well-ordered.*

Theorem 1.37. *Let $\{\alpha_i\}_{i\in\omega_0}$ be a countable set of countable ordinal numbers; that is, each $\alpha_i < \omega_1$. Then there is an ordinal β such that $\alpha_i < \beta$ for each i and $\beta < \omega_1$.*

Theorem 1.38. *For any countable set of countable ordinals $\{\alpha_i\}_{i\in\omega_0}$, there is a countable limit ordinal γ such that for every ordinal $\beta < \gamma$, there exists an α_i such that $\beta < \alpha_i < \gamma$.*

Theorem 1.39. *Let A and B be unbounded sets of ordinals in ω_1, that is, for every ordinal $\delta \in \omega_1$, there is an ordinal $\alpha \in A$ such that $\delta < \alpha$ and an ordinal $\beta \in B$ such that $\delta < \beta$. Then there exists a limit ordinal γ in ω_1 such that γ is a limit of ordinals from A and is also a limit of ordinals from B.*

Chapter 2

Topological Spaces: Fundamentals

Theorem 2.1. *Let $\{U_i\}_{i=1}^n$ be a* finite *collection of open sets in a topological space $(X, \mathcal{T})$. Then $\bigcap_{i=1}^n U_i$ is open.*

Exercise 2.2. *Why does your proof not prove the false statement that the infinite intersection of open sets is necessarily open?*

Theorem 2.3. *A set U is open in a topological space $(X, \mathcal{T})$ if and only if for every point $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$.*

Exercise 2.4. *Verify that $\mathcal{T}_{\text{std}}$ is a topology on $\mathbb{R}^n$; in other words, it satisfies the four conditions of the definition of a topology.*

Exercise 2.5. *Verify that the discrete, indiscrete, finite complement, and countable complement topologies are indeed topologies for any set X.*

Exercise 2.6. *Describe some of the open sets you get if $\mathbb{R}$ is endowed with the topologies described above (standard, discrete, indiscrete, finite complement, and countable complement). Specifically, identify sets that demonstrate the differences among these topologies, that is, find sets that are open in some topologies but not in others. For each of the topologies, determine if the interval $(0, 1)$ is an open set in that topology.*

Exercise 2.7. *Give an example of a topological space and a collection of open sets in that topological space that show that the* infinite *intersection of open sets need not be open.*

Exercise 2.8. *Let $X = \mathbb{R}$ and $A = (1, 2)$. Verify that 0 is a limit point of A in the indiscrete topology and the finite complement topology, but not in the standard topology nor the discrete topology on $\mathbb{R}$.*

Theorem 2.9. *Suppose $p \notin A$ in a topological space $(X, \mathcal{T})$. Then p is* not *a limit point of A if and only if there exists a neighborhood U of p such that $U \cap A = \varnothing$.*

Exercise 2.10. *If p is an isolated point of a set A in a topological space X, then there exists an open set U such that $U \cap A = \{p\}$.*

Exercise 2.11. *Give examples of sets A in various topological spaces $(X, \mathcal{T})$ with*

1. *a limit point of A that is an element of A;*
2. *a limit point of A that is not an element of A;*
3. *an isolated point of A;*
4. *a point not in A that is not a limit point of A.*

Exercise 2.12.

1. *Which sets are closed in a set X with the discrete topology?*
2. *Which sets are closed in a set X with the indiscrete topology?*
3. *Which sets are closed in a set X with the finite complement topology?*
4. *Which sets are closed in a set X with the countable complement topology?*

Theorem 2.13. *For any topological space $(X, \mathcal{T})$ and $A \subset X$, $\overline{A}$ is closed. That is, for any set A in a topological space, $\overline{\overline{A}} = \overline{A}$.*

Theorem 2.14. *Let $(X, \mathcal{T})$ be a topological space. Then the set A is closed if and only if $X - A$ is open.*

Theorem 2.15. *Let $(X, \mathcal{T})$ be a topological space, and let U be an open set and A be a closed subset of X. Then the set $U - A$ is open and the set $A - U$ is closed.*

Theorem 2.16. *Let $(X, \mathcal{T})$ be a topological space. Then:*

$i)$ $\varnothing$ *is closed.*

$ii)$ X is closed.

$iii)$ The union of finitely many closed sets is closed.

$iv)$ Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of closed subsets in $(X, \mathcal{T})$. Then $\bigcap_{\alpha \in \lambda} A_\alpha$ is closed.

Exercise 2.17. *Give an example to show that the union of infinitely many closed sets in a topological space may be a set that is not closed.*

Exercise 2.18. *Give examples of topological spaces and sets in them that:*

1. *are closed, but not open;*
2. *are open, but not closed;*
3. *are both open and closed;*
4. *are neither open nor closed.*

Exercise 2.19. *State whether each of the following sets are open, closed, both, or neither.*

1. *In $\mathbb{Z}$ with the finite complement topology:* $\{0, 1, 2\}$, $\{$*prime numbers*$\}$, $\{n \mid |n| \geq 10\}$.
2. *In $\mathbb{R}$ with the standard topology:* $(0, 1), (0, 1], [0, 1], \{0, 1\}, \{1/n \mid n \in \mathbb{N}\}$.
3. *In $\mathbb{R}^2$ with the standard topology:* $\{(x, y) \mid x^2+y^2 = 1\}$, $\{(x, y) \mid x^2+y^2 > 1\}$, $\{(x, y) \mid x^2+y^2 \geq 1\}$.

Theorem 2.20. *For any set A in a topological space X, the closure of A equals the intersection of all closed sets containing A, that is,*

$$\overline{A} = \bigcap_{B \supset A, B \in \mathcal{C}} B$$

where $\mathcal{C}$ is the collection of all closed sets in X.

Exercise 2.21. *Pick several different subsets of $\mathbb{R}$, and find their closures in:*

1. *the discrete topology;*
2. *the indiscrete topology;*
3. *the finite complement topology;*
4. *the standard topology.*

Theorem 2.22. *Let A and B be subsets of a topological space X. Then*

1. $A \subset B$ *implies* $\overline{A} \subset \overline{B}$.
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Exercise 2.23. *Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of subsets of a topological space X. Then is the following statement true?*

$$\overline{\bigcup_{\alpha \in \lambda} A_\alpha} = \bigcup_{\alpha \in \lambda} \overline{A}_\alpha.$$

Exercise 2.24. *In $\mathbb{R}^2$ with the standard topology, describe the limit points and closure of each of the following two sets:*

1. $S = \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \mid x \in (0,1)\right\}$. *The closure of the set S is called the* topologist's sine curve.

2. $C = \{(x,0) \mid x \in [0,1]\} \cup \bigcup_{n=1}^{\infty} \left\{\left(\frac{1}{n}, y\right) \mid y \in [0,1]\right\}$. *The closure of the set C is called the* topologist's comb.

Exercise 2.25. *In the standard topology on $\mathbb{R}$, there exists a non-empty subset C of the closed unit interval $[0,1]$ that is closed, contains no non-empty open interval, and where no point of C is an isolated point.*

Theorem 2.26. *Let A be a subset of a topological space X. Then p is an interior point of A if and only if there exists an open set U with $p \in U \subset A$.*

Exercise 2.27. *Show that a set U is open in a topological space X if and only if every point of U is an interior point of U.*

Theorem 2.28. *Let A be a subset of a topological space X. Then $Int(A)$, $Bd(A)$ and $Int(X - A)$ are disjoint sets whose union is X.*

Exercise 2.29. *Pick several different subsets of $\mathbb{R}$, and for each one, find its interior and boundary using:*

1. *the discrete topology;*

2. *the indiscrete topology;*

3. *the finite complement topology;*

4. *the standard topology.*

Theorem 2.30. *Let A be a subset of the topological space X and let p be a point in X. If $\{x_i\}_{i \in \mathbb{N}} \subset A$ and $x_i \to p$, then p is in the closure of A.*

Theorem 2.31. *In standard topology on $\mathbb{R}^n$, if p is a limit point of a set A, then there is a sequence of points in A that converge to p.*

Exercise 2.32. *Find an example of a topological space and a convergent sequence in that space, where the limit of the sequence is not unique.*

Exercise 2.33.

1. *Consider sequences in $\mathbb{R}$ with the finite complement topology. Which sequences converge? To what value(s) do they converge?*

2. *Consider sequences in $\mathbb{R}$ with the countable complement topology. Which sequences converge? To what value(s) do they converge?*

Chapter 3

Bases, Subspaces, Products: Creating New Spaces

Theorem 3.1. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if:*

1. *$\mathcal{B} \subset \mathcal{T}$, and*

2. *for each set U in $\mathcal{T}$ and point p in U there is a set V in $\mathcal{B}$ such that $p \in V \subset U$.*

Exercise 3.2.

1. *Let $\mathcal{B}_1 = \{(a, b) \subset \mathbb{R} \mid a, b \in \mathbb{Q}\}$. Show that $\mathcal{B}_1$ is a basis for the* standard topology *on $\mathbb{R}$.*

2. *Let $\mathcal{B}_2 = \{(a, b) \cup (c, d) \subset \mathbb{R} \mid a < b < c < d$ are distinct irrational numbers$\}$. Show that $\mathcal{B}_2$ is also a basis for the* standard topology *on $\mathbb{R}$.*

Theorem 3.3. *Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a basis for some topology on X if and only if:*

1. *each point of X is in some element of $\mathcal{B}$, and*

2. *if U and V are sets in $\mathcal{B}$ and p is a point in $U \cap V$, there is a set W in $\mathcal{B}$ such that $p \in W \subset (U \cap V)$.*

Exercise 3.4. *Show that the basis proposed above for the lower limit topology is in fact a basis.*

Theorem 3.5. *Every open set in $\mathbb{R}_{\text{std}}$ is an open set in $\mathbb{R}_{\text{LL}}$, but not vice versa.*

Exercise 3.6. *Give an example of two topologies on $\mathbb{R}$ such that neither is finer than the other, that is, the two topologies are not comparable.*

Exercise 3.7. *Check that the collection of sets that we specify as a basis in the Double Headed Snake actually forms the basis for a topology.*

Exercise 3.8. *In the Double Headed Snake, show that every point is a closed set; however, it is impossible to find disjoint open sets U and V such that $0' \in U$ and $0'' \in V$.*

Exercise 3.9.

1. *In the topological space $\mathbb{R}_{\text{har}}$, what is the closure of the set $H = \{1/n\}_{n\in\mathbb{N}}$?*
2. *In the topological space $\mathbb{R}_{\text{har}}$, what is the closure of the set $H^- = \{-1/n\}_{n\in\mathbb{N}}$?*
3. *Is it possible to find disjoint open sets U and V in $\mathbb{R}_{\text{har}}$ such that $0 \in U$ and $H \subset V$?*

Exercise 3.10.

1. *In $\mathbb{H}_{\text{bub}}$, what is the closure of the set of rational points on the x-axis?*
2. *In $\mathbb{H}_{\text{bub}}$, which subsets of the x-axis are closed sets?*
3. *In $\mathbb{H}_{\text{bub}}$, let A be a countable set on the x-axis and let z be a point on the x-axis not in A. Then there exist disjoint open sets U and V such that $A \subset U$ and $z \in V$. (Do you need the countability hypothesis on A?)*
4. *In $\mathbb{H}_{\text{bub}}$, let A and B be countable sets on the x-axis such that A and B are disjoint. Then there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.*
5. *In $\mathbb{H}_{\text{bub}}$, let A be the rational numbers and let B be the irrational numbers. Do there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$?*

Exercise 3.11. *Check that the arithmetic progressions form a basis for a topology on $\mathbb{Z}$.*

Theorem 3.12. *There are infinitely many primes.*

Exercise 3.13. *A basis for a topology is also a subbasis for that topology.*

Theorem 3.14. *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ is a subbasis for $\mathcal{T}$ if and only if*

1. *$\mathcal{S} \subset \mathcal{T}$, and*
2. *for each set U in $\mathcal{T}$ and point p in U there is a finite collection $\{V_i\}_{i=1}^n$ of elements of $\mathcal{S}$ such that*

$$p \in \bigcap_{i=1}^{n} V_i \subset U\ .$$

Exercise 3.15. *Let $\mathcal{S}$ be the following collection of subsets of $\mathbb{R}$: $\{x \mid x < a$ for some $a \in \mathbb{R}\}$ and $\{x \mid a < x$ for some $a \in \mathbb{R}\}$. Then $\mathcal{S}$ is a subbasis for $\mathbb{R}$ with the standard topology.*

Theorem 3.16. *Suppose X is a set and $\mathcal{S}$ is a collection of subsets of X. Then $\mathcal{S}$ is a subbasis for some topology on X if and only if every point of X is in some element of $\mathcal{S}$.*

Exercise 3.17. *Let $\mathcal{S}$ be the following collection of subsets of $\mathbb{R}$: $\{x \mid x < a$ for some $a \in \mathbb{R}\}$ and $\{x \mid a \leq x$ for some $a \in \mathbb{R}\}$. For what topology on $\mathbb{R}$ is $\mathcal{S}$ a subbasis?*

Exercise 3.18. *Let X be a set totally ordered by $<$. Let $\mathcal{S}$ be the collection of sets of the following forms*

$$\{x \in X \mid x < a\} \quad \text{or} \quad \{x \in X \mid a < x\}$$

for $a \in X$. Then $\mathcal{S}$ forms a subbasis for the order topology on X.

Exercise 3.19. *Verify that the order topology on $\mathbb{R}$ with the usual $<$ order is the standard topology on $\mathbb{R}$.*

Exercise 3.20. *In the lexicographically ordered square find the closures of the following subsets:*

$$\begin{aligned}
A &= \left\{ \left(\frac{1}{n}, 0\right) \mid n \in \mathbb{N} \right\}. \\
B &= \left\{ \left(1 - \frac{1}{n}, \frac{1}{2}\right) \mid n \in \mathbb{N} \right\}. \\
C &= \{(x, 0) \mid 0 < x < 1\}. \\
D &= \left\{ \left(x, \frac{1}{2}\right) \mid 0 < x < 1 \right\}. \\
E &= \left\{ \left(\frac{1}{2}, y\right) \mid 0 < y < 1 \right\}.
\end{aligned}$$

Exercise 3.21. *Assume that $\mathbb{N}$ has the usual order. Let $\mathbb{N}^\omega$ denote the Cartesian product of a countable number of copies of the space $\mathbb{N}$. It can be endowed with the dictionary order in a natural way. Show that $\mathbb{N}^\omega$ with the dictionary order topology is uncountable, is not well-ordered, and any set that does not have a least element does have a limit point.*

Theorem 3.22. *Consider the topological space ω_1 consisting of all ordinals less than ω_1, the first uncountable ordinal, with the order topology. Let A be an infinite set of ordinals in ω_1. Then there is an ordinal $\beta < \omega_1$ that is a limit point of A.*

Theorem 3.23. *Let A and B be unbounded closed sets in the topological space ω_1. Then $A \cap B \neq \varnothing$.*

Theorem 3.24. *Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$. Then the collection of sets $\mathcal{T}_Y$ is in fact a topology on Y.*

Exercise 3.25. *Consider $Y = [0, 1)$ as a subspace of $\mathbb{R}_{\text{std}}$. In Y, is the set $[1/2, 1)$ open, closed, neither, or both?*

Exercise 3.26. *Consider a subspace Y of the topological space X. Is every subset $U \subset Y$ that is open in Y also open in X?*

Theorem 3.27. *Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if there is a set $D \subset X$, closed in $(X, \mathcal{T})$, such that $C = D \cap Y$.*

Corollary 3.28. *Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if* $\mathrm{Cl}_X(C) \cap Y = C$.

Theorem 3.29. *Let $(X, \mathcal{T})$ be a topological space, and (Y, T_Y) be a subspace. If $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{B}_Y = \{B \cap Y | B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$.*

Exercise 3.30. *Consider the following subspaces of the lexicographically ordered square:*

1. $D = \left\{\left(x, \frac{1}{2}\right) \mid 0 < x < 1\right\}$.
2. $E = \left\{\left(\frac{1}{2}, y\right) \mid 0 < y < 1\right\}$.
3. $F = \{(x, 1) \mid 0 < x < 1\}$.

As sets they are all lines. Describe their relative topologies, especially noting any connections to topologies you have seen already.

Exercise 3.31. *Verify that the collection of basic open sets above satisfies the conditions of Theorem 3.3, thus confirming that this collection is the basis for a topology.*

Exercise 3.32. *Draw examples of basic and arbitrary open sets in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ using the standard topology on $\mathbb{R}$. Find (i) an open set in $\mathbb{R} \times \mathbb{R}$ that is not the product of open sets, and (ii) a closed set in $\mathbb{R} \times \mathbb{R}$ that is not the product of closed sets.*

Exercise 3.33. *Is the product of closed sets closed?*

Exercise 3.34. *Show that the product topology on $X \times Y$ is the same as the topology generated by the subbasis of inverse images of open sets under the projection functions, that is the subbasis is $\{\pi_X^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_Y^{-1}(V) \mid V \text{ open in } Y\}$.*

Exercise 3.35. *Using the standard topology on $\mathbb{R}$, is the product topology on $\mathbb{R} \times \mathbb{R}$ the same as the standard topology on $\mathbb{R}^2$?*

Exercise 3.36. *A* basis *for the* product topology *on $\Pi_{\alpha \in \lambda} X_\alpha$ is the collection of all sets of the form $\Pi_{\alpha \in \lambda} U_\alpha$ where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ for all but finitely many α.*

Exercise 3.37. *Let $\mathcal{T}$ be the topology on 2^X with basis generated by the subbasis $\mathcal{S}$.*

1. *Every basic open set in 2^X is both open and closed.*

2. *Show that if a collection of subbasic open sets of 2^X has the property that every point of 2^X lies in at least one of those subbasic open sets, then there are two subbasic open sets in that collection such that every point of 2^X lies in one of those two subbasic sets.*

3. *Show that if a collection of basic open sets of 2^X has the property that every point of 2^X lies in at least one of those basic open sets, then there are a finite number of basic open sets in that collection such that every point of 2^X lies in one of those basic sets.*

Exercise 3.38. *In the product space $2^{\mathbb{R}}$, what is the closure of the set Z consisting of all elements of $2^{\mathbb{R}}$ that are 0 on every rational coordinate, but may be 0 or 1 on any irrational coordinate? Equivalently, thinking of $2^{\mathbb{R}}$ as subsets of $\mathbb{R}$, what is the closure of the set Z consisting of all subsets of $\mathbb{R}$ that do not contain any rational?*

Exercise 3.39. *Find a subset A of $2^{\mathbb{R}}$ and a limit point x of A such that no sequence in A converges to x. For an even greater challenge, determine whether you can find such an example if A is countable.*

Exercise 3.40. *Let $\mathbb{R}^\omega$ be the countable product of copies of $\mathbb{R}$. So every point in $\mathbb{R}^\omega$ is a sequence $(x_1, x_2, x_3, ...)$. Let $A \subset \mathbb{R}^\omega$ be the set consisting of all points with only positive coordinates. Show that in the product topology, $\mathbf{0} = (0, 0, 0, ...)$ is a limit point of the set A, and there is a sequence of points in A converging to $\mathbf{0}$. Then show that in the box topology, $\mathbf{0} = (0, 0, 0, ...)$ is a limit point of the set A, but there is no sequence of points in A converging to $\mathbf{0}$.*

Exercise 3.41. *Show that the set $2^{\mathbb{N}}$ in the box topology is a discrete space, whereas the set $2^{\mathbb{N}}$ in the product topology has no isolated points.*

Chapter 4

Separation Properties: Separating This from That

Theorem 4.1. *A space $(X, \mathcal{T})$ is T_1 if and only if every point in X is a closed set.*

Exercise 4.2. *Let X be a space with the finite complement topology. Show that X is T_1.*

Exercise 4.3. *Show that $\mathbb{R}_{\text{std}}$ is Hausdorff.*

Exercise 4.4. *Show that $\mathbb{H}_{\text{bub}}$ is regular.*

Exercise 4.5. *Show that $\mathbb{R}_{\text{LL}}$ is normal.*

Exercise 4.6.

1. *Consider $\mathbb{R}^2$ with the standard topology. Let $p \in \mathbb{R}^2$ be a point not in a closed set A. Show that $\inf\{d(a,p) \mid a \in A\} > 0$. (Recall that $\inf E$ is the greatest lower bound of a set of real numbers E.)*
2. *Show that $\mathbb{R}^2$ with the standard topology is regular.*
3. *Find two disjoint closed subsets A and B of $\mathbb{R}^2$ with the standard topology such that*

$$\inf\{d(a,b) \mid a \in A \text{ and } b \in B\} = 0.$$

4. *Show that $\mathbb{R}^2$ with the standard topology is normal.*

Theorem 4.7.

1. *A T_2-space (Hausdorff) is a T_1-space.*
2. *A T_3-space (regular and T_1) is a Hausdorff space, that is, a T_2-space.*
3. *A T_4-space (normal and T_1) is regular and T_1, that is, a T_3-space.*

Theorem 4.8. *A topological space X is regular if and only if for each point p in X and open set U containing p there exists an open set V such that $p \in V$ and $\overline{V} \subset U$.*

Theorem 4.9. *A topological space X is normal if and only if for each closed set A in X and open set U containing A there exists an open set V such that $A \subset V$, and $\overline{V} \subset U$.*

Theorem 4.10. *A topological space X is normal if and only if for each pair of disjoint closed sets A and B, there are disjoint open sets U and V such that $A \subset U$, $B \subset V$, and $\overline{U} \cap \overline{V} = \varnothing$.*

Theorem 4.11 (The Incredible Shrinking Theorem)**.** *A topological space X is normal if and only if for each pair of open sets U, V such that $U \cup V = X$, there exist open sets U', V' such that $\overline{U'} \subset U$ and $\overline{V'} \subset V$, and $U' \cup V' = X$.*

Exercise 4.12.

1. *Describe an example of a topological space that is T_1 but not T_2.*
2. *Describe an example of a topological space that is T_2 but not T_3.*
3. *Describe an example of a topological space that is T_3 but not T_4.*

Exercise 4.13. *Construct a table, listing our previous examples of topological spaces as column titles, and listing the separation properties as row titles. In each box, answer the question of whether the example of the column has the property of the row. Here are the spaces to use as column titles:*

1. $\mathbb{R}_{std}$
2. $\mathbb{R}^n_{std}$
3. *indiscrete topology*
4. *discrete topology*
5. *finite complement topology*
6. *countable complement topology*
7. *lower limit topology,* $\mathbb{R}_{\mathrm{LL}}$
8. *double headed snake,* $\mathbb{R}_{+00}$
9. $\mathbb{R}_{har}$
10. *Sticky Bubble Topology,* $\mathbb{H}_{\mathrm{bub}}$
11. *arithmetic progression topology,* $\mathbb{Z}_{arith}$
12. *lexicographically ordered square*
13. 2^X

Here are the properties to use as row titles:

1. T_1
2. *Hausdorff*
3. *regular*
4. *normal*

Exercise 4.14. *Show that* $\mathbb{H}_{\text{bub}}$ *is not normal.*

Theorem 4.15. *Order topologies are* T_1*, Hausdorff, regular, and normal.*

Theorem 4.16. *Let* X *and* Y *be Hausdorff. Then* $X \times Y$ *is Hausdorff.*

Theorem 4.17. *Let* X *and* Y *be regular. Then* $X \times Y$ *is regular.*

Exercise 4.18. *Show that* $\mathbb{R}_{\text{LL}} \times \mathbb{R}_{\text{LL}}$ *is not normal. It may help to consider the "negative diagonal" line* L*.*

Theorem 4.19. *Every Hausdorff space is hereditarily Hausdorff.*

Theorem 4.20. *Every regular space is hereditarily regular.*

Exercise 4.21.

1. *Prove that the space* $2^{\mathbb{R}}$ *is normal.*
2. *Prove that if you remove a single point from* $2^{\mathbb{R}}$*, the resulting subspace is not normal.*

Exercise 4.22. *(Walking the Tychonoff Plank, or Mutiny on the Boundary)*

1. *Show that the Tychonoff Plank is normal.*
2. *Show that the Tychonoff Plank minus the single point* (ω_0, ω_1) *is not normal.*

Theorem 4.23. *Let* A *be a closed subset of a normal space* X*. Then* A *is normal when given the relative topology.*

Exercise 4.24.

1. *Prove that for any set* X*,* 2^X *is normal. (This part is not really different from showing that* $2^{\mathbb{R}}$ *is normal, which you did in a previous exercise.)*
2. *Recall that there is a one-to-one correspondence between the points of* 2^X *and subsets of* X*, as follows: recall that each point of* 2^X *is a function* $f : X \to \{0,1\}$*, so* $f^{-1}(1)$ *is a subset of* X*. Let* $C \subset 2^X$ *consist of those points that take on the value* 1 *on only a countable set of coordinates, that is,* C *is the set of functions* $f : X \to \{0,1\}$*, for which* $f^{-1}(1)$ *is countable. Prove that* C *with the subspace topology is normal.*

Exercise 4.25. *Let Y be a subspace of a topological space X, and let A and B be two disjoint closed subsets of Y in the subspace topology. Show that both $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$, where the closures are taken in X.*

Theorem 4.26. *The space X is a completely normal space if and only if X is hereditarily normal.*

Exercise 4.27.

1. *Recall that $\mathbb{R}$ is an order topology. Find a subset of $\mathbb{R}$ where the subspace topology is not the order topology on the subset.*
2. *Find a line in the lexicographically ordered square whose relative topology is the discrete topology on this line, but this is not the order topology on the subset.*
3. *Notice that $\mathbb{R}_{LL}$ is not an order topology. Find a line in the lexicographically ordered square whose relative topology is the lower limit topology.*

Theorem 4.28. *Order topologies are hereditarily normal.*

Theorem 4.29 (The Normality Lemma)**.** *Let A and B be subsets of a topological space X and let $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ be two collections of open sets such that*

1. *$A \subset \bigcup_{i\in\mathbb{N}} U_i$,*
2. *$B \subset \bigcup_{i\in\mathbb{N}} V_i$,*
3. *for each i in $\mathbb{N}$, $\overline{U}_i \cap B = \varnothing$ and $\overline{V}_i \cap A = \varnothing$.*

Then there exist open sets U and V such that $A \subset U$, $B \subset V$, and $U \cap V = \varnothing$.

Theorem 4.30. *If X is normal and $C = \cup_{i\in\mathbb{N}} K_i$ is the union of closed sets K_i in X, then the subspace C is normal.*

Theorem 4.31. *Suppose a space X is regular and countable. Then X is normal.*

Theorem 4.32. *Suppose a space X is regular and has a countable basis. Then X is normal.*

Theorem 4.33. *Let X be a perfectly normal space. Then X is completely normal.*

Chapter 5

Countable Features of Spaces: Size Restrictions

Exercise 5.1. *Show that A is dense in X if and only if every non-empty open set of X contains a point of A.*

Exercise 5.2. *Show that $\mathbb{R}_{std}$ is separable. With which of the topologies on $\mathbb{R}$ that you have studied is $\mathbb{R}$ not separable?*

Exercise 5.3. *Add 'separable' as a new property in your chart, and complete your chart by deciding which of the spaces we've studied are separable.*

Exercise 5.4. *Find a separable space that contains a subspace that is not separable in the subspace topology.*

Theorem 5.5. *If X and Y are separable spaces, then $X \times Y$ is separable.*

Theorem 5.6. *The space $2^{\mathbb{R}}$ is separable.*

Exercise 5.7. *Let $\{X_\beta\}_{\beta\in\mu}$ be a collection of separable spaces where $|\mu| \leq 2^{\omega_0}$, then $\prod_{\beta\in\mu} X_\beta$ is separable.*

Exercise 5.8. *If X is a separable, Hausdorff space, then $|X| \leq \left|2^{2^{\mathbb{N}}}\right|$.*

Theorem 5.9. *Let X be a 2^{nd} countable space, then X is separable.*

Exercise 5.10.

1. *The space $\mathbb{R}_{std}$ is 2^{nd} countable (and hence separable).*
2. *The space $\mathbb{R}_{LL}$ is separable but not 2^{nd} countable.*
3. *The space $\mathbb{H}_{\mathrm{bub}}$ is separable but not 2^{nd} countable.*

Theorem 5.11. *Every uncountable set in a 2^{nd} countable space has a limit point.*

Exercise 5.12. *A 2^{nd} countable space is hereditarily 2^{nd} countable.*

Exercise 5.13. *If X and Y are 2nd countable spaces, then $X \times Y$ is 2nd countable.*

Theorem 5.14. *Let X be a 2^{nd} countable space. Then X is 1^{st} countable.*

Theorem 5.15. *If X is a topological space, $p \in X$, and p has a countable neighborhood basis, then p has a nested countable neighborhood basis.*

Exercise 5.16.

1. *The space $\mathbb{R}_{LL}$ is 1^{st} countable.*
2. *The space $\mathbb{H}_{\mathrm{bub}}$ is 1^{st} countable.*
3. *The space $2^{\mathbb{R}}$ is not 1^{st} countable.*

Exercise 5.17. *You may as well extend your table of spaces and properties by adding new rows for the properties 1^{st} countable and 2^{nd} countable and determining those properties for each of your spaces.*

Theorem 5.18. *Suppose x is a limit point of the set A in a 1^{st} countable space X. Then there is a sequence of points $\{a_i\}_{i \in \mathbb{N}}$ in A that converges to x.*

Exercise 5.19. *A 1^{st} countable space is hereditarily 1^{st} countable.*

Exercise 5.20. *If X and Y are 1st countable spaces, then $X \times Y$ is 1st countable.*

Exercise 5.21. *Show that the real line with the standard topology is Souslin.*

Theorem 5.22. *A separable space has the Souslin property.*

Theorem 5.23. *For any set X, the topological space 2^X has the Souslin property.*

Exercise 5.24. *Find a Souslin space that is not separable.*

Theorem. *Let $\{X_\beta\}_{\beta \in \mu}$ be a collection of separable spaces, then $\prod_{\beta \in \mu} X_\beta$ is Souslin.*

Chapter 6

Compactness: The Next Best Thing to Being Finite

Theorem 6.1. *Let X be a finite topological space. Then X is compact.*

Theorem 6.2. *Let C be a compact subset of $\mathbb{R}_{std}$. Then C has a maximum point, that is, there is a point $m \in C$ such that for every $x \in C$, $x \leq m$.*

Theorem 6.3. *If X is a compact space, then every infinite subset of X has a limit point.*

Corollary 6.4. *If X is compact and E is a subset of X with no limit point, then E is finite.*

Theorem 6.5. *A space X is compact if and only if every collection of closed sets with the finite intersection property has a non-empty intersection.*

Theorem 6.6. *A space X is compact if and only if for any open set U in X and any collection of closed sets $\{K_\alpha\}_{\alpha\in\lambda}$ such that $\bigcap_{\alpha\in\lambda} K_\alpha \subset U$, there exist a finite number of the K_α's whose intersection lies in U.*

Exercise 6.7. *If A and B are compact subsets of X, then $A \cup B$ is compact. Suggest and prove a generalization.*

Theorem 6.8. *Let A be a closed subspace of a compact space. Then A is compact.*

Theorem 6.9. *Let A be a compact subspace of a Hausdorff space X. Then A is closed.*

Exercise 6.10. *Construct an example of a compact subset of a topological space that is not closed.*

Exercise 6.11. *Must the intersection of two compact sets be compact? Add hypotheses, if necessary. Extend any theorems you discover, if possible.*

Theorem 6.12. *Every compact, Hausdorff space is normal.*

Theorem 6.13. *Let $\mathcal{B}$ be a basis for a space X. Then X is compact if and only if every cover of X by basic open sets in $\mathcal{B}$ has a finite subcover.*

Theorem 6.14. *For any $a \leq b$ in $\mathbb{R}$, the subspace $[a, b]$ is compact.*

Heine-Borel Theorem 6.15. *Let A be a subset of $\mathbb{R}_{std}$. Then A is compact if and only if A is closed and bounded.*

Exercise 6.16. *Consider the rationals $\mathbb{Q}$ with the subspace topology inherited from $\mathbb{R}$. Find a set A in $\mathbb{Q}$ that is closed and bounded but not compact.*

Theorem 6.17. *Every compact subset C of $\mathbb{R}$ contains a maximum in the set C, i.e., there is an $m \in C$ such that for any $x \in C$, $x \leq m$.*

Theorem 6.18 (The tube lemma)**.** *Let $X \times Y$ be a product space with Y compact. If U is an open set of $X \times Y$ containing the set $x_0 \times Y$, then there is some open set W in X containing x_0 such that U contains $W \times Y$ (called a "tube" around $x_0 \times Y$).*

Theorem 6.19. *Let X and Y be compact spaces. Then $X \times Y$ is compact.*

Heine-Borel Theorem 6.20. *Let A be a subset of $\mathbb{R}^n$ with the standard topology. Then A is compact if and only if A is closed and bounded.*

Alexander Subbasis Theorem 6.21. *Let $\mathcal{S}$ be a subbasis for a space X. Then X is compact if and only if every subbasic open cover has a finite subcover.*

Exercise 6.22. *Use the Alexander Subbasis Theorem to prove that the space 2^X is compact for every X.*

Tychonoff's Theorem 6.23. *Any product of compact spaces is compact.*

Exercise 6.24. *Consider the set $[0, 1]^\omega$ and show that the Tychonoff Theorem is not true if the box topology is used instead of the product topology.*

Theorem 6.25. *Every countably compact and Lindelöf space is compact.*

Theorem 6.26. *Let X be a T_1 space. Then X is countably compact if and only if every infinite subset of X has a limit point.*

Theorem 6.27. *If X is a Lindelöf space, then every uncountable subset of X has a limit point.*

Exercise 6.28. *Formulate and prove theorems about Lindelöf and countably compact spaces analogous to the theorems you proved relating compactness with collections of closed sets with the finite intersection property.*

Theorem 6.29. *If A is a closed subspace of a countably compact (respectively, Lindelöf) space, then A is countably compact (respectively, Lindelöf).*

Theorem 6.30. *Every regular, Lindelöf space is normal.*

Theorem 6.31. *Let $\mathcal{B}$ be a basis for a space X. Then X is Lindelöf if and only if every cover of X by basic open sets in $\mathcal{B}$ has a countable subcover.*

Corollary 6.32. *Every 2^{nd} countable space is Lindelöf.*

Exercise 6.33. *Can you think of a topological space in which every countable open cover by basic open sets has a finite subcover and yet not every countable open cover has a finite subcover?*

Exercise 6.34. *Show that $\mathbb{R}_{LL}$ is Lindelöf, but $\mathbb{R}_{LL} \times \mathbb{R}_{LL}$ is not Lindelöf.*

Theorem 6.35. *The space ω_1 of countable ordinals is countably compact but not compact.*

Theorem 6.36. *The space $\omega_1 + 1$, which includes all countable ordinals together with the ordinal ω_1, is compact.*

Exercise 6.37. *Extend your table of spaces and properties by adding new rows for the properties compact, Lindelöf, and countably compact and determining those properties for each of your spaces.*

Theorem 6.38. *Let $\mathcal{B} = \{B_\alpha\}_{\alpha\in\lambda}$ be a locally finite collection of subsets of a space X. Then*

$$\overline{(\bigcup_{\alpha\in\lambda} B_\alpha)} = \bigcup_{\alpha\in\lambda} \overline{B}_\alpha.$$

Theorem 6.39. *Let A be a closed subspace of a paracompact space. Then A is paracompact.*

Theorem 6.40. *Every paracompact space is normal.*

Theorem 6.41. *Every regular, T_1, Lindelöf space is paracompact.*

Chapter 7

Continuity: When Nearby Points Stay Together

Theorem 7.1. *Let X and Y be topological spaces and let $f : X \to Y$ be a function. Then the following are equivalent:*

1. *The function f is continuous.*
2. *For every closed set K in Y, the inverse image $f^{-1}(K)$ is closed in X.*
3. *For every limit point p of a set A in X, the image $f(p)$ belongs to $\overline{f(A)}$.*
4. *For every $x \in X$ and open set V containing $f(x)$, there is an open set U containing x such that $f(U) \subset V$.*

Theorem 7.2. *Let X, Y be topological spaces and $y_0 \in Y$. The constant map $f : X \to Y$ defined by $f(x) = y_0$ is continuous.*

Theorem 7.3. *Let $X \subset Y$ be topological spaces. The inclusion map $i : X \to Y$ defined by $i(x) = x$ is continuous.*

Theorem 7.4. *Let $f : X \to Y$ be a continuous map between topological spaces, and let A be a subset of X. Then the restriction map $f|_A : A \to Y$ defined by $f|_A(a) = f(a)$ is continuous.*

Theorem 7.5. *A function $f : \mathbb{R}_{std} \to \mathbb{R}_{std}$ is continuous if and only if for every point x in $\mathbb{R}$ and $\varepsilon > 0$, there is a $\delta > 0$ such that for every $y \in \mathbb{R}$ with $d(x, y) < \delta$, then $d(f(x), f(y)) < \varepsilon$.*

Theorem 7.6. *Let X be a 1^{st} countable space and Y be a topological space. Then a function $f : X \to Y$ is continuous if and only if for each convergent sequence $x_n \to x$ in X, $f(x_n)$ converges to $f(x)$ in Y.*

Theorem 7.7. *Let X be a space with a dense set D, and let Y be Hausdorff. Let $f : X \to Y$ and $g : X \to Y$ be continuous functions such that for every d in D, $f(d) = g(d)$. Then for all x in X, $f(x) = g(x)$.*

Theorem 7.8. *The cardinality of the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ is the same as the cardinality of $\mathbb{R}$.*

Theorem 7.9. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then their composition $g \circ f : X \to Z$ is continuous.*

Theorem 7.10 (pasting lemma)**.** *Let $X = A \cup B$, where A, B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions that agree on $A \cap B$. Then the function $h : A \cup B \to Y$ such that $h = f$ on A and $h = g$ on B is continuous.*

Theorem 7.11 (pasting lemma)**.** *Let $X = A \cup B$, where A, B are open in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions which agree on $A \cap B$. Then the function $h : A \cup B \to Y$ such that $h = f$ on A and $h = g$ on B is continuous.*

Exercise 7.12. *Is the pasting lemma true when A and B in the preceding theorems are arbitrary sets?*

Theorem 7.13. *Let $f : X \to Y$ be a function and let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if for every open set B in $\mathcal{B}$, $f^{-1}(B)$ is open in X.*

Theorem 7.14. *Let $f : X \to Y$ be a function and let $\mathcal{B}$ be a subbasis for Y. Then f is continuous if and only if for every open set B in $\mathcal{B}$, $f^{-1}(B)$ is open in X.*

Theorem 7.15. *If X is compact, and $f : X \to Y$ is continuous and surjective, then Y is compact.*

Theorem 7.16. *If X is Lindelöf and $f : X \to Y$ is continuous and surjective, then Y is Lindelöf.*

Theorem 7.17. *If X is countably compact and $f : X \to Y$ is continuous and surjective, then Y is countably compact.*

Theorem 7.18. *Let D be a dense set of a topological space X and let $f : X \to Y$ be continuous and surjective. Then $f(D)$ is dense in Y.*

Corollary 7.19. *Let X be a separable space and let $f : X \to Y$ be continuous and surjective. Then Y is separable.*

Exercise 7.20.

1. *Find an open function that is not continuous.*
2. *Find a closed function that is not continuous.*
3. *Find a continuous function that is neither open nor closed.*
4. *Find a continuous function that is open but not closed.*

5. *Find a continuous function that is closed but not open.*

Theorem 7.21. *If X is normal and $f : X \to Y$ is continuous, surjective, and closed, then Y is normal.*

Theorem 7.22. *If $\{B_\alpha\}_{\alpha \in \lambda}$ is a basis for X and $f : X \to Y$ is continuous, surjective, and open, then $\{f(B_\alpha)\}_{\alpha \in \lambda}$ is a basis for Y.*

Corollary 7.23. *If X is 2^{nd} countable and $f : X \to Y$ is continuous, surjective, and open, then Y is 2^{nd} countable.*

Theorem 7.24. *Let X be compact and Y be Hausdorff. Then any continuous function $f : X \to Y$ is closed.*

Theorem 7.25. *Let X be compact and 2^{nd} countable and let Y be Hausdorff. If $f : X \to Y$ is continuous and surjective, then Y is 2^{nd} countable.*

Theorem 7.26. *Being homeomorphic is an equivalence relation on topological spaces.*

Exercise 7.27. *Let a and b be points in $\mathbb{R}^1$ with $a < b$. Show that (a, b) with the subspace topology from $\mathbb{R}^1_{std}$ is homeomorphic to $\mathbb{R}^1_{std}$.*

Theorem 7.28. *If $f : X \to Y$ is continuous, the following are equivalent:*

a) f is a homeomorphism.

b) f is a closed bijection.

c) f is an open bijection.

Theorem 7.29. *Suppose $f : X \to Y$ is a continuous bijection where X is compact and Y is Hausdorff. Then f is a homeomorphism.*

Exercise 7.30. *Construct some examples to show why the compactness and Hausdorff assumptions in the previous theorem are necessary.*

Corollary 7.31. *Let X be a compact space and let Y be Hausdorff. If $f : X \to Y$ is a continuous, injective map, then f is an embedding.*

Theorem 7.32. *Let X and Y be topological spaces. The projection maps π_X, π_Y on $X \times Y$ are continuous, surjective, and open.*

Theorem 7.33. *Let X and Y be topological spaces. The product topology on $X \times Y$ is the coarsest topology on $X \times Y$ that makes the projection maps π_X, π_Y on $X \times Y$ continuous.*

Exercise 7.34. *Find an example of X and Y that shows that the projection map $\pi_X : X \times Y \to X$ is not necessarily a closed map.*

Theorem 7.35. *Let X and Y be topological spaces. For every $y \in Y$, the subspace $X \times \{y\}$ of $X \times Y$ is homeomorphic to X.*

Theorem 7.36. *Let X, Y, and Z be topological spaces. A function $g : Z \to X \times Y$ is continuous if and only if $\pi_X \circ g$ and $\pi_Y \circ g$ are both continuous.*

Exercise 7.37. *What about maps* out of *a product space, i.e., $f : X \times Y \to Z$? Do you think f is continuous if it is continuous in each coordinate?*

Theorem 7.38. *Let $\prod_{\alpha \in \lambda} X_\alpha$ be the product of topological spaces $\{X_\alpha\}_{\alpha \in \lambda}$. The projection map $\pi_\beta : \prod_{\alpha \in \lambda} X_\alpha \to X_\beta$ is a continuous, surjective, and open map.*

Theorem 7.39. *The product topology is the coarsest (smallest) topology on $\prod_{\alpha \in \lambda} X_\alpha$ that makes each projection map continuous.*

Theorem 7.40. *Let $\prod_{\alpha \in \lambda} X_\alpha$ be the product of topological spaces $\{X_\alpha\}_{\alpha \in \lambda}$ and let Z be a topological space. A function $g : Z \to \prod_{\alpha \in \lambda} X_\alpha$ is continuous if and only if $\pi_\beta \circ g$ is continuous for each β in λ.*

Exercise 7.41. *Let $\mathbb{R}^\omega$ be the countably infinite product of $\mathbb{R}$ with itself. Let $f : \mathbb{R} \to \mathbb{R}^\omega$ be defined by $f(x) := (x, x, x, ...)$. Then f is continuous if $\mathbb{R}^\omega$ is given the product topology, but not if given the box topology. (This strange result once again shows why the box topology would be a poor choice as the standard topology for infinite products.)*

Theorem 7.42. *The Cantor set is homeomorphic to the product $\prod_{n \in \mathbb{N}} \{0, 1\}$ where $\{0, 1\}$ has the discrete topology.*

Exercise 7.43. *The cylinder C from our example above did not need to be embedded in $\mathbb{R}^3$ to be defined; it could have been defined as an identification space of $X = [0, 1] \times [0, 1]$, using the partition whose sets are either singletons or pairs:*

$$C^* = \Big\{ \{(x, y)\} : x \in (0, 1), y \in [0, 1] \Big\} \cup \Big\{ \{(0, y) \cup (1, y)\} : y \in [0, 1] \Big\}.$$

What is the identification map $f : X \to C^*$? What is a basis for the topology on C^*?

Exercise 7.44. *A **Möbius band** is obtained by taking a strip of paper X and gluing two opposite sides with a "twist". Sometimes this gluing is notated by drawing X with arrows on two parallel sides that point in opposite directions. Construct a Möbius band explicitly as an identification space of $X = [0, 8] \times [0, 1]$.*

Exercise 7.45. *A* ***torus*** *is the surface of a doughnut. Construct a torus explicitly as:*

1. *an identification space of C, the cylinder.*

2. *an identification space of $X = [0,1] \times [0,1]$.*

3. *an identification space of $\mathbb{R}^2$.*

Exercise 7.46. *Describe the 2-dimensional sphere (the boundary of a 3-dimensional ball in $\mathbb{R}^3$) as an identification space of two discs in $\mathbb{R}^2$ by drawing a figure.*

Theorem 7.47. *The quotient topology actually defines a topology.*

Theorem 7.48. *Let X be a topological space, Y be a set, and $f : X \to Y$ be a surjective map. The quotient topology on Y is the finest (largest) topology that makes f continuous.*

Theorem 7.49. *Let X and Y be topological spaces. A surjective, continuous map $f : X \to Y$ that is an open map is a quotient map.*

Theorem 7.50. *Let X and Y be topological spaces. A surjective, continuous map $f : X \to Y$ that is a closed map is a quotient map.*

Exercise 7.51. *Show with examples that not all quotient maps are open maps, and not all quotient maps are closed maps.*

Exercise 7.52. *Is $\pi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\pi(x, y) = x$ a quotient map?*

Theorem 7.53. *Let $f : X \to Y$ be a quotient map. Then a map $g : Y \to Z$ is continuous if and only if $g \circ f$ is continuous.*

Exercise 7.54. *Let the cylinders C^* and C be defined as at the beginning of this Section. Prove that C^* is homeomorphic to C by constructing a map $h : C^* \to C$ and showing it is a continuous bijection from a compact space into a Hausdorff space.*

Exercise 7.55. *Suppose X is a subspace of $\mathbb{R}^n$ for some n. View $\mathbb{R}^n$ as a subset of $\mathbb{R}^{n+1}$ in the usual way (that is, $\mathbb{R}^n$ is the space of the first n coordinates of $\mathbb{R}^{n+1}$ where the final coordinate is 0). Choose a point $x_0 \in \mathbb{R}^{n+1} - \mathbb{R}^n$. Let C be the subspace of R^{n+1} consisting of the union of all the line segments from x_0 to points in X. Show that C is homeomorphic to the cone over X as defined above, thus justifying the name "cone."*

Lemma 7.56. *Let A and B be disjoint closed sets in a normal space X. Then for each rational $r \in [0,1]$, there exists an open set U_r such that $A \subset U_0$, $B \subset (X - U_1)$, and for $r < s$, $\overline{U_r} \subseteq U_s$.*

Urysohn's Lemma 7.57. *A topological space X is normal if and only if for each pair of disjoint closed sets A and B in X, there exists a continuous function $f : X \to [0,1]$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$.*

Lemma 7.58. *Let X be a normal space, and let A be a closed subset of X. Let $f : A \to [0,1]$ be a continuous function and let $r \in (0,1)$. Then there exist disjoint open sets U_r and V_r such that $f^{-1}([0,r)) \subset U_r$ and $f^{-1}((r,1]) \subset V_r$. Or equivalently, there exists an open set U_r such that $f^{-1}([0,r)) \subset U_r$ and $\overline{U_r} \cap f^{-1}((r,1]) = \varnothing$.*

Tietze Extension Theorem 7.59. *A space X is normal if and only if for every closed set $A \subset X$ and continuous function $f : A \to [0,1]$, there exists a continuous function $F : X \to [0,1]$ such that $F(x) = f(x)$ for every $x \in A$.*

Theorem 7.60. *A space X is normal if and only if for every closed set $A \subset X$ and continuous function $f : A \to (0,1)$, there exists a continuous function $F : X \to (0,1)$ such that $F(x) = f(x)$ for every $x \in A$.*

Theorem 7.61. *A space X is normal if and only if for every closed set $A \subset X$ and continuous function $f : A \to [0,1)$, there exists a continuous function $F : X \to ([0,1)$ such that $F(x) = f(x)$ for every $x \in A$.*

Theorem 7.62. *A space X is normal if and only if for every closed set $A \subset X$ and continuous function $f : A \to [0,1] \times [0,1]$, there exists a continuous function $F : X \to [0,1] \times [0,1]$ such that $F(x) = f(x)$ for every $x \in A$.*

Theorem 7.63. *A space X is normal if and only if for every closed set $A \subset X$ and continuous function $f : A \to \prod_{\alpha\in\lambda}[0,1]_\alpha$, where each $[0,1]_\alpha$ is a copy of $[0,1]$ in the usual topology, there exists a continuous function $F : X \to \prod_{\alpha\in\lambda}[0,1]_\alpha$ such that $F(x) = f(x)$ for every $x \in A$.*

Theorem 7.64. *Let X be a normal space and let A be a closed subspace of X homeomorphic to $[0,1]$ with the usual topology. Then there exists a continuous function $r : X \to A$ such that for every $x \in A$, $r(x) = x$.*

Theorem 7.65. *Let X be a normal space and let A be a closed subspace of X homeomorphic to S^1 with the usual topology. Then there exists an open set U containing A and a continuous function $r : U \to A$ such that for every $x \in A$, $r(x) = x$.*

Exercise 7.66. *Think of (many) other possible alternatives to S^1 in the preceding theorem that would allow you to draw the same conclusion.*

Theorem 7.67. *Suppose X is perfectly normal. Then for each pair of disjoint closed sets A and B in X, there exists a continuous function $f : X \to [0,1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$.*

Theorem 7.68. *Every perfectly normal space is completely normal.*

Theorem 7.69. *Let X be a normal, T_1 space. Then X is homeomorphic to a subspace of $\prod_{\alpha\in\lambda}[0,1]_\alpha$ for some λ, where each factor is the unit interval with the standard topology.*

Scholium 7.70. *A space X is completely regular and T_1 if and only if X can be embedded in $\prod_{\alpha\in\lambda}[0,1]_\alpha$ for some λ, where each factor is the unit interval with the standard topology.*

Theorem 7.71. *Given a locally finite open cover $\{U_\alpha\}_{\alpha\in\lambda}$ of a normal, T_1 space X, there is a collection of corresponding continuous functions $\phi_\alpha : X \to [0,1]$ such that (i) each ϕ_α is zero outside U_α, and (ii) the ϕ_α pointwise add to 1. The collection $\{\phi_\alpha\}_{\alpha\in\lambda}$ is called a* ***partition of unity****.*

Chapter 8

Connectedness: When Things Don't Fall into Pieces

Theorem 8.1. *The following are equivalent:*

1. X *is connected.*
2. *there is no continuous function* $f : X \to \mathbb{R}_{std}$ *such that* $f(X) = \{0, 1\}$.
3. X *is not the union of two disjoint non-empty separated sets.*
4. X *is not the union of two disjoint non-empty closed sets.*
5. *the only subsets of* X *that are both closed and open in* X *are the empty set and* X *itself.*
6. *for every pair of points* p *and* q *and every open cover* $\{U_\alpha\}_{\alpha\in\lambda}$ *of* X *there exist a finite number of the* U_α*'s,* $\{U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, ..., U_{\alpha_n}\}$ *such that* $p \in U_{\alpha_1}$, $q \in U_{\alpha_n}$ *and for each* $i < n$, $U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \varnothing$.

Exercise 8.2. *Which of the following spaces are connected?*

1. $\mathbb{R}$ *with the discrete topology?*
2. $\mathbb{R}$ *with the indiscrete topology?*
3. $\mathbb{R}$ *with the finite complement topology?*
4. $\mathbb{R}_{LL}$*?*
5. $\mathbb{Q}$ *as a subspace of* $\mathbb{R}_{std}$*?*
6. $\mathbb{R} - \mathbb{Q}$ *as a subspace of* $\mathbb{R}_{std}$*?*

Theorem 8.3. *The space* $\mathbb{R}_{std}$ *is connected.*

Theorem 8.4. *Let A, B be separated subsets of a space X. If C is a connected subset of $A \cup B$, then either $C \subset A$ or $C \subset B$.*

Theorem 8.5. *Let $\{C_\alpha\}_{\alpha\in\lambda}$ be a collection of connected subsets of X and E be another connected subset of X such that for each α in λ, $E \cap C_\alpha \neq \varnothing$. Then $E \cup \left(\bigcup_{\alpha\in\lambda} C_\alpha\right)$ is connected.*

Theorem 8.6. *Let C be a connected subset of the topological space X. If D is a subset of X such that $C \subset D \subset \overline{C}$, then D is connected.*

Exercise 8.7. *Show that the closure of the topologist's sine curve in $\mathbb{R}^2_{std}$ is connected.*

Theorem 8.8. *Let X be a connected space, C a connected subset of X, and $X - C = A \mid B$. Then $A \cup C$ and $B \cup C$ are each connected.*

Theorem 8.9. *For topological spaces X and Y, $X \times Y$ is connected if and only if each of X and Y is connected.*

Theorem 8.10. *For spaces $\{X_\alpha\}_{\alpha\in\lambda}$, $\prod_{\alpha\in\lambda} X_\alpha$ is connected if and only if for each α in λ, X_α is connected.*

Exercise 8.11. *Show that the box product of countably infinitely many copies of $\mathbb{R}_{std}$ is not connected.*

Theorem 8.12. *Let $f : X \to Y$ be a continuous, surjective function. If X is connected, then Y is connected.*

Theorem 8.13. *(Intermediate Value Theorem) Let $f : \mathbb{R}_{std} \to \mathbb{R}_{std}$ be a continuous map. If $a, b \in \mathbb{R}$ and r is a point of $\mathbb{R}$ such that $f(a) < r < f(b)$ then there exists a point c in (a, b) such that $f(c) = r$.*

Theorem 8.14. *Let X be a countable, regular, T_1 space with more than one point. Then X is not connected.*

Exercise 8.15. *Show that Bing's Sticky Foot Topology is a countable, connected, Hausdorff space.*

Theorem 8.16. *If X is a normal, T_1 space with more than one point and $|X| < |\mathbb{R}|$, then X is not connected.*

Exercise 8.17. *Let A be a countable subset of $\mathbb{R}^n$ for $n \geq 2$. Show that $\mathbb{R}^n - A$ is connected. In fact, if the cardinality of A is any cardinality less than the cardinality of $\mathbb{R}$, then $\mathbb{R}^n - A$ will still be connected. Actually, for any two points p and q in $\mathbb{R}^n - A$, p can be connected to q by two intersecting straight line segments in $\mathbb{R}^n - A$.*

Theorem 8.18. *Each component of X is connected, closed, and not contained in any strictly larger connected subset of X.*

Theorem 8.19. *The set of components of a space X is a partition of X.*

Lemma 8.20. *Let X be a topological space and let $\{H_\alpha\}_{\alpha\in\lambda}$ be the set of subsets of X that are both open and closed. Then the following are equivalent:*

1. *For every two components A and B of X, there exists a separation of X into two disjoint closed sets such that A is in one and B is in the other.*

2. *For every component A of X, $\bigcap\{H_\alpha \mid A \subset H_\alpha\} = A$.*

Lemma 8.21. *Let X be a compact space and let U be an open set in X. Let $\{H_\alpha\}_{\alpha\in\lambda}$ be closed subsets of X such that $\bigcap_{\alpha\in\lambda} H_\alpha \subset U$. Then there exist a finite number of the H_α's whose intersection lies in U.*

Lemma 8.22. *Let A and B be components of a compact, Hausdorff space X. Then $X = H \mid K$ where $A \subset H$ and $B \subset K$.*

Theorem 8.23. *Let X be a compact, Hausdorff space. Let X^* be the partition of X into its components. Then the identification space X^* is a compact, Hausdorff space.*

Theorem 8.24. *Let A and B be closed subsets of a compact, Hausdorff space X such that no component intersects both A and B. Then $X = H \mid K$ where $A \subset H$ and $B \subset K$.*

Theorem 8.25. *Let U be a proper, open subset of a continuum X. Then each component of $\overline{U}$ contains a point of ∂U, the boundary of U. (Recall: $\partial U = \overline{U} - U$.)*

Theorem 8.26. *Let U be a proper, open subset of a continuum X. Then each component of U has a limit point on ∂U.*

Theorem 8.27. *No continuum X is the union of a countable number (> 1) of disjoint, non-empty closed subsets.*

Exercise 8.28. *Show that in Figure* **??**, *X is connected and is the union of a countable number of disjoint closed sets.*

Theorem 8.29. *Let $\{C_i\}_{i\in\omega}$ be a collection of continua such that for each i, $C_{i+1} \subset C_i$. Then $\bigcap_{i\in\omega} C_i$ is a continuum.*

Theorem 8.30. *Let $\{C_\alpha\}_{\alpha\in\lambda}$ be a collection of continua indexed by a well-ordered set λ such that if $\alpha < \beta$, then $C_\beta \subset C_\alpha$. Then $\bigcap_{\alpha\in\lambda} C_\alpha$ is a continuum.*

Lemma 8.31. *Let X be a continuum, p be a point of X, and $X - \{p\} = H \mid K$. Then $H \cup \{p\}$ is a continuum and if $q \neq p$ is a non-separating point of $H \cup \{p\}$, then q is a non-separating point of X.*

Theorem 8.32. *Let X be a separable continuum with more than one point. Then X has at least two non-separating points.*

Theorem 8.33. *Let X be a continuum with more than one point. Then X has at least two non-separating points.*

Theorem 8.34. *The bucket handle continuum is indecomposable.*

Theorem 8.35. *A path connected space is connected.*

Exercise 8.36. *The flea and comb space is connected but not path connected.*

Exercise 8.37. *The closure of the topologist's sine curve is connected but not path connected.*

Theorem 8.38. *The product of path connected spaces is path connected.*

Exercise 8.39.

1. *What are the path components of the closure of the topologist's sine curve?*
2. *What are the path components of the closure of the topologist's comb?*

Exercise 8.40. *Must every non-empty open connected subset U of $\mathbb{R}^n$ be path connected?*

Theorem 8.41. *Let p and q be two points in a Hausdorff space X such that there exists a continuous function $f : [0,1] \to X$ with $f(0) = p$ and $f(1) = q$. Then there exists an embedding $h : [0,1] \to X$ with $h(0) = p$ and $h(1) = q$.*

Theorem 8.42. *The following are equivalent:*

1. *X is locally connected.*
2. *X has a basis of connected open sets.*
3. *For each $x \in X$ and open set U with $x \in U$, the component of x in U is open.*
4. *For each $x \in X$ and open set U with $x \in U$, there is a connected set C such that $x \in$ Int $C \subset C \subset U$.*
5. *For each $x \in X$ and open set U with $x \in U$, there is an open set V containing x and $V \subset$ (the component of x in U).*

Exercise 8.43.

1. *Show that the closure of the topologist's comb is not locally connected.*
2. *Construct a space that is connected but not locally connected at any point.*
3. *Find an example of a space that is locally connected but not connected.*

Theorem 8.44. *The product of two locally connected spaces is locally connected.*

Exercise 8.45.

1. *Find an example of an infinite number of locally connected spaces where the infinite product space is not connected.*

2. *Prove that an arbitrary box product of locally connected spaces is locally connected.*

Theorem 8.46. *Let X be a locally connected space and let $f : X \to Y$ be a continuous, surjective, closed or open map. Then Y is locally connected.*

Exercise 8.47. *Construct an example of a locally connected space X and a continuous, surjective function $f : X \to Y$ such that Y is not locally connected.*

Theorem 8.48. *A locally path connected space is locally connected.*

Theorem 8.49. *The following are equivalent:*

1. *X is locally arcwise connected.*

2. *For each $x \in X$ and open set U with $x \in U$, there is an arcwise connected open set V such that $x \in V \subset U$.*

3. *X has a basis of connected, arcwise connected open sets.*

Theorem 8.50. *A Hausdorff space X is a Peano Continuum if and only if it is the image of $[0, 1]$ under a continuous map.*

Theorem 8.51. *Let $f : [0, 1] \to X$ be a continuous surjective map where X is Hausdorff. Then X is locally arcwise connected. Equivalently, every Peano Continuum is locally arcwise connected.*

Theorem 8.52. *Let X be a 0-dimensional, T_1 space. Then X is totally disconnected.*

Exercise 8.53. *Create a Hausdorff space that is totally disconnected but is not 0-dimensional.*

Theorem 8.54. *The standard Cantor set is precisely those real numbers in $[0, 1]$ that can be written using only 0's or 2's in their ternary (that is, base 3) expansion.*

Exercise 8.55. *Show that every real number in $[0, 2]$ is the sum of two numbers in the standard Cantor set.*

Exercise 8.56. *Let C be the Cantor set. Create a continuous function $f : C \to [0, 1]$ that is surjective.*

Exercise 8.57. *Let C be the Cantor set. Create an embedding $h : C \to [0, 1] \times [0, 1]$ such that for every $x \in [0, 1]$, $(\{x\} \times [0, 1]) \cap h(C) \neq \varnothing$.*

Exercise 8.58. *Let C be the Cantor set. Create an embedding $h : C \to [-1, 1] \times [-1, 1] - \{(0, 0)\}$ such that every ray from $(0, 0)$ straight out to infinity intersects $h(C)$.*

Exercise 8.59. *Let C be the Cantor set. Create an embedding $h : C \to [0, 1] \times [0, 1]$ such that for every continuous function $f : [0, 1] \to [0, 1]$, $G_f \cap h(C) \neq \varnothing$, where G_f is the graph of f.*

Exercise 8.60. *Let C be the Cantor set, let $h : C \to \mathbb{R}^2$ be an embedding, and let p and q be points in $\mathbb{R}^2 - h(C)$. Show that you can find a polygonal path from p to q in $\mathbb{R}^2 - h(C)$.*

Theorem 8.61. *Let C be the standard Cantor set and let $h : C \to \mathbb{R}^2$ be an embedding. Then there exists a homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that for every $x \in C$, $H(h(x)) = x$.*

Theorem 8.62. *Let C be the standard Cantor set. There exists an embedding $h : C \to \mathbb{R}^3$ such that no homeomorphism $H : \mathbb{R}^3 \to \mathbb{R}^3$ exists where $H(h(C)) = C$.*

Theorem 8.63. *Let C be the standard Cantor set and let X be a 2^{nd} countable, compact, Hausdorff space. Then there exists a continuous, surjective function $f : C \to X$.*

Chapter 9

Metric Spaces: Getting Some Distance

Exercise 9.1. *Verify that the following are all metrics on* $\mathbb{R}^n$.

1. *The* ***Euclidean metric*** *on* $\mathbb{R}^n$ *is defined by* $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.
2. *The* ***box metric*** *on* $\mathbb{R}^n$ *is defined by* $d(\mathbf{x}, \mathbf{y}) = \max_i\{|x_i - y_i|\}$.
3. *The* ***taxi-cab metric*** *on* $\mathbb{R}^n$ *is defined by* $d(\mathbf{x}, \mathbf{y}) = \sum_i\{|x_i - y_i|\}$.

Show that when $n \geq 2$*, these metrics are different.*

Exercise 9.2. *Let* X *be a compact topological space. Let* $\mathcal{C}(X)$ *denote the set of continuous functions* $f : X \to \mathbb{R}$. *We can endow* $\mathcal{C}(X)$ *with a metric:*

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|$$

and this distance is also sometimes denoted $\|f - g\|$. *Check that* d *is a well-defined metric on* $\mathcal{C}(X)$.

Theorem 9.3. *Let* d *be a metric on the set* X. *Then the collection of all open balls*

$$\mathcal{B} = \{B(p, \epsilon) = \{y \in X | d(p, y) < \epsilon\} \text{ for every } p \in X \text{ and every } \epsilon > 0\}$$

forms a basis for a topology on X.

Exercise 9.4. *On* $\mathbb{R}^n$*, show that the Euclidean metric, box metric, and taxi-cab metric generate the same topology as the product topology on* n *copies of* $\mathbb{R}_{std}$.

Exercise 9.5. *Now find a metric on* $\mathbb{R}^n$ *that does* not *induce the product topology on* n *copies of* $\mathbb{R}_{std}$.

Theorem 9.6. *For any metric space* (X, d)*, there exists a metric* $\bar{d}$ *such that* d *and* $\bar{d}$ *generate the same topology, yet for each* $x, y \in X$*,* $\bar{d}(x, y) < 1$.

Theorem 9.7. *If* X *is a metric space and* $Y \subset X$*, then* Y *is a metric space.*

Theorem 9.8. *A metric space is Hausdorff, regular, and normal.*

Theorem 9.9. *A metric space is completely normal and perfectly normal.*

Theorem 9.10. *A metric space is a 1^{st} countable space.*

Theorem 9.11. *In a metric space X, the following are equivalent:*

1. *X is separable,*

2. *X is 2^{nd} countable,*

3. *X is Lindelöf,*

4. *every uncountable set in X has a limit point,*

Exercise 9.12. *If you've read about the Souslin property in Section* **??**, *then a fifth property can be added to the above theorem: a metric space X has the Souslin property if and only if it has the other properties mentioned in Theorem 9.11.*

Theorem 9.13. *Let (X, d) and (Y, e) be metric spaces. Then $X \times Y$ is a metric space.*

Theorem 9.14. *Let $\{(X_i, d_i)\}_{i\in\omega_0}$ be a countable collection of metric spaces. Then $\prod_{i\in\omega_0} X_i$ is metrizable.*

Exercise 9.15. *Show that if $\{X_\alpha\}_{\alpha\in\lambda}$ is an uncountable collection of non-degenerate spaces, then $\prod_{\alpha\in\lambda} X_\alpha$ is not metrizable.*

Exercise 9.16. *Consider the set $\mathbb{R}^\omega$ with the box topology, and show that it is not metrizable.*

Theorem 9.17. *A metric space is compact if and only if it is countably compact.*

Theorem 9.18. *A metric space is compact if and only if every infinite subset of X has a limit point.*

Theorem 9.19. *A function f from a metric space (X, d_X) to a metric space (Y, d_Y) is continuous at the point x (in the topological sense) if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $y \in X$, if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$. The function f is continuous if and only if it is continuous at every point $x \in X$.*

Exercise 9.20. *Give an example of a continuous function from $\mathbb{R}^1$ to $\mathbb{R}^1$ with the standard topology that is not uniformly continuous.*

Theorem 9.21. *Let $f : X \to Y$ be a continuous function from a compact metric space to a metric space Y. Then f is uniformly continuous.*

Exercise 9.22. *Find a sequence of continuous functions $f_i : [0,1] \to [0,1]$ ($i \in \mathbb{N}$) such that for each point $x \in [0,1]$, the points $f_i(x)$ converge to a point p_x in $[0,1]$ and yet the function $L : [0,1] \to [0,1]$ defined by $L(x) = p_x$ is not continuous.*

Theorem 9.23. *Let X be a topological space and let Y be a compact metric space. If a sequence of continuous functions $f_i : X \to Y$ converges uniformly, then $f : X \to Y$ defined by $f(x) = \lim f_i(x)$ for each $x \in X$ exists and is continuous.*

Lebesgue Number Theorem 9.24. *Let $\{U_\alpha\}_{\alpha \in \lambda}$ be an open cover of a compact set A in a metric space X. Then there exists a $\delta > 0$ such that for every point p in A, $B(p, \delta) \subset U_\alpha$ for some α. This number δ is called a **Lebesgue number** of the cover.*

Theorem 9.25. *Let $\gamma : [0,1] \to X$ be a **path**: a continuous map from $[0,1]$ into the space X. Given an open cover $\{U_\alpha\}$ of X, show that $[0,1]$ can be divided into N intervals of the form $I_i = [\frac{i-1}{N}, \frac{i}{N}]$ such that each $\gamma(I_i)$ lies completely in one set of the cover.*

Exercise 9.26.

1. *The space $\mathbb{R}^n$ is complete.*
2. *There is a metric that generates the standard topology on $\mathbb{R}^1$ that is not a complete metric.*

Theorem 9.27. *Let X be a compact metric space. Then every metric for X is a complete metric.*

Theorem 9.28 (The Baire Category Theorem)**.** *Let X be a complete metric space and $\{U_i\}_{i \in \mathbb{N}}$ be a collection of dense open sets. Then $\bigcap_{i \in \mathbb{N}} U_i$ is a dense set.*

Theorem 9.29 (The Baire Category Theorem)**.** *Let X be a complete metric space. Then X is not the union of countably many nowhere dense sets.*

Theorem 9.30. *Let X be a topological space and let Y be a complete metric space. If a sequence of continuous functions $f_i : X \to Y$ converges uniformly, then $f : X \to Y$ defined by $f(x) = \lim f_i(x)$ for each $x \in X$ exists and is continuous.*

Theorem 9.31. *If X and Y are complete metric spaces, then $X \times Y$ is complete.*

Theorem 9.32. *Every closed subset of a complete metric space is complete.*

Theorem 9.33. *Let U be an open subset of a complete metric space X. Then U is topologically complete, that is, there is a complete metric on U that generates the relative topology of U.*

Theorem 9.34. *If $\{X_i\}_{i \in \mathbb{N}}$ is a collection of complete metric spaces, then $\prod_{i \in \mathbb{N}} X_i$ is complete.*

Theorem 9.35. *Let $\{U_i\}_{i \in \mathbb{N}}$ be a countable collection of open sets in a complete space X. Then $Y = \bigcap_{i \in \mathbb{N}} U_i$ is complete.*

Theorem 9.36. *Let X be a complete space. Then $Y \subset X$ is complete if and only if there exists a countable collection of open sets $\{U_i\}_{i\in\mathbb{N}}$ such that $Y = \bigcap_{i\in\mathbb{N}} U_i$.*

Theorem 9.37. *A Hausdorff space X is a Peano Continuum if and only if X is the image of $[0,1]$ under a continuous, surjective function.*

Theorem 9.38. *A Peano Continuum is path connected and locally path connected.*

Theorem 9.39. *An open, connected subset of a Peano Continuum is path connected.*

Theorem 9.40. *Let X be a metric continuum with exactly two non-separating points. Then X is homeomorphic to $[0,1]$.*

Theorem 9.41. *Let X be a non-degenerate metric continuum where no point separates X but every pair of points separates X. Then X is homeomorphic to $\mathbb{S}^1$.*

Theorem 9.42. *Let X be a metric continuum with more than one point where no pair of points separates X, but every subset of X homeomorphic to $\mathbb{S}^1$ separates X. Then X is homeomorphic to $\mathbb{S}^2$.*

Exercise 9.43.

1. *Is the space $\mathbb{R}$ with the discrete topology metrizable?*
2. *Is the space $\mathbb{R}_{LL}$ metrizable?*

Exercise 9.44. *Take your chart of examples and properties and add metric space as an example and add metrizable as a property and fill in the chart.*

Urysohn's Metrization Theorem 9.45. *Every regular, T_1, 2^{nd} countable space is metrizable.*

Theorem 9.46. *Let X be a compact Hausdorff space that is 2^{nd} countable. Then X is metrizable.*

Theorem 9.47. *Let X be a compact metric space, Y be a Hausdorff space, and $f : X \to Y$ be a continuous, surjective function. Then Y is a compact metric space.*

Theorem 9.48. *Let X be a Hausdorff space, and let C be the standard Cantor set. Then X is a compact metric space if and only if there exists a continuous surjective function $f : C \to X$.*

Theorem 9.49. *Every separable metric space can be embedded in a countable product of $[0,1]$'s.*

Exercise 9.50. *Show that a hedgehog is a metric space where the distance between two points can be described as taking the distance from one point to the 0 on its spine and then adding the distance out to the second point on the other point's spine.*

Theorem 9.51. *The countable product of hedgehogs is metrizable.*

Theorem 9.52. *A regular space with a σ-discrete basis is normal. In fact, given a discrete collection of closed sets $\{C_\alpha\}_{\alpha\in\lambda}$, there exists a discrete collection of open sets $\{U_\alpha\}_{\alpha\in\lambda}$ such that for each α, $C_\alpha \subset U_\alpha$.*

Theorem 9.53. *A regular, T_1 space X with a σ-discrete basis is metrizable.*

Lemma 9.54. *Let $\{U_i\}_{i\in\mathbb{N}}$ be a countable open cover of a metric space X. For each point $x \in X$ let $m(x)$ be the natural number such that $x \in U_i$ but $x \notin U_j$ for $j < i$. Then for every $n \in \mathbb{N}$ there exists a discrete collection of closed sets $\{C_{i,n}\}$ such that*

1. *for each i, $C_{i,n} \subset U_i$;*
2. *for each $x \in C_{i,n}$, $B(x, \frac{1}{n}) \subset U_i$;*
3. *for each i, $C_{i,n}$ does not intersect U_j for $j < i$; and*
4. *for each i, $C_{i,n}$ contains every point $x \in U_i$ for which $m(x) = i$ and for which $d(x, X - U_i) > \frac{1}{n}$.*

Then $\bigcup C_{i,n} = X$ and for each n, the collection of $\frac{1}{3n}$ neighborhoods of the $C_{i,n}$'s, that is,

$$\left\{ \bigcup_{x\in C_{i,n}} B(x, \frac{1}{3n}) \right\}_{i\in\mathbb{N}},$$

is a discrete collection of open sets.

Lemma 9.55. *Let $\{U_\alpha\}_{\alpha\in\lambda}$ be an open cover of a metric space X where the index set λ is well-ordered. For each point $x \in X$ let $m(x)$ be the ordinal number α such that $x \in U_\alpha$ but $x \notin U_\beta$ for $\beta < \alpha$. Then for every $n \in \mathbb{N}$ there exists a discrete collection of closed sets $\{C_{\alpha,n}\}$ such that*

1. *for each α, $C_{\alpha,n} \subset U_\alpha$;*
2. *for each $x \in C_{\alpha,n}$, $B(x, \frac{1}{n}) \subset U_\alpha$;*
3. *for each α, $C_{\alpha,n}$ does not intersect U_β for $\beta < \alpha$; and*
4. *for each α, $C_{\alpha,n}$ contains every point $x \in U_\alpha$ for which $m(x) = \alpha$ and for which $d(x, X - U_\alpha) > \frac{1}{n}$.*

Then $\bigcup C_{\alpha,n} = X$ and for each n, the collection of $\frac{1}{3n}$ neighborhoods of the $C_{\alpha,n}$'s, that is,

$$\left\{ \bigcup_{x\in C_{\alpha,n}} B(x, \frac{1}{3n}) \right\}_{\alpha\in\lambda},$$

is a discrete collection of open sets.

The Bing Metrization Theorem 9.56. *A regular, T_1 space X is metrizable if and only if X has a σ-discrete basis.*

Theorem 9.57. *A regular space with a σ-locally finite basis is normal. In fact, given a discrete collection of closed sets $\{C_\alpha\}_{\alpha\in\lambda}$, there exists a discrete collection of open sets $\{U_\alpha\}_{\alpha\in\lambda}$ such that for each α, $C_\alpha \subset U_\alpha$.*

The Nagata-Smirnov Metrization Theorem 9.58. *A regular, T_1 space X is metrizable if and only if X has a σ-locally finite basis.*

Lemma 9.59. *Let X be a space with a σ-locally finite basis $\{\{B_{\alpha,n}\}_{\alpha\in\lambda_i}\}_{n\in\mathbb{N}}$. Let $\{U_\alpha\}_{\alpha\in\lambda}$ be a locally finite collection of open sets in a space X where the index set λ is well-ordered. (In the application, this collection of U_α's will be one of the locally finite collections of basis elements.) For each point $x \in \bigcup_{\alpha\in\lambda} U_\alpha$ let $m(x)$ be the ordinal number α such that $x \in U_\alpha$ but $x \notin U_\beta$ for $\beta < \alpha$. Then for every $n \in N$ there exists a discrete collection of closed sets $\{C_{\alpha,n}\}$ such that*

1. *for each α, $C_{\alpha,n} \subset U_\alpha$;*
2. *for each α, $C_{\alpha,n}$ does not intersect U_β for $\beta < \alpha$; and*
3. *for each α, $C_{\alpha,n}$ contains every point $x \in U_\alpha$ for which $m(x) = \alpha$ and for which $x \in \{B_{\alpha,n}\}$.*

Then

1. *for each n, $\{C_{\alpha,n}\}_{\alpha\in\lambda}$ is a discrete collection of closed sets;*
2. *$\bigcup_{\alpha\in\lambda;n\in\mathbb{N}} C_{\alpha,n} = \bigcup_{\alpha\in\lambda} U_\alpha$; and*
3. *for each n, there exists a discrete collection of open sets $\{V_{\alpha,n}\}_{\alpha\in\lambda}$ such that for each $\alpha \in \lambda$, $C_{\alpha,n} \subset V_{\alpha,n} \subset \overline{V_{\alpha,n}} \subset U_\alpha$.*

Lemma 9.60. *Let $\{B_i\}_{i\in\mathbb{N}}$ be a countable basis of a regular space X. Let $\{U_\alpha\}_{\alpha\in\lambda}$ be an open cover of X. Let $\{C_i\}_{i\in\mathbb{N}}$ be the set of all B_i's such that each C_i lies in some U_α in the open cover. Then $\{C_i\}_{i\in\mathbb{N}}$ is an open refinement of the open cover $\{U_\alpha\}_{\alpha\in\lambda}$; however, it is not locally finite. Let $\{D_i\}_{i\in\mathbb{N}}$ be the set of all B_i's such that each $\overline{D_i}$ is a subset of some C_k. For each $i \in \mathbb{N}$ let $E_i = C_i - \bigcup\{\overline{D_j} | j < i$ and $\overline{D_j} \subset C_k$ for some $k < i\}$. Then $\{E_i\}_{i\in\mathbb{N}}$ is a locally finite refinement of $\{U_\alpha\}_{\alpha\in\lambda}$.*

Theorem 9.61. *Metric spaces are paracompact.*

Part II

Algebraic and Geometric Topology

Chapter 11

Classification of 2-Manifolds: Organizing Surfaces

Exercise 11.1. *Show that the torus T^2 is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$.*

Exercise 11.2. *For a given number of holes, demonstrate that the n-holed torus where the holes are lined up is homeomorphic to an n-holed torus where the holes are arranged in a circle.*

Exercise 11.3.

1. *Show that $\mathbb{R}\mathrm{P}^2 \cong \mathbb{S}^2/\langle x \sim -x \rangle$, that is, the projective plane is homeomorphic to the 2-sphere with diametrically opposite points identified.*
2. *Show that $\mathbb{R}\mathrm{P}^2$ is also homeomorphic to a disk with two edges on its boundary (called a **bigon**), identified as indicated in Figure 11.1.*

Figure 11.1: $\mathbb{R}\mathrm{P}^2$.

3. *Show that the Klein bottle can be realized as a square with certain edges identified.*

Theorem 11.4. *Suppose M is a compact, connected 1-manifold. Then M is triangulable. That is, M is homeomorphic to a subset C of $\mathbb{R}^n$ consisting of a finite collection of straight line segments where any two segments of C are either disjoint or meet at an endpoint of each.*

Exercise 11.5. *Provide a complete classification of compact, connected 1-manifolds. That is, describe a collection of topological spaces such that every compact, connected 1-manifold is homeomorphic to one member of the collection.*

Exercise 11.6. *Provide a complete classification of compact 1-manifolds.*

Theorem 11.7. *Every compact* 2*-manifold is triangulable, that is, it is homeomorphic to a subset C of $\mathbb{R}^n$ consisting of a finite collection $T = \{\sigma_i\}_{i=1}^k$ of rectilinear triangles (a fancy word for a rectilinear triangle is a* 2*-simplex) where each pair of triangles are disjoint or they meet in one vertex of each or they share a single edge. Since the space C is homeomorphic to a* 2*-manifold, each edge of each triangle making up C is shared by exactly two triangles, and around each vertex is a circle of triangles whose union is a disk.*

Exercise 11.8. *The boundary of a tetrahedron is naturally triangulated with a triangulation T consisting of four* 2*-simplexes, having six edges and four vertices.*

1. *On the boundary of a tetrahedron draw the first and second barycentric subdivisions of T.*
2. *Locate the edges of the four triangles in T.*
3. *Draw the regular neighborhood of the union of all the edges of T.*
4. *Draw the regular neighborhood of a single edge of a triangle in T.*

Exercise 11.9. *In the second barycentric subdivisions of a triangulation of the torus (Figure* **??***), find regular neighborhoods of various subsets of the edges.*

Exercise 11.10. *Consider the triangulation of the torus in Figure* **??***. Describe those graphs created from edges in the* 1*-skeleton of T that have regular neighborhoods homeomorphic to a disk.*

Theorem 11.11. *Let M^2 be a compact, triangulated* 2*-manifold with triangulation T. Let S be a tree whose edges are* 1*-simplices in the* 1*-skeleton of T. Then $N(S)$, the regular neighborhood of S, is homeomorphic to $\mathbb{D}^2$.*

Theorem 11.12. *Let M^2 be a compact, triangulated* 2*-manifold with triangulation T. Let S be a tree equal to a union of 'edges' in the dual* 1*-skeleton of T. Then $\cup\{\sigma_j'' \mid \sigma_j'' \in T''$ and $\sigma_j'' \cap S \neq \varnothing\}$ is homeomorphic to $\mathbb{D}^2$.*

Theorem 11.13. *Let M^2 be a connected, compact, triangulated* 2*-manifold with triangulation T. Let S be a tree in the* 1*-skeleton of T. Let S' be the subgraph of the dual* 1*-skeleton of T whose 'edges' do not intersect S. Then S' is connected.*

Theorem 11.14. *Let M^2 be a connected, compact, triangulated* 2*-manifold. Then $M^2 = D_0 \cup D_1 \cup \left(\bigcup_{i=1}^k H_i\right)$ where D_0, D_1, and each H_i is homeomorphic to $\mathbb{D}^2$, $\operatorname{Int} D_0 \cap D_1 = \varnothing$, the H_i's are disjoint, $\bigcup_{i=1}^k \operatorname{Int} H_i \cap (D_0 \cup D_1) = \varnothing$, and for each i, $H_i \cap D_1$ equals* 2 *disjoint arcs each arc on the boundary of each of H_i and D_1.*

Theorem 11.15. *Let M^2 be a connected, compact, triangulated* 2*-manifold. Then:*

1. *There is a disk D_0 in M^2 such that $M^2 - (\mathrm{Int}\, D_0)$ is homeomorphic to the following subset of $\mathbb{R}^3$: a disk D_1 with a finite number of disjoint strips, H_i for $i \in \{1, \dots n\}$, attached to boundary of D_1 where each strip has no twist or a $1/2$ twist. (See Figure 11.2.)*

2. *Furthermore, the boundary of the disk with strips, $D_1 \cup \left(\bigcup_{i=1}^{k} H_i\right)$, is connected.*

Figure 11.2: A disk with four handles attached.

Exercise 11.16. *In the conclusion of the previous theorem, any strip H_i divides the boundary of D_1 into two arcs, e_i^1 and e_i^2, where H_i is* not *attached, that is, the two arcs that make up $(D_1 \bigcap H_i)$ are disjoint from the two arcs e_i^1 and e_i^2 except at their endpoints. Show that if a strip H_j is attached to D_1 with no twists, then there must be a strip H_k that is attached to both e_j^1* and *e_j^2.*

Theorem 11.17. *Let M^2 be a connected, compact, triangulated 2-manifold. Then there is a disk D_0 in M^2 such that $M^2 - \mathrm{Int}\, D_0$ is homeomorphic to a disk D_1 with strips attached as follows: first come a finite number of strips with $1/2$ twist each of whose attaching arcs are consecutive along $\mathrm{Bd}\, D_1$, and next come a finite number of pairs of untwisted strips, each pair with attaching arcs entwined as pictured with the four arcs from each pair consecutive along $\mathrm{Bd}\, D_1$.*

Theorem 11.18. *Let X be the union of a disk with three strips attached as follows: a disk E_0 with one strip attached with a $1/2$ twist with its attaching arcs consecutive along $\mathrm{Bd}\, E_0$ and one pair of untwisted strips with attaching arcs entwined as pictured with the four arcs consecutive along $\mathrm{Bd}\, E_0$. Let Y again be a union of a disk with three strips attached, but the three are attached differently. The set Y consists of a disk E_1 with three strips with a $1/2$ twist each whose attaching arcs are consecutive along $\mathrm{Bd}\, E_1$. Then X is homeomorphic to Y.*

Figure 11.3: These spaces are homeomorphic.

Theorem 11.19. *Let M^2 be a connected, compact, triangulated 2-manifold. Then there is a disk D_0 in M^2 such that $M^2 - \mathrm{Int}\, D_0$ is homeomorphic to one of the following:*

a) a disk D_1,

b) a disk D_1 with k $\frac{1}{2}$-twisted strips with consecutive attaching arcs, or

c) a disk D_1 with k pairs of untwisted strips, each pair in entwining position with the four attaching arcs from each pair consecutive.

Theorem 11.20. *Suppose M_1 and M_2 are compact, triangulated, connected 2-manifolds and M is a connected sum of M_1 and M_2 (that is, we can select triangulations of M_1 and M_2, apply the process above and arrive at a space homeomorphic to M). Then M is a compact, connected, triangulable 2-manifold.*

Exercise 11.21. *Suppose M is a compact, connected, triangulated 2-manifold. What is $\mathbb{S}^2 \# M$?*

Exercise 11.22. *Sketch $\overset{n}{\underset{i=1}{\#}} \mathbb{T}^2$.*

Theorem 11.23 (Classification of compact, connected 2-manifolds)**.** *Any connected, compact, triangulated 2-manifold is homeomorphic to the 2-sphere $\mathbb{S}^2$, a connected sum of tori, or a connected sum of projective planes.*

Exercise 11.24. *Identify the following spaces and give justification.*

(a) (b) (c)

Theorem 11.25. *Let P be a polygonal presentation. Then P is a 2-manifold.*

Theorem 11.26. *Suppose M is a compact, connected, triangulable 2-manifold. Then M is homeomorphic to a polygonal presentation.*

Theorem 11.27. *Let P be a polygonal presentation. Then P is a compact, connected, triangulable 2-manifold.*

Theorem 11.28. *Let $Abb^{-1}C$ be a string of $2n$ letters where each letter occurs twice, neglecting superscripts (and there is at least one pair other than b and b^{-1}). Then the 2-manifold obtained from the word $Abb^{-1}C$ is homeomorphic to that obtained from AC.*

Theorem 11.29. *Suppose P is a polygonal presentation not homeomorphic to $\mathbb{S}^2$. Then there is a homeomorphic polygonal presentation where all the vertices are in the same equivalence class, that is, all the vertices are identified to each other.*

Theorem 11.30. *Suppose P is a polygonal presentation not homeomorphic to $\mathbb{S}^2$. Then P is homeomorphic to a polygonal presentation where all the vertices are identified and for every pair of edges with the same orientation, the two edges of that pair are consecutive.*

Theorem 11.31. *Suppose P is a polygonal presentation not homeomorphic to $\mathbb{S}^2$. Then P is homeomorphic to a polygonal presentation where all the vertices are identified, every pair of edges with the same orientation are consecutive, and all other edges are grouped in disjoint sets of two intertwined pairs following the pattern $aba^{-1}b^{-1}$.*

Theorem 11.32. *If A and C are (possibly empty) words, then the polygonal presentation $Aaba^{-1}b^{-1}ccC$ is homeomorphic to that represented by $AddeeffC$.*

Theorem 11.33. *Any compact, connected, triangulated 2-manifold M is homeomorphic to the polygonal presentation given by one of the following words: aa^{-1}, $a_1a_1 \ldots a_na_n$ (where $n \geq 1$) or $a_1a_2a_1^{-1}a_2^{-1} \ldots a_{n-1}a_na_{n-1}^{-1}a_n^{-1}$ (where $n \geq 2$ is even).*

Exercise 11.34. *Suppose that we have two compact, connected 2-manifolds represented by the words w_1 and w_2, respectively. Suppose in addition that w_1 and w_2 have no letters in common. What can you say about the 2-manifold corresponding to the concatenated word w_1w_2 in terms of the connected sum?*

Exercise 11.35. *Re-state Theorem 11.32 above in the case that A and C are empty, in terms of connected sum.*

Theorem 11.36 (Classification of compact, connected 2-manifolds)**.** *Any compact, connected, triangulated 2-manifold is homeomorphic to exactly one of the following:*

1. $\mathbb{S}^2$,
2. *a connected sum of n tori, and*
3. *a connected sum of n projective planes.*

Exercise 11.37. *Describe heuristically a strategy by which you would define a consistent clockwise direction on the standard embedding of the 2-sphere in $\mathbb{R}^3$. What is the relevant property?*

Exercise 11.38. *Show that the induced orientation is well defined; in other words, that it is independent of the choice of positive equivalence class representative for the original 2-simplex.*

Theorem 11.39. *Show that the following are equivalent for a 2-manifold M.*

1. *Every triangulation of M is not orientable (that is if K_T is a simplicial complex to which M is homeomorphic, K_T is not orientable).*
2. *M admits a triangulation that is not orientable.*
3. *M admits a triangulation that contains a collection of simplices whose union is homeomorphic to the Möbius band.*
4. *M admits an embedding of a Möbius band.*
5. *There is a map $F : \mathbb{S}^1 \times [0,1] \to M$ such that $F(\cdot, t)$ is an embedding for each t and such that $F(\cdot, 1) = F(r(\cdot), 0)$, where r is a reflection map of $\mathbb{S}^1$ about some line through its center.*

Theorem 11.40. *Let $M_1, \ldots, M_n$ be connected, compact, triangulated 2-manifolds. Let M be a connected sum of $M_1, \ldots, M_n$. Then M is orientable if and only if M_i is orientable for each $i \in \{1, \ldots, n\}$.*

Exercise 11.41. *State and prove which compact, connected, triangulated 2-manifolds are orientable and which are not.*

Exercise 11.42. *Calculate the Euler characteristic of the following spaces.*

1. $\mathbb{S}^2$
2. $\mathbb{T}^2$
3. $\mathbb{K}^2$
4. $\mathbb{R}\mathrm{P}^2$

Lemma 11.43. *Suppose M_1 and M_2 are compact 2-manifolds. If $M_1 \# M_2$ is* any *choice for the connect sum of M_1 and M_2, then $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$.*

Exercise 11.44.

1. *Calculate the Euler characteristic of* $\mathop{\#}\limits_{i=1}^{n} \mathbb{R}\mathrm{P}^2$.
2. *Calculate the Euler characteristic of* $\mathop{\#}\limits_{i=1}^{n} \mathbb{T}^2$.

Theorem 11.45. *The combination of Euler characteristic and orientability is a complete invariant of compact, connected 2-manifolds.*

Exercise 11.46. *Identify the following 2-manifolds as a sphere, a connected sum of n tori (specifying n), or a connected sum of n projective planes (specifying n).*

a. $\mathbb{T}\#\mathbb{R}\mathrm{P}$

b. $\mathbb{K}\#\mathbb{R}\mathrm{P}$

c. $\mathbb{R}\mathrm{P}\#\mathbb{T}\#\mathbb{K}\#\mathbb{R}\mathrm{P}$

d. $\mathbb{K}\#\mathbb{T}\#\mathbb{T}\#\mathbb{R}\mathrm{P}\#\mathbb{K}\#\mathbb{T}$

Exercise 11.47. *Identify the surface obtained by identifying the edges of the decagon as indicated in Figure 11.4.*

Figure 11.4: The decagon with edges identified in pairs.

Exercise 11.48. *Notice that the edge (or boundary) of a Möbius band is a simple close curve. Construct a space by gluing a disk to the Möbius band along their respective boundaries. Show that this space is homeomorphic to the projective plane.*

Chapter 12

Fundamental Group: Capturing Holes

Theorem (Fundamental Theorem of Algebra). *A polynomial $p(z) = a_n z^n + \ldots + a_1 z + a_0$ with complex coefficients and degree $n > 1$ has at least one root.*

Exercise 12.1. *A polynomial $p(x) = a_n x^n + \ldots + a_1 x + a_0$ with real coefficients where $a_n \neq 0$ and n is odd has at least one real root.*

Theorem 12.2. *Given topological spaces X and Y with $S \subset X$, homotopy relative to S is an equivalence relation on the set of all continuous functions from X to Y. In particular, if $S = \varnothing$, homotopy is an equivalence relation on the set of all continuous functions from X to Y.*

Theorem 12.3. *If α, α', β, and β' are paths in a space X such that $\alpha \sim \alpha'$, $\beta \sim \beta'$, and $\alpha(1) = \beta(0)$, then $\alpha \cdot \beta \sim \alpha' \cdot \beta'$.*

Theorem 12.4. *Given paths α, β, and γ where the following products are defined, then $(\alpha \cdot \beta) \cdot \gamma \sim \alpha \cdot (\beta \cdot \gamma)$ and $([\alpha] \cdot [\beta]) \cdot [\gamma] = [\alpha] \cdot ([\beta] \cdot [\gamma])$.*

Theorem 12.5. *Let α be a path with $\alpha(0) = x_0$. Then $\alpha \cdot \alpha^{-1} \sim e_{x_0}$, where e_{x_0} is the constant path at x_0.*

Theorem 12.6. *The fundamental group $\pi_1(X, x_0)$ is a group. The identity element is the class of homotopically trivial loops based at x_0.*

Theorem 12.7. *If X is path connected, then $\pi_1(X, p) \cong \pi_1(X, q)$ for any points $p, q \in X$.*

Corollary 12.8. *Suppose X is a topological space and there is a path between points p and q in X. Then $\pi_1(X, p)$ is isomorphic to $\pi_1(X, q)$.*

Exercise 12.9. *Let α be a loop into a topological space X. Then $\alpha = \beta \circ \omega|_{[0,1]}$ where ω is the standard wrapping map and β is some continuous function from $\mathbb{S}^1$ into X. This relationship gives a correspondence between loops in X and continuous maps from $\mathbb{S}^1$ into X.*

Theorem 12.10. *Let X be a topological space and let p be a point in X. Then a loop $\alpha = \beta \circ \omega|_{[0,1]}$ (where ω is the standard wrapping map and β is a continuous function from $\mathbb{S}^1$ into X) is homotopically trivial if and only if β can be extended to a continuous function from the unit disk $\mathbb{D}^2$ to X.*

Theorem 12.11. *Show the following (1 denotes the trivial group):*

1. $\pi_1([0,1]) \cong 1$.
2. $\pi_1(\mathbb{R}^n) \cong 1$ *for* $n \geq 1$.
3. $\pi_1(X) \cong 1$, *if X is a convex set in $\mathbb{R}^n$.*
4. $\pi_1(X) \cong 1$, *if X is a cone.*
5. $\pi_1(X) \cong 1$ *if X is a star-like space in $\mathbb{R}^n$ (a subset X of $\mathbb{R}^n$ is called **star-like** if there is a fixed point $x_0 \in X$ such that for any $y \in X$, the line segment between x_0 and y lies in X; a five pointed 'star' is an example of a star-like space that is not convex).*

Exercise 12.12. *Show the following:*

1. $\pi_1(\mathbb{S}^0, 1) \cong 1$ *where $\mathbb{S}^0$ is the zero-dimensional sphere $\{-1, 1\}$, the set of points unit distance from the origin in $\mathbb{R}^1$.*
2. $\pi_1(\mathbb{S}^2) \cong 1$.
3. $\pi_1(\mathbb{S}^n) \cong 1$ *for* $n \geq 3$.

Exercise 12.13. *Show that the cone over the Hawaiian earring is simply connected. Can you generalize your insight?*

Theorem 12.14.

1. *Any loop $\alpha : [0,1] \to \mathbb{S}^1$ with $\alpha(0) = 1$ can be written $\alpha = \omega \circ \widetilde{\alpha}$, where $\widetilde{\alpha} : [0,1] \to \mathbb{R}^1$ satisfies $\widetilde{\alpha}(0) = 0$ and ω is the standard wrapping map.*
2. *If $\alpha : [0,1] \to \mathbb{S}^1$ is a loop, then $\widetilde{\alpha}(1)$ is an integer.*
3. *Loops α_1 and α_2 are equivalent in $\mathbb{S}^1$ if and only if $\widetilde{\alpha_1}(1) = \widetilde{\alpha_2}(1)$.*
4. $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.

Theorem 12.15. *Let (X, x_0), (Y, y_0) be path connected spaces. Then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

via the canonical map that takes a loop γ in $X \times Y$ to $(p \circ \gamma, q \circ \gamma)$ where $p : X \times Y \to X$ and $q : X \times Y \to Y$ are the projection maps.

Exercise 12.16. *Find:*

1. $\pi_1(X)$ *where* X *is a solid torus.*

2. $\pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$

3. $\pi_1(\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2)$

4. $\pi_1(X)$*, where* X *is a direct product of* k_n *copies of* $\mathbb{S}^n$*, with* $k_n = 0$ *for* n *sufficiently large.*

Exercise 12.17. *The fundamental group of the torus* $\pi_1(\mathbb{T}^2)$ *is* $\mathbb{Z}^2$*. Moreover, if* μ *is a meridian and* λ *is a longitude, then* $\{[\mu], [\lambda]\}$ *is a* $\mathbb{Z}$*-basis for* $\pi_1(\mathbb{T}^2)$*.*

Exercise 12.18. *Check that for a continuous function* $f : X \to Y$*, the induced homomorphism* f_* *is well-defined (that is, the image of an equivalence class is independent of the chosen representative). Show that it is indeed a group homomorphism.*

Theorem 12.19. *The following are true:*

1. *If* $f : (X, x_0) \to (Y, y_0)$ *and* $g : (Y, y_0) \to (Z, z_0)$ *are continuous maps, then* $(g \circ f)_* = g_* \circ f_*$*.*

2. *If* $id : (X, x_0) \to (X, x_0)$ *is the identity map, then* $id_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ *is the identity homomorphism.*

Theorem 12.20. *If* $h : X \to Y$ *is a homeomorphism then*

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, h(x_0))$$

is a group isomorphism. Thus homeomorphic path-connected spaces have isomorphic fundamental groups.

Theorem 12.21. *Fix a torus with chosen meridian* μ *and longitude* λ*. Suppose* $p, q \in \mathbb{Z}$*. Then there is a homeomorphism of the torus to itself which takes a representative of the class* $q[\mu] + p[\lambda] \in \pi(\mathbb{T}^2)$ *to* μ *if and only if* p *and* q *are relatively prime.*

Theorem 12.22. *If* $f, g : (X, x_0) \to (Y, y_0)$ *are continuous functions and* f *is homotopic to* g *relative to* x_0*, then* $f_* = g_*$*.*

Lemma 12.23. *Homotopy equivalence of spaces is an equivalence relation.*

Theorem 12.24. *If* $f : X \to Y$ *is a homotopy equivalence and* $y_0 = f(x_0)$*, then* $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ *is an isomorphism. In particular, if* $X \sim Y$*, then* $\pi_1(X) \cong \pi_1(Y)$*.*

Exercise 12.25. *Show that for* $n \geq 0$*,* $\mathbb{R}^{n+1} - \{0\}$ *can be strong deformation retracted onto* $\mathbb{S}^n$*.*

Lemma 12.26. *If A is a strong deformation retract of X, then A and X are homotopy equivalent.*

Theorem 12.27. $\mathbb{R}^2$ *is not homeomorphic to* $\mathbb{R}^n$ *for any* $n \neq 2$.

Exercise 12.28. *Let x and y be two points in $\mathbb{R}^2$. Show that $\mathbb{R}^2 - \{x, y\}$ strong deformation retracts onto the figure eight. In addition, show that $\mathbb{R}^2 - \{x, y\}$ strong deformation retracts onto a theta space.*

Theorem 12.29. *If $r : X \to A$ is a strong deformation retraction and $a \in A$, then $\pi_1(X, a) \cong \pi_1(A, a)$.*

Exercise 12.30. *Calculate the fundamental group of the following spaces.*

1. *An annulus.*

2. *A cylinder.*

3. *The Möbius Band.*

4. *An open 3-ball with a diameter removed.*

Exercise 12.31. *Find an example of a space X with a subspace A such that if $i : A \to X$ is the inclusion map, $i_* : \pi_1(A) \to \pi_1(X)$ is not injective.*

Theorem 12.32. *Let A be a retract of X via the inclusion $i : A \hookrightarrow X$ and retraction $r : X \hookrightarrow A$. Then for $a \in A$, $i_* : \pi_1(A, a) \to \pi_1(X, a)$ is injective and $r_* : \pi_1(X, a) \to \pi_1(A, a)$ is surjective.*

Theorem 12.33 (No Retraction Theorem for $\mathbb{D}^2$)**.** *There is no retraction from $\mathbb{D}^2$ to its boundary.*

Theorem 12.34 (Brouwer Fixed Point Theorem for $\mathbb{D}^2$)**.** *Let $f : \mathbb{D}^2 \to \mathbb{D}^2$ be a continuous map. Then there is some $x \in \mathbb{D}^2$ for which $f(x) = x$.*

Lemma 12.35. *A space is contractible if and only if it is homotopy equivalent to a point.*

Theorem 12.36. *A contractible space is simply connected.*

Theorem 12.37. *A retract of a contractible space is contractible.*

Corollary 12.38. *The house with two rooms is contractible.*

Corollary 12.39. *The Dunce's Hat is contractible.*

Theorem 12.40. *Let $X = U \cup V$, where U and V are open and path connected and $U \cap V$ is path-connected, simply connected, and non-empty. Then $\pi_1(X)$ is isomorphic to the free product of $\pi_1(U)$ and $\pi_1(V)$, that is, $\pi_1(X) \cong \pi_1(U) * \pi_1(V)$.*

Question 12.41. *Let X be the bouquet of n circles. What is $\pi_1(X)$?*

Exercise 12.42. *Find a path-connected space X with open, path-connected subsets U and V of X such that $X = U \cup V$ such that U and V are both simply connected, but X is not simply connected. Conclude that the hypothesis that $U \cap V$ is path connected is necessary.*

Lemma 12.43. *Let $X = U \cup V$, where U and V are open and $U \cap V$ is path connected, and let $p \in U \cap V$. Then any element of $\pi_1(X, p)$ has a representative $\alpha_1\beta_1\alpha_2\beta_2 \cdots \alpha_n\beta_n$, where each α_i is a loop in U based at p and each β_i is a loop in V based at p.*

Theorem 12.44. *Let X be a wedge of two cones over two Hawaiian earrings, where they are identified at the points of tangency of the circles of each Hawaiian earring, as in Figure* **??**. *Then $\pi_1(X) \ncong 1$.*

Theorem 12.45. *Let $X = U \cup V$ where U, V are open, path connected, and simply connected and $U \cap V$ is nonempty and path connected. Then X is simply connected.*

Theorem 12.46. *Let $X = U \cup V$ where U, V are open and path connected and $U \cap V$ is path connected, $x \in U \cap V$, and $\pi_1(U, x) \cong 1$. Let $i : U \cap V \to V$ be the inclusion map. Then*

$$\pi_1(X, x) \cong \frac{\pi_1(V, x)}{N}$$

where N is the smallest normal subgroup of $\pi_1(V, x)$ containing the subgroup $i_*(\pi_1(U \cap V, x))$.

Theorem 12.47 (Van Kampen's Theorem)**.** *Let $X = U \cup V$ where U, V are open and path connected and $U \cap V$ is path connected and $x \in U \cap V$. Let $i : U \cap V \to U$ and $j : U \cap V \to V$ be the inclusion maps. Then*

$$\pi_1(X, x) \cong \frac{\pi_1(U, x) * \pi_1(V, x)}{N}$$

where N is the smallest normal subgroup containing $\{i_*(\alpha)j_*(\alpha^{-1})\}_{\alpha \in \pi_1(U \cap V, x)}$ (so N contains elements created by taking a finite sequence of products and conjugates starting with elements of the form $i_*(\alpha)j_*(\alpha^{-1})$).

Theorem 12.48 (Van Kampen's Theorem; group presentations version)**.** *Let $X = U \cup V$ where U, V are open and path connected and $U \cap V$ is path connected and $x \in U \cap V$. Let $i : U \cap V \to U$ and $j : U \cap V \to V$ be the inclusion maps. Suppose $\pi_1(U, x) = \langle g_1, \ldots, g_n | r_1, \ldots, r_m \rangle$, $\pi_1(V, x) = \langle h_1, \ldots, h_t | s_1, \ldots, s_u \rangle$ and $\pi_1(U \cap V, x) = \langle k_1, \ldots, k_v | t_1, \ldots, t_w \rangle$ then*

$$\begin{aligned} \pi_1(X, x) &= \langle g_1, \ldots, g_n, h_1, \ldots, h_t \mid \quad r_1, \ldots, r_m, s_1, \ldots, s_u, \\ &\qquad\qquad\qquad\qquad\qquad i_*(k_1) = j_*(k_1), \ldots, i_*(k_v) = j_*(k_v) \rangle . \end{aligned}$$

Exercise 12.49. *Let P be a polygonal representation of a compact, connected 2-manifold such that the all the vertices of P are identified in the corresponding quotient. Give a presentation for $\pi_1(P)$.*

Figure 12.1: The unknot.

Exercise 12.50. *Give presentations of the fundamental groups for our canonical polygonal presentations of* $\mathop{\#}\limits_{i=1}^{n} \mathbb{T}^2$ *and* $\mathop{\#}\limits_{i=1}^{n} \mathbb{RP}^2$.

Theorem 12.51. *Each 2-manifold in the following infinite list is topologically different from all the others on the list:* $\mathbb{S}^2$, $\mathop{\#}\limits_{i=1}^{n} \mathbb{RP}^2$, *and* $\mathop{\#}\limits_{i=1}^{n} \mathbb{T}^2$.

Theorem 12.52. *Suppose G is a finitely presented group. Then there exists a 2-complex (K,T) such that $\pi_1(K) \cong G$.*

Theorem 12.53. *The fundamental group of the Hawaiian earring is not finitely generated. In fact, it is not countably generated.*

Lemma 12.54. *If $p, q \in \mathbb{N}$ are relatively prime, the line described in the above process will eventually intersect the upper-right vertex of the square. Moreover, the line will not intersect itself until it does.*

Lemma 12.55. *Let p and q be relatively prime integers and let $\rho_{p,q}$ be the simple closed curve constructed above. Then there is a homeomorphism of the square (with the standard identifications made for the torus) that takes $\rho_{p,q}$ to our canonical meridian.*

Theorem 12.56. *For $p, q \in \mathbb{Z}$ relatively prime, the lens space $L(p,q)$ is triangulable.*

Exercise 12.57. *Let $p, q \in \mathbb{Z}$ be relatively prime. Calculate the fundamental group of the Lens space $L(p,q)$.*

Lemma 12.58. *Every loop in M_K is homotopic in M_K to a product of a_i's. In other words, the loops $\{a_i\}$ generate $\pi_1(M_K)$.*

Lemma 12.59. *At every crossing, such as that illustrated in Figure* **??**, *the following relation holds: $acb^{-1} = c$ or $acb^{-1}c^{-1} = 1$.*

Theorem 12.60. *Let K be a knot in $\mathbb{S}^3$ and let $\{a_i\}$ be the set of loops consisting of one loop for each arc in a knot projection of K as described above. Then $\pi_1(M_K) = \{a_1, a_2, \ldots, a_n | a_i a_j a_k^{-1} a_j^{-1}$ where there is one relation of the form $a_i a_j a_k^{-1} a_j^{-1}$ for each crossing in the knot projection$\}$.*

Exercise 12.61. *Find the fundamental group of the complement of the unknot (See Figure 12.1).*

Exercise 12.62. *Find the fundamental group of the complement of the trefoil knot.*

Exercise 12.63. *Find the fundamental group of the complement of the figure-8 knot, shown in Figure* **??**.

Exercise 12.64. *The collection of homotopy classes of continuous maps of the type $f : (\mathbb{D}^n, \partial\mathbb{D}^n) \to (X, x_0)$, with the product defined above, forms a group.*

Theorem 12.65. *Homotopy equivalent spaces have the same homotopy groups.*

Chapter 13

Covering Spaces: Layering It On

Theorem 13.1. *Let $(\widetilde{X}, p)$ be a covering space of X. If $x, y \in X$, then $|p^{-1}(x)| = |p^{-1}(y)|$.*

Exercise 13.2.

1. *Describe two non-homeomorphic 2-fold covers of the Klein bottle.*
2. *Describe all non-homeomorphic 2-fold covers of the figure eight.*
3. *Describe all non-homeomorphic 3-fold covers of the figure eight.*

Theorem 13.3. *Let $(\mathbb{R}^1, \omega)$ be the standard wrapping map covering of $\mathbb{S}^1$. Then any path $f : [0,1] \to \mathbb{S}^1$ has a lift $\widetilde{f} : [0,1] \to \mathbb{R}^1$.*

Theorem 13.4. *If $(\widetilde{X}, p)$ is a cover of X, Y is connected, and $\widetilde{f}, \widetilde{g} : Y \to \widetilde{X}$ are continuous functions such that $p \circ \widetilde{f} = p \circ \widetilde{g}$, then $\{y \mid \widetilde{f}(y) = \widetilde{g}(y)\}$ is empty or all of Y.*

Theorem 13.5. *Let $(\widetilde{X}, p)$ be a cover of X and let f be a path in X. Then for each $x_0 \in \widetilde{X}$ such that $p(x_0) = f(0)$, there exists a unique lift $\widetilde{f}$ of f satisfying $\widetilde{f}(0) = x_0$.*

Exercise 13.6. *Let p be a k-fold covering of $\mathbb{S}^1$ by itself and α a loop in $\mathbb{S}^1$ which when lifted to $\mathbb{R}^1$ by the standard lift satisfies $\widetilde{\alpha}(0) = 0$ and $\widetilde{\alpha}(1) = n$. What are the conditions on n under which α will lift to a loop?*

Theorem 13.7 (Homotopy Lifting Lemma)**.** *Let $(\widetilde{X}, p)$ be a cover of X and α, β be two paths in X. If $\widetilde{\alpha}$, $\widetilde{\beta}$ are lifts of α, β satisfying $\widetilde{\alpha}(0) = \widetilde{\beta}(0)$, then $\widetilde{\alpha} \sim \widetilde{\beta}$ if and only if $\alpha \sim \beta$.*

Theorem 13.8. *If $(\widetilde{X}, p)$ is a cover of X, then p_* is a monomorphism (i.e. an injective homomorphism) from $\pi_1(\widetilde{X})$ into $\pi_1(X)$.*

Theorem 13.9. *Let $(\widetilde{X}, p)$ be a cover of X, α a loop in X, and $\widetilde{x}_0 \in \widetilde{X}$ such that $p(\widetilde{x}_0) = \alpha(0)$. Then α lifts to a loop based at $\widetilde{x}_0$ if and only if $[\alpha] \in p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

Figure 13.1: A covering of the figure eight.

Exercise 13.10. *Recast a proof of the fact that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ using the language of covering spaces.*

Theorem 13.11. *Let $(\widetilde{X}, p)$ be a covering space of X and let $x_0 \in X$. Fix $\widetilde{x}_0 \in p^{-1}(x_0)$. Then a subgroup H of $\pi_1(X, x_0)$ is in $\{p_*(\pi_1(\widetilde{X}, \widetilde{x}))\}_{p(\widetilde{x})=x_0}$ if and only if H is a conjugate of $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

Theorem 13.12. *Let $(\widetilde{X}, p)$ be a covering space of X. Choose $x \in X$, then $|p^{-1}(x)| = [\pi_1(X) : p_*(\pi_1(\widetilde{X}))]$, where the equation has the obvious interpretation if either side is infinite.*

Exercise 13.13. *Give a covering space of $\mathbb{S}^1$ that corresponds to a subgroup of index 3. If p is the covering map, describe p_*.*

Theorem 13.14. *Let $(\widetilde{X}, p)$ be a covering space of X and $\widetilde{x}_0 \in \widetilde{X}$, $x_0 \in X$ with $p(\widetilde{x}_0) = x_0$. Also let $f : Y \to X$ be continuous where Y is connected and locally path connected and $y_0 \in Y$ such that $f(y_0) = x_0$. Then there is a lift $\widetilde{f} : Y \to \widetilde{X}$ such that $p \circ \widetilde{f} = f$ and $f(y_0) = \widetilde{x}_0$ if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Furthermore, $\widetilde{f}$ is unique.*

Exercise 13.15. *Let $X = \mathbb{S}^1$, $\widetilde{X} = \mathbb{R}$, $(\widetilde{X}, \omega)$ be the covering space of X given by the standard wrapping map, and Y as in Figure* **??**. *When does a map $f : Y \to X$ not have a lift? Why is this example here?*

Exercise 13.16. *Show that $\pi_2(\mathbb{T}^2) = 0$, i.e., every map of a sphere $\mathbb{S}^2$ into $\mathbb{T}^2$ is null homotopic.*

Theorem 13.17. *Let $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ be covering spaces of X. Let $\widetilde{x}_1 \in \widetilde{X}_1$ and $\widetilde{x}_2 \in \widetilde{X}_2$ such that $p_1(\widetilde{x}_1) = p_2(\widetilde{x}_2)$. Then there is a cover isomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ with $f(\widetilde{x}_1) = \widetilde{x}_2$ if and only if $p_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.*

Exercise 13.18. *What is $\mathcal{C}(\widetilde{X}, p)$ for the covering space of the figure eight shown in Figure 13.1?*

Theorem 13.19. *If $(\widetilde{X}, p)$ is a covering space of X and $f \in \mathcal{C}(\widetilde{X}, p)$, then $f = Id_{\widetilde{X}}$ if and only if f has a fixed point.*

Exercise 13.20. *Consider the second three-fold covering space of the figure eight discussed in Exercise 13.18. Find an element of $p_*(\pi_1(\widetilde{X}))$ which, when conjugated, is not in $p_*(\pi_1(\widetilde{X}))$. Conclude that the covering space is not regular.*

Theorem 13.21. *If $(\widetilde{X}, p)$ is a regular covering space of X and $x_1, x_2 \in \widetilde{X}$ such that $p(x_1) = p(x_2)$, then there exists a unique $h \in \mathcal{C}(\widetilde{X}, p)$ such that $h(x_1) = x_2$.*

Exercise 13.22. *Do such covering transformations necessarily exist in irregular covering spaces?*

Theorem 13.23. *A covering space is regular if and only if for every loop in the base space either all its lifts are loops or all its lifts are paths that are not loops.*

Exercise 13.24. *Find a covering space $p : \widetilde{X} \to X$ and generators $e_1, \ldots, e_n$ of $\pi_1(X)$ such that each e_i satisfies the criteria of the previous theorem but the cover is not regular.*

Exercise 13.25.

1. *Describe all regular 3-fold covering spaces of the figure eight.*
2. *Describe all irregular 3-fold covering spaces of the figure eight.*
3. *Describe all regular 3-fold covering spaces of the bouquet of 3 circles.*

Theorem 13.26. *Let $(\widetilde{X}, p)$ be a regular covering space of X. Then $\mathcal{C}(\widetilde{X}, p) \cong \pi_1(X)/p_*(\pi_1(\widetilde{X}))$. In particular, $\mathcal{C}(\widetilde{X}, p) \cong \pi_1(X)$ if $\widetilde{X}$ is simply connected.*

Exercise 13.27. *Observe that the standard wrapping map is a regular covering map of $\mathbb{S}^1$ by $\mathbb{R}^1$. Describe the covering transformations for this covering space. Describe the covering map that maps $\mathbb{R}^2$ to the torus $\mathbb{T}^2$ and describe the covering transformations for this covering space.*

Theorem 13.28 (Existence of covering spaces)**.** *Let X be connected, locally path connected, and semi-locally simply connected. Then for every $G < \pi_1(X, x_0)$ there is a covering space $(\widetilde{X}, p)$ of X and $\widetilde{x}_0 \in \widetilde{X}$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = G$. Furthermore, $(\widetilde{X}, p)$ is unique up to isomorphism.*

Corollary 13.29. *Let X be connected, locally path connected, and semi-locally simply connected. Then there is a one-to-one correspondence between the subgroups of $\pi_1(X)$ and the collection of isomorphism classes of covering spaces of X where the covering space $p : \tilde{X} \to X$ corresponds to $p_*(\pi_1(\widetilde{X}))$.*

Corollary 13.30. *Every connected, locally path connected, semi-locally simply connected space admits a unique universal covering space.*

Exercise 13.31. *Find a universal cover $\widetilde{X}$ for each of the Klein bottle, the torus, and the projective plane. In each case, show explicitly that $\mathcal{C}(\widetilde{X}, p) \cong \pi_1(X)$.*

Theorem 13.32. *A finite tree is contractible.*

Theorem 13.33. *Let G be a finite graph, and T be a maximal tree in G. Then if $\{e_1, \ldots e_n\}$ is the set of edges that are not in T, $\pi_1(G) = F_n$, the free group on n generators; and there is a system of generators that are in one-to-one correspondence with the edges $\{e_1, \ldots e_n\}$.*

Lemma 13.34. *Let X be the bouquet of n circles. Every finite cover of X is homeomorphic to a finite graph.*

Theorem 13.35. *Let F_n be the free group on n letters. Then every subgroup of F_n of finite index is isomorphic to a free group on finitely many letters.*

Lemma 13.36. *Suppose that G is a graph and that $K \subset G$ is compact. Then K is contained in a finite subgraph of G.*

Theorem 13.37. *Every tree is simply connected.*

Theorem 13.38. *Let G be a graph, then $\pi_1(G)$ is free.*

Exercise 13.39. *Show that the free group of rank 2 has finite index subgroups that are isomorphic to free groups of arbitrarily large rank.*

Lemma 13.40. *Let X be the bouquet of n circles. Then every cover of X is homeomorphic to a graph.*

Corollary 13.41 (Nielsen-Schreier Theorem)**.** *Let F_n be the free group on n letters. A subgroup of F_n is always free.*

Exercise 13.42. *Describe a regular k-fold cover $\widetilde{X}$ of a bouquet of n-circles. What (in terms of k and n) is the rank of the free group $\pi_1(\widetilde{X})$? What does this insight tell us about the normal subgroups of finite index of the free group on n letters?*

Exercise 13.43.

1. *Let F be a free group on n letters. Let $G < F$ be of finite index k and contain 7 free generators. What can the value of n be?*

2. *Let F be a free group on n letters. Let $G < F$ be of finite index k and contain 4 free generators. What can the value of n be?*

3. *Let F be a free group on n letters. Let $G < F$ be of finite index k and contain 24 free generators. What can the value of n be?*

Theorem 13.44. *Let F be a 2-manifold and $(\widetilde{F}, p)$ be a covering space of F. Then $\widetilde{F}$ is a 2-manifold.*

Theorem 13.45. *Let F be a compact connected surface and $p_n : \widetilde{F} \to F$ be an n-fold covering of F (for $n < \infty$). Then $\widetilde{F}$ is a compact surface and $\chi(\widetilde{F}) = n\chi(F)$. Moreover, if F is orientable, then $\widetilde{F}$ is as well.*

Exercise 13.46.

1. *Describe all non-homeomorphic 3-fold covers of the Klein bottle.*

2. *Describe all non-homeomorphic 2-fold covers of $\mathbb{T}^2 \# \mathbb{T}^2$.*

3. *Describe all non-homeomorphic 3-fold covers of* $\mathbb{T}^2 \# \mathbb{T}^2 \# \mathbb{T}^2$.

4. *Describe all non-homeomorphic 3-fold covers of* $\mathbb{R}\mathrm{P}^2$.

Exercise 13.47. *Given a compact, connected 2-manifold and a natural number* n*, describe all non-homeomorphic* n*-fold covers of that surface.*

Chapter 14

Manifolds, Simplexes, Complexes, and Triangulability: Building Blocks

Exercise 14.1. *Show that the standard n-ball and the standard n-cube are homeomorphic spaces and each is compact and connected.*

Exercise 14.2. *Show that for $n \geq 1$, the n-sphere is compact and connected.*

Exercise 14.3. *Consider $\mathbb{S}^0$, $\mathbb{S}^1$, and $\mathbb{S}^2$. Is any pair of them homeomorphic? If not, are there properties that allow you to distinguish them?*

Theorem 14.4. *For a separable, metric space M^n, the following are equivalent:*

1. *M^n is an n-manifold;*
2. *for each point $p \in M^n$, p has a neighborhood base of open sets each homeomorphic to the interior of an n-ball;*
3. *for every point $p \in M^n$, $p \in U$ where U is an open set homeomorphic to $\mathbb{R}^n$.*

Exercise 14.5. *If you are comfortable with ordinal numbers, construct a topological space where every point has an open set containing it that is homeomorphic to $\mathbb{R}^1$, and yet the space is not metrizable. You might call your space the **long line**.*

Exercise 14.6. *Show that a locally Euclidean space is Hausdorff and second countable if and only if it is separable and metrizable.*

Exercise 14.7. *Show that $\mathbb{S}^n$ is an n-manifold.*

Theorem 14.8. *If M is an n-manifold and U is an open subset of M, then U is also an n-manifold.*

Theorem 14.9. *If M is an m-manifold and N is an n-manifold, then $M \times N$ is an $(m+n)$-manifold.*

Theorem 14.10. *Let M^n be an n-dimensional manifold with boundary. Then ∂M^n is an $(n-1)$-manifold.*

Exercise 14.11. *Show that if σ is a simplex and τ is one of its faces, then $\tau \subset \sigma$.*

Exercise 14.12. *Show that an n-simplex is homeomorphic to a closed n-dimensional ball.*

Exercise 14.13. *Exhibit a collection of simplices that satisfies condition (1) but not (2) in the definition of a simplicial complex.*

Exercise 14.14. *Let K be the simplicial complex in $\mathbb{R}^2$:*

$$K = \{\sigma, e_1, e_2, e_3, e_4, e_5, v_1, v_2, v_3, v_4\}$$

where $\sigma = \{(0,0)(0,1)(1,0)\}$, $e_1 = \{(0,0)(0,-1)\}$, $e_2 = \{(0,-1)(1,0)\}$, $e_3 = \{(0,0)(0,1)\}$, $e_4 = \{(0,1)(1,0)\}$, $e_5 = \{(1,0)(0,0)\}$, $v_1 = \{(0,0)\}$, $v_2 = \{(0,1)\}$, $v_3 = \{(1,0)\}$, and $v_4 = \{(0,-1)\}$. Draw K and its underlying space.

Exercise 14.15. *Show that the space shown in Figure* **??** *is triangulable by exhibiting a simplicial complex whose underlying space it is homeomorphic to.*

Exercise 14.16. *For each n, $\mathbb{S}^n$ is triangulable.*

Theorem 14.17. *A simplicial map from K to L is determined by the images of the vertices of K.*

Theorem 14.18. *A composition of simplicial maps is a simplicial map.*

Theorem 14.19. *If two complexes are simplicially homeomorphic, then there are 1-1 correspondences between their k-simplices for each $k \geq 0$.*

Theorem 14.20. *A simplicial map $f : K \to L$ is continuous as a map on the underlying spaces. In particular, simplicially homeomorphic complexes have homeomorphic underlying spaces.*

Theorem 14.21. *A composition of PL maps is PL. A PL homeomorphism is an equivalence relation.*

Theorem 14.22. *PL homeomorphic complexes are homeomorphic as topological spaces.*

Exercise 14.23. *Let K is a complex consisting of the boundary of a triangle (three vertices and three edges) and L be an isomorphic complex. Both $|K|$ are $|L|$ are topologically circles. There is a continuous map that takes the circle $|K|$ and winds it twice around the circle $|L|$; however, show that there is no simplicial map from K to L that winds the circle $|K|$ twice around the circle $|L|$.*

Exercise 14.24. *How many n-simplices are there in the first barycentric subdivision of an n-simplex?*

Exercise 14.25. *Convince yourself that the barycentric subdivision of a complex K is, in fact, a subdivision of K.*

Theorem 14.26. *Let K be a finite simplicial complex and let a_n be the maximum among the diameters of simplices in $\mathsf{sd}^n K$. Then $\lim_{n\to\infty} a_n = 0$.*

Exercise 14.27. *The star of a vertex v in a complex K is an open set of $|K|$, and the collection of all vertex stars covers $|K|$.*

Exercise 14.28. *If the simplex $\sigma = \{v_0, \dots, v_k\}$ in K is the minimal face of a point $x \in |K|$, then $x \in \mathrm{St}(v_0) \cap \cdots \cap \mathrm{St}(v_k)$.*

Theorem 14.29. *Suppose K and L are simplicial complexes. Then a continuous function $f : |K| \to |L|$ satisfies the star condition with respect to K and L if and only if f has a simplicial approximation $g : K \to L$.*

Theorem 14.30. *If $g, g' : K \to L$ are both simplicial approximations to a continuous function $f : |K| \to |L|$, then for any point $x \in |K|$, if σ is the minimal face of x in K, the point $f(x)$ and the simplices $g(\sigma)$ and $g'(\sigma)$ all lie in a single simplex of L.*

Theorem 14.31. *Let K and L be simplicial complexes. If $f : |K| \to |L|$ has a simplicial approximation $g : K \to L$, then f is homotopic to $g : |K| \to |L|$.*

Theorem 14.32. *Suppose K and L are finite simplicial complexes and $f : |K| \to |L|$ is a continuous function between their underlying spaces. Then there exists $m \geq 1$ such that the function $f : |\mathsf{sd}^m K| \to |L|$ satisfies the star condition with respect to $\mathsf{sd}^m K$ and L.*

Theorem 14.33. *Suppose K and L are simplicial complexes and $f : |K| \to |L|$ is a continuous function between their underlying spaces. Then there exists $m \geq 1$ such that f has a simplicial approximation $g : \mathsf{sd}^m K \to L$.*

Theorem 14.34 (Simplicial Approximation Theorem)**.** *Let K and L be simplicial complexes, let $f : |K| \to |L|$ be a continuous function between their underlying spaces, and let $\epsilon > 0$. Then there exist $m, n \geq 1$ and a simplicial map $g : \mathsf{sd}^n K \to \mathsf{sd}^m L$ such that f is homotopic to g and for every $x \in |K|$, $d(f(x), g(x)) < \epsilon$.*

Theorem 14.35. *Let K be a subdivision of a 1-simplex σ. Label every vertex of K with a 0 or a 1 such that one of the two vertices of σ is labeled with a 0 and the other is labeled with a 1. Then there is a 1-simplex τ in K such that one vertex of τ is labeled 0 and the other vertex of τ is labeled 1.*

Theorem 14.36. *Let K be a subdivision of a 2-simplex σ. Label every vertex of K with 0, 1, or 2 such that the three vertices of σ are labeled with different numbers. Then there is a 2-simplex τ in K such that its vertices are labeled with all different numbers.*

Theorem 14.37 (Sperner's Lemma)**.** *Let K be a subdivision of a n-simplex σ. Label every vertex of K with one of $\{0, 1, \ldots, n\}$ such that the $(n+1)$ vertices of σ are labeled with different numbers. Then there is an n-simplex τ in K such that its vertices are labeled with all different numbers.*

Theorem 14.38. *Let σ be an n-simplex with boundary $\partial\sigma$. There does not exist a continuous function $r : \sigma \to \partial\sigma$ such that for every $x \in \partial\sigma$, $r(x) = x$.*

Theorem 14.39 (n-dimensional Brouwer Fixed Point Theorem)**.** *Let σ^n be an n-simplex. For every continuous function $f : \sigma^n \to \sigma^n$ there exists a point $x \in \sigma^n$ such that $f(x) = x$.*

Theorem 14.40. *Let $h : \mathbb{S}^1 \to \mathbb{R}^2$ be an embedding that is a polygon, that is, $h(\mathbb{S}^1)$ consists of a finite number of straight line intervals. Then $h(\mathbb{S}^1)$ separates $\mathbb{R}^2$ into two components and each point of $h(\mathbb{S}^1)$ is a limit point of each component.*

Theorem 14.41. *Let $h : \mathbb{S}^1 \to \mathbb{R}^2$ be an embedding that is a polygon, that is, $h(\mathbb{S}^1)$ consists of a finite number of straight line intervals. Then there is a homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H(h(S^1))$ is the unit circle.*

Theorem 14.42. *Let A and B be disjoint closed subsets of $[0,1] \times [0,1]$ such that $A \cap ([0,1] \times \{0,1\} \cup \{1\} \times [0,1]) = \varnothing$ and $B \cap ([0,1] \times \{0,1\} \cup \{0\} \times [0,1]) = \varnothing$. Then there exists a path in $[0,1] \times [0,1]$ from $(1/2, 0)$ to $(1/2, 1)$ that misses $A \cup B$.*

Theorem 14.43. *Suppose $h : [0,1] \to \mathbb{R}^2$ is an embedding and suppose p and q are points in $\mathbb{R}^2$ not contained in $h([0,1])$. Then there exists a path $f : [0,1] \to \mathbb{R}^2$ such that $f(0) = p$, $f(1) = q$, and $f([0,1]) \cap h([0,1]) = \varnothing$.*

Theorem 14.44. *Suppose $g, h : [0,1] \to \mathbb{R}^2$ are embeddings such that $g([0,1])$ is a straight line segment, $g(0) = h(0)$, $g(1) = h(1)$, and $g((0,1)) \cap h((0,1)) = \varnothing$. Then $g([0,1]) \cup h([0,1])$ separates $\mathbb{R}^2$ into two components and each point of $g([0,1]) \cup h([0,1])$ is a limit point of each component.*

Theorem 14.45. *Let $h : \mathbb{S}^1 \to \mathbb{R}^2$ be an embedding. Then $h(\mathbb{S}^1)$ separates $\mathbb{R}^2$ into two components and each point of $h(\mathbb{S}^1)$ is a limit point of each component.*

Exercise 14.46.

1. *Let C_0 be a disk with two holes. Construct a subset C_1 of C_0 such that C_1 is also homeomorphic to a disk with two holes, and for which each point $x \in C_1$ is within distance 1 of points in each of the three components of $\mathbb{R}^2 - C_1$.*

2. *Construct a continuum $C \subset \mathbb{R}^2$ such that $\mathbb{R}^2 - C$ has three components and each point $x \in C$ is a limit point of each component of $\mathbb{R}^2 - C$.*

3. *Construct a continuum $C \subset \mathbb{R}^2$ such that $\mathbb{R}^2 - C$ has infinitely many components and each point $x \in C$ is a limit point of each component of $\mathbb{R}^2 - C$.*

Theorem 14.47. *Let $h : \mathbb{S}^1 \to \mathbb{R}^2$ be an embedding. Let p be a point in the bounded component of $\mathbb{R}^2 - h(\mathbb{S}^1)$ and let $\epsilon > 0$. Then there exists an embedding $g : \mathbb{S}^1 \to \mathbb{R}^2$ such that $g(\mathbb{S}^1)$ is a polygonal simple closed curve in the bounded component of $\mathbb{R}^2 - h(\mathbb{S}^1)$, $g(\mathbb{S}^1)$ lies in the ϵ-neighborhood of $h(\mathbb{S}^1)$, and p is in the bounded component of $\mathbb{R}^2 - g(\mathbb{S}^1)$.*

Theorem 14.48. *Let $h : \mathbb{S}^1 \to \mathbb{R}^2$ be an embedding. Let $\epsilon > 0$ and let A be an arc on $\mathbb{S}^1$ with endpoints a and b such that the diameter of $h(A)$ is less than ϵ and let p be a point in the bounded component of $\mathbb{R}^2 - h(\mathbb{S}^1)$. Then there exists an embedding $g : \mathbb{S}^1 \to \mathbb{R}^2$ such that $g(\mathbb{S}^1)$ is a polygonal simple closed curve in the bounded component of $\mathbb{R}^2 - h(\mathbb{S}^1)$, p is in the bounded component of $\mathbb{R}^2 - g(\mathbb{S}^1)$, and for every $x \in A$, $d(g(x), h(x)) < \epsilon$.*

Theorem 14.49. *Let $h : \mathbb{S}^1 \to \mathbb{R}^2$ be an embedding, let U be the bounded component of $\mathbb{R}^2 - h(\mathbb{S}^1)$, and let D be the closed unit ball in $\mathbb{R}^2$. Then there is a homeomorphism $H : (U \cup h(\mathbb{S}^1) \to D$.*

Theorem 14.50. *Let $h : \mathbb{S}^1 \to \mathbb{R}^2$ be an embedding. Then there is a homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H(h(\mathbb{S}^1))$ is the unit circle.*

Theorem 14.51. *Let $h : [0,1] \to \mathbb{R}^2$ be an embedding of $[0,1]$ in the plane and let $\epsilon > 0$. Then there exists an embedding $g : [0,1] \to \mathbb{R}^2$ such that $h(0) = g(0)$, $h(1) = g(1)$, and for every $x \in [0,1]$, $d(h(x), g(x)) < \epsilon$.*

Theorem 14.52. *Let $f, g : [0,1] \to \mathbb{R}^2$ be two embeddings of $[0,1]$ in the plane such that $f(0) = g(0)$ and $f(1) = g(1)$. Let $\epsilon > 0$. Suppose for every $x \in [0,1]$, $d(f(x), g(x)) < \epsilon$. Then there exists a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that for every $t \in [0,1]$, $h(f(t)) = g(t)$ and for every $x \in \mathbb{R}^2$, $d(x, h(x)) < \epsilon$.*

Theorem 14.53. *Every compact 2-manifold is triangulable, that is, it is homeomorphic to a subset C of $\mathbb{R}^n$ consisting of a finite collection $T = \{\sigma_i\}_{i=1}^k$ of (rectilinear) 2-simplices where each pair of 2-simplices are disjoint or they meet in one vertex of each or they share a single edge. Since the space C is homeomorphic to a 2-manifold, each edge of each 2-simplex making up C is shared by exactly two triangles, and around each vertex is a circle of triangles whose union is a disk.*

Theorem 14.54. *Every 2-manifold is triangulable.*

Lemma 14.55. *Let M_1 and M_2 be two rectilinearly triangulated 2-manifolds in $\mathbb{R}^n$. Let $h : M_1 \to M_2$ be a topological homeomorphism. Then there exists a homeomorphism $g : M_1 \to M_2$ such that the image of every edge in the triangulation of M_1 is a polyhedral arc in M_2.*

Theorem 14.56. *Let M_1 and M_2 be two rectilinearly triangulated 2-manifolds in $\mathbb{R}^n$. Let $h : M_1 \to M_2$ be a topological homeomorphism. Then there exists a homeomorphism $g : M_1 \to M_2$ such that the image of every triangle in a triangulation of M_1 is a rectilinear triangle in M_2.*

Theorem 14.57. *Any two triangulations of a compact 2-manifold are equivalent.*

Theorem 14.58. *The Euler characteristic is well-defined for compact 2-manifolds.*

Theorem 14.59. *Orientability is well-defined for compact 2-manifolds.*

Chapter 15

Simplicial $\mathbb{Z}_2$-Homology: Physical Algebra

Exercise 15.1. *Check that* $\mathsf{C}_n(K)$ *is an abelian group.*

Exercise 15.2. *Verify that* ∂ *is a homomorphism, and use the definition to compute the* $\mathbb{Z}_2$*-boundary of* $\sigma_1 + \sigma_2$ *in Figure* **??**.

Exercise 15.3. *Explore:*

1. *Which* 2*-chains of Figure* **??** *are cycles?*
2. *Which* 1*-chains of Figure* **??** *are cycles?*
3. *Which* 1*-chains of Figure* **??** *are boundaries?*
4. *Which* 0*-chains of Figure* **??** *are cycles?*
5. *Which* 0*-chains of Figure* **??** *are boundaries?*

Theorem 15.4. *Both* $\mathsf{Z}_n(K)$ *and* $\mathsf{B}_n(K)$ *are subgroups of* $\mathsf{C}_n(K)$*. Moreover,*

$$\partial \circ \partial = 0,$$

in other words, $\partial_n \circ \partial_{n+1} = 0$ *for each index* $n \geq 0$*. Hence,* $\mathsf{B}_n(K) \subset \mathsf{Z}_n(K)$*.*

Exercise 15.5. *List all the equivalence classes of* 0*-cycles,* 1*-cycles and* 2*-cycles in the complex in Figure* **??**.

Exercise 15.6. *List all the equivalence classes of* 0*-cycles,* 1*-cycles and* 2*-cycles in a triangulated 2-sphere with its standard triangulation as the faces of a 3-simplex.*

Theorem 15.7. *If* K *is a one-point space,* $\mathsf{H}_n(K) \cong 0$ *for* $n \geq 0$ *and* $\mathsf{H}_0(K) \cong \mathbb{Z}$*.*

Theorem 15.8. *If K is connected, then $\mathsf{H}_0(K)$ is isomorphic to $\mathbb{Z}_2$. If K has r connected components, then $\mathsf{H}_0(K)$ is isomorphic to $\mathbb{Z}_2^r$.*

Exercise 15.9. *Let K be a triangulation of a 3-dimensional ball that consists of a 3-simplex together with its faces. Compute $\mathsf{H}_n(K)$ for each n.*

Exercise 15.10. *Let K be a triangulation of a 2-sphere that consists of the proper faces of a 3-simplex. Compute $\mathsf{H}_n(K)$ for each n.*

Theorem 15.11. *For x seeing K, and σ a simplex of K,*

$$\partial\, \mathsf{Cone}_x(\sigma) + \mathsf{Cone}_x(\partial\sigma) = \sigma.$$

Corollary 15.12. *For any complex K and x seeing K, the complex $x * K$ is acyclic.*

Theorem 15.13. *The complex K consisting of an n-simplex together with all its faces is acyclic.*

Exercise 15.14. *Let $f : K \to L$ be a simplicial map. Carefully write out the definition of the natural induced map from n-chains of K to n-chains of L: $f_{\#n} : \mathsf{C}_n(K) \to \mathsf{C}_n(L)$ and show that it is a homomorphism.*

Exercise 15.15. *If the simplicial map $f : K \to L$ maps an n-simplex σ to an $(n-1)$-simplex τ, what is $f_{\#n}(\sigma)$?*

Theorem 15.16. *Let $f : K \to L$ be a simplicial map, and let $f_\#$ be the induced map $f_\# : \mathsf{C}_n(K) \to \mathsf{C}_n(L)$. Then for any chain $c \in \mathsf{C}_n(K)$,*

$$\partial(f_\#(c)) = f_\#(\partial(c)).$$

In other words, the diagram:

$$\begin{array}{ccc} \mathsf{C}_n(K) & \xrightarrow{f_\#} & \mathsf{C}_n(L) \\ \partial\big\downarrow & & \big\downarrow\partial \\ \mathsf{C}_{n-1}(K) & \xrightarrow{f_\#} & \mathsf{C}_{n-1}(L) \end{array}$$

commutes.

Theorem 15.17. *Let $f : K \to L$ be a simplicial map. Then the induced homomorphism $f_* : \mathsf{H}_n(K) \to \mathsf{H}_n(L)$ is a well-defined homomorphism.*

Exercise 15.18. *Let K be a complex comprising the proper faces of a hexagon: six edges and six vertices $v_0, \ldots, v_5$. Let L be the complex comprising the proper faces of a triangle: three edges and three vertices w_0, w_1, w_2. Let f be a simplicial map that sends v_i to $w_{(i \mod 3)}$. Compute the homology groups of K and L and describe the simplicial map f and the induced homomorphism f_*.*

Exercise 15.19. *Suggest a homomorphism from* $\mathsf{C}_n(K)$ *to* $\mathsf{C}_n(\mathsf{sd}\, K)$ *that commutes with* ∂*. Could its induced homomorphism on homology be an inverse for* λ_**?*

Theorem 15.20. *The subdivision operator commutes with the boundary operator, that is, if* c *is a chain in* K*, then* $\mathsf{SD}(\partial c) = \partial\, \mathsf{SD}(c)$*.*

Exercise 15.21. *Show that* $\lambda_\# \circ \mathsf{SD} = \mathsf{id}$*, the identity map on* $C_n(K)$*, and therefore* $\lambda_* \circ \mathsf{SD}_* = \mathsf{id}_*$*, the identity map on* $H_n(K)$*.*

Exercise 15.22. *Show that* $\mathsf{SD} \circ \lambda_\#$ *and* id*, the identity map on* $C_n(\mathsf{sd}\, K)$*, induce the same homomorphism on homology.*

Exercise 15.23. *If* $\sigma \in \mathsf{sd}\, K$ *is contained in* $\tau \in K$*, then* $\mathsf{SD} \circ \lambda_\#(\sigma)$ *and* $\mathsf{id}(\sigma)$ *both lie inside* τ*.*

Theorem 15.24. *Let* K *be a simplicial complex. Then* $\mathsf{H}_n(K)$ *is isomorphic to* $\mathsf{H}_n(\mathsf{sd}\, K)$*. In fact, if the simplicial map* $\lambda : \mathsf{sd}\, K \to K$ *is defined by taking each vertex in* $\mathsf{sd}\, K$ *to any vertex of the simplex in* K *of which it is the barycenter, then*

$$\lambda_* : \mathsf{H}_n(K) \to \mathsf{H}_n(\mathsf{sd}\, K)$$

is an isomorphism. Also, the induced homomorphism of the subdivision operator

$$\mathsf{SD}_* : \mathsf{H}_n(K) \to \mathsf{H}_n(\mathsf{sd}\, K)$$

is an isomorphism and is the inverse of λ_**.*

Theorem 15.25. *Let* $\mathsf{sd}^\ell K$ *and* $\mathsf{sd}^m K$ *be barycentric subdivisions of* K*. Suppose* $g : \mathsf{sd}^\ell K \to L$ *and* $h : \mathsf{sd}^m K \to L$ *are simplicial approximations to a continuous function* $f : |K| \to |L|$*. Then* $g_* \circ \mathsf{SD}_*^\ell : \mathsf{H}_n(K) \to \mathsf{H}_n(L)$ *is the same homomorphism as* $h_* \circ \mathsf{SD}_*^m$*.*

Lemma 15.26. *If* K*,* L*, and* M *are simplicial complexes and* $f : |K| \to |L|$ *and* $g : |L| \to |M|$ *are continuous maps, then* $(g \circ f)_* = g_* \circ f_*$*.*

Lemma 15.27. *If* $i : |K| \to |K|$ *is the identity map, then* i_* *is the identity homomorphism on each homology group.*

Theorem 15.28. *Let* K *and* L *be simplicial complexes. If* $f : |K| \to |L|$ *is a homeomorphism, then* f *induces an isomorphism between the* $\mathbb{Z}_2$*-homology groups of* K *and* L*.*

Theorem 15.29. *Let* K *and* L *be simplicial complexes. If* $f : |K| \to |L|$ *is a homotopy equivalence, then* f *induces an isomorphism between the* $\mathbb{Z}_2$*-homology groups of* K *and* L*.*

Corollary 15.30. *If* K *is a strong deformation retract of* L*. Then* K *and* L *have isomorphic* $\mathbb{Z}_2$*-homologies.*

Exercise 15.31. *If K is a finite simplicial complex, verify that the intersection of two subcomplexes of K is a subcomplex.*

Exercise 15.32. *Note that a cycle in $A \cap B$ is still a cycle in A, B, and K. Then answer:*

1. *Can a trivial cycle in $A \cap B$ be non-trivial in A?*
2. *Can a non-trivial cycle in $A \cap B$ be trivial in A?*
3. *Can a non-trivial cycle in $A \cap B$ that's also non-trivial in A and in B be trivial in K?*

Theorem 15.33. *Let K be a finite simplicial complex and A and B be subcomplexes such that $K = A \cup B$. If α, β are k-cycles in A and B respectively, and if $\alpha \sim_{\mathbb{Z}_2} \beta$ in K, then there is a k-cycle c in $A \cap B$ such that $\alpha \sim_{\mathbb{Z}_2} c$ in A and $\beta \sim_{\mathbb{Z}_2} c$ in B.*

Theorem 15.34. *Let K be a finite simplicial complex and A and B be subcomplexes such that $K = A \cup B$. Let z be a k-cycle in K. Then there exist k-chains α and β in A and B respectively such that:*

1. *$z = \alpha + \beta$ and*
2. *$\partial\alpha = \partial\beta$ is a $(n-1)$-cycle c in $A \cap B$.*

Furthermore, if $z = \alpha' + \beta'$, a sum of n-chains in A and B respectively, and $c' = \partial\alpha' = \partial\beta'$ is a $(n-1)$-cycle, then c' is homologous to c in $A \cap B$.

Exercise 15.35. *Let K be a simplicial complex and A and B be subcomplexes such that $K = A \cup B$. Construct natural homomorphisms ϕ, ψ, δ among the groups below and show that $\psi \circ \phi = 0$ and $\delta \circ \psi = 0$.*

1. *$\phi : \mathsf{H}_n(A \cap B) \to \mathsf{H}_n(A) \oplus \mathsf{H}_n(B)$.*
2. *$\psi : \mathsf{H}_n(A) \oplus \mathsf{H}_n(B) \to \mathsf{H}_n(K)$.*
3. *$\delta : \mathsf{H}_n(K) \to \mathsf{H}_{n-1}(A \cap B)$.*

Theorem 15.36 ($\mathbb{Z}_2$ **Mayer-Vietoris).** *Let K be a finite simplicial complex and A and B be subcomplexes such that $K = A \cup B$. The sequence*

$$\cdots \to \mathsf{H}_n(A \cap B) \to \mathsf{H}_n(A) \oplus \mathsf{H}_n(B) \to \mathsf{H}_n(K) \to \mathsf{H}_{n-1}(A \cap B) \to \dots$$

using the homomorphisms ϕ, ψ, δ above, is exact.

Exercise 15.37. *Let C, D, E be groups, and arrows represent homomorphisms below.*

1. *$0 \to C \xrightarrow{\phi} D$ is exact at C if and only if ϕ is one-to-one.*

2. $D \overset{\psi}{\to} E \to 0$ *is exact at* E *if and only if* ψ *is onto.*

3. $0 \to C \overset{\phi}{\to} D \to 0$ *is exact if and only if* ϕ *is an isomorphism.*

Exercise 15.38. *Compute the* $\mathbb{Z}_2$*-homology groups for each complex* K *below.*

1. *The bouquet of* k *circles (the union of* k *circles identified at a point).*

2. *A wedge of a* 2*-sphere and a circle (the two spaces are glued at one point).*

3. *A* 2*-sphere union its equatorial disk.*

4. *A double solid torus.*

Exercise 15.39. *Compute the* $\mathbb{Z}_2$*-homology groups of a torus using Mayer-Vietoris in two different ways (with two different decompositions).*

Exercise 15.40. *Use the Mayer-Vietoris Theorem to compute* $\mathsf{H}_n(M)$ *for every compact, triangulated* 2*-manifold* M*. What compact, triangulated* 2*-manifolds are not distinguished from one another by* $\mathbb{Z}_2$*-homology? What does* $\mathsf{H}_2(M)$ *tell you?*

Exercise 15.41. *Let* $p, q \in \mathbb{Z}$ *be relatively prime. Calculate* $\mathsf{H}_n(L(p,q))$*, the homology of the lens space* $L(p,q)$*.*

Exercise 15.42. *Use the Mayer-Vietoris Theorem to compute* $\mathsf{H}_n(K)$ *for the complexes* K *pictured in Figure 15.1.*

Figure 15.1: Two interesting spaces.

Exercise 15.43. *Use the Mayer-Vietoris Theorem to find the* Z_2*-homology groups for each of the following spaces.*

1. $\mathbb{S}^n$*.*

2. *A cone over a finite simplicial complex* K*.*

3. *A suspension over a finite simplicial complex* K *(that is, the finite simplicial complex created by gluing two cones over* K *along* K*).*

4. $\mathbb{R}\mathrm{P}^n$ *(which is* $\mathbb{S}^n$ *with antipodal points identified).*

Theorem 15.44. *Let K be a simplicial complex where $T = \{\sigma_i\}_{i=1}^k$. Then*

$$K = \bigsqcup_{i=1}^{k} \mathrm{Int}(\sigma_i),$$

where $\sqcup$ denotes disjoint union.

Exercise 15.45. *Let K be a 3-simplex with triangulation shown (a tetrahedron). Find an open cell decomposition of K with one vertex, one open 2-cell, and one open 3-cell. This example shows that it is not necessary to have every dimension less then the dimension of K represented.*

Theorem 15.46. *Both $\mathsf{Z}_n^c(K^c)$ and $\mathsf{B}_n^c(K^c)$ are subgroups of $\mathsf{C}_n^c(K^c)$. Moreover, $\mathsf{B}_n^c(K^c) \subset \mathsf{Z}_n^c(K^c)$.*

Theorem 15.47. *Let K^c be an open cell decomposition of the finite simplicial complex K. Then for each n, the obvious homomorphism $\mathsf{H}_n^c(K^c) \to \mathsf{H}_n(K)$ is an isomorphism.*

Exercise 15.48. *For each space below, describe a triangulation K and an open cell decomposition K^c. Then use cellular homology to compute $\mathsf{H}_n(K)$ for each n:*

1. *The sphere.*
2. *The torus.*
3. *The projective plane.*
4. *The Klein bottle.*
5. *The double torus.*
6. *Any compact, connected, triangulated 2-manifold.*
7. *The Möbius band.*
8. *The annulus.*
9. *Two (hollow) triangles joined at a vertex.*

Exercise 15.49. *What is $\mathsf{H}_n(\mathbb{S}^k)$ for $n = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$?*

Exercise 15.50. *What is $\mathsf{H}_n(\mathbb{T})$ for $n = 0, 1, 2, \ldots$ for a solid torus $\mathbb{T}$?*

Chapter 16

Applications of $\mathbb{Z}_2$-Homology: A Topological Superhero

Theorem 16.1 (No Retraction Theorem). *Let M^n be a connected triangulated n-manifold with $\partial M^n \neq \varnothing$. Then there is no retraction $r : M^n \to \partial M^n$, i.e., no continuous function $r : M^n \to \partial M^n$ such that for each $x \in \partial M^n$, $r(x) = x$.*

Theorem 16.2 (n-dimensional Brouwer Fixed Point Theorem). *Let $\mathcal{B}^n$ be the n-dimensional ball. For every continuous function $f : \mathcal{B}^n \to \mathcal{B}^n$ there exists a point $x \in \mathcal{B}^n$ such that $f(x) = x$.*

Lemma 16.3. *Let M^n be a triangulated, connected n-manifold. Let $f : M^n \to M^n$ be a simplicial map. Then $f_* : \mathsf{H}_n(M^n) \to \mathsf{H}_n(M^n)$ is surjective if and only if $f_\#(M^n) = M^n$.*

Theorem 16.4. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an antipode preserving continuous map (that is, for every $x \in \mathbb{S}^1$, $f(-x) = -f(x)$. Then $f_* : \mathsf{H}_1(\mathbb{S}^1) \to \mathsf{H}_1(\mathbb{S}^1)$ is surjective.*

Theorem 16.5 (Borsuk-Ulam Theorem for $\mathbb{S}^2$). *Let $f : \mathbb{S}^2 \to \mathbb{R}^2$ be a continuous map. Then there exists an $x \in \mathbb{S}^2$ such that $f(-x) = f(x)$.*

Theorem 16.6. *Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be an antipode preserving map (that is, for every $x \in \mathbb{S}^n$, $f(-x) = -f(x)$). Then $f_* : \mathsf{H}_n(\mathbb{S}^n) \to \mathsf{H}_n(\mathbb{S}^n)$ is surjective.*

Theorem 16.7 (Borsuk-Ulam). *Let $f : \mathbb{S}^n \to \mathbb{R}^n$ be a continuous function. Then there is an $x \in \mathbb{S}^n$ such that $f(-x) = f(x)$.*

Theorem 16.8 (Ham Sandwich Theorem). *Let $A_1, A_2, ..., A_n$ be measurable sets of finite measure in $\mathbb{R}^n$. Then there exists an $(n-1)$-dimensional hyperplane H in $\mathbb{R}^n$ that simultaneously cuts each A_i in half.*

Exercise 16.9. *Draw a sequence of pictures to demonstrate a sequence of elastic moves with no tricks, no cutting, and no gluing that takes the left hand picture of Figure* **??** *and turns it into the right hand picture.*

Theorem 16.10. *If $m \neq n$ then $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$.*

Theorem 16.11 (Invariance of Domain or Invariance of Dimension). *A space cannot be both an n-manifold and an m-manifold if $n \neq m$.*

Exercise 16.12. *Describe a continuous function from $[0,1]$ to $[0,1]$ that is nowhere differentiable.*

Theorem 16.13. *Let B be the set of all nowhere differentiable continuous functions from $[0,1]$ to $[0,1]$. Then B is the intersection of countably many dense open sets in $\mathcal{C}$.*

Exercise 16.14. *Describe an embedding of $[0,1]$ into the plane that has infinite length. In fact, you might choose the graph of a differentiable function.*

Exercise 16.15. *Describe an embedding of $[0,1]$ into the unit square and two points x and y in the unit square not on the embedded arc such that to connect x to y by a polygonal path missing the embedded arc requires a polygonal path of length at least a mile.*

Lemma 16.16. *Let $h : [0,1] \to \mathbb{R}^2$ be an embedding and let p and q be points in $\mathbb{R}^2 - h([0,1])$. If p and q are connected in $\mathbb{R}^2 - h([0,\frac{1}{2}])$ and p and q are connected in $\mathbb{R}^2 - h([\frac{1}{2},1])$, then p and q are connected in $\mathbb{R}^2 - h([0,1])$.*

Theorem 16.17. *Let $h : [0,1] \to \mathbb{R}^2$ be an embedding. Then $h([0,1])$ does not separate $\mathbb{R}^2$.*

Lemma 16.18. *For any natural number n, let $h : [0,1] \to \mathbb{R}^n$ be an embedding and let p and q be points in $\mathbb{R}^n - h([0,1])$. If p and q are connected in $\mathbb{R}^n - h([0,\frac{1}{2}])$ and p and q are connected in $\mathbb{R}^n - h([\frac{1}{2},1])$, then p and q are connected in $\mathbb{R}^n - h([0,1])$.*

Theorem 16.19. *For any natural number n, let $h : [0,1] \to \mathbb{R}^n$ be an embedding. Then $h([0,1])$ does not separate $\mathbb{R}^n$.*

Lemma 16.20. *For any natural number n, let $h : [0,1] \to \mathbb{R}^n$ be an embedding and let Z be a $\mathbb{Z}_2$ 1-cycle in $\mathbb{R}^n - h([0,1])$. If Z bounds a 2-chain in $\mathbb{R}^n - h([0,\frac{1}{2}])$ and Z bounds a 2-chain in $\mathbb{R}^n - h([\frac{1}{2},1])$, then Z bounds a 2-chain in $\mathbb{R}^n - h([0,1])$.*

Theorem 16.21. *For any natural number n, let $h : [0,1] \to \mathbb{R}^n$ be an embedding and let Z be a $\mathbb{Z}_2$ 1-cycle in $\mathbb{R}^n - h([0,1])$. Then Z bounds a 2-chain in $\mathbb{R}^n - h([0,1])$.*

Theorem 16.22. *For any natural numbers n and k with $k < n$, let $h : [0,1] \to \mathbb{S}^n$ be an embedding and let Z be a $\mathbb{Z}_2$ k-cycle in $\mathbb{S}^n - h([0,1])$. Then Z bounds a $(k+1)$-chain in $\mathbb{S}^n - h([0,1])$.*

Theorem 16.23 (Jordan-Brouwer Separation Theorem). *Let $h : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a topological embedding. Then $h(\mathbb{S}^{n-1})$ separates $\mathbb{S}^n$ into precisely two components and is the boundary of each.*

Theorem 16.24 (Two Chains Theorem). *Let $f : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a simplicial map. Then there exist $\mathbb{Z}_2$ n-chains A^n and C^n such that $\partial(A^n) = \partial(C^n) = f_{\#}(\mathbb{S}^{n-1})$ and $A^n \cup C^n = \mathbb{S}^n$.*

Theorem 16.25 (Absolute Neighborhood Retract Theorem). *For every k, $\mathbb{S}^k$ is an absolute neighborhood retract.*

Theorem 16.26. *Let $h : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a topological embedding and let U be a neighborhood of $h : \mathbb{S}^{n-1}$ that retracts to it. Then $U \neq \mathbb{S}^n$.*

Corollary 16.27. *Let $h : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a topological embedding and let U be a neighborhood of $h : \mathbb{S}^{n-1}$ with retraction $r : U \to h(\mathbb{S}^{n-1})$. Then there exists an open set $V \subset U$ such that for every point $v \in V$, the straight line between v and $r(v)$ is contained in U.*

Lemma 16.28. *Let $h : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a topological embedding. Then there exists an $\epsilon > 0$ such that if $f : \mathbb{S}^{n-1} \to \mathbb{S}^n$ is a simplicial map such that $d(f(x), h(x)) < \epsilon$ for all $x \in \mathbb{S}^{n-1}$, then $f_{\#}(\mathbb{S}^{n-1})$ does not bound an n-chain in the ϵ-neighborhood of $h(\mathbb{S}^{n-1})$.*

Lemma 16.29. *Let $h : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a topological embedding and let $\epsilon > 0$. Then there exists a $\delta > 0$ such that if $f, g : \mathbb{S}^{n-1} \to \mathbb{S}^n$ are simplicial maps such that $d(f(x), h(x)) < \delta$ and $d(g(x), h(x)) < \delta$ for all $x \in \mathbb{S}^{n-1}$, then $f_{\#}(\mathbb{S}^{n-1})$ and $g_{\#}(\mathbb{S}^{n-1})$ bound an n-chain in the ϵ-neighborhood of $h(\mathbb{S}^{n-1})$.*

Lemma 16.30. *Let $h : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a topological embedding, U be a neighborhood that retracts to $h(\mathbb{S}^{n-1})$, let $f : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a simplicial map such that for each point $x \in \mathbb{S}^{n-1}$, the straight line segment between $h(x)$ and $f(x)$ lies in U. Let A^n and C^n be $\mathbb{Z}_2$ n-chains such that $\partial(A^n) = \partial(C^n) = f_{\#}(\mathbb{S}^{n-1})$ and $A^n \cup C^n = \mathbb{S}^n$. Then there exists a point $a \in (A - U)$ and a point $c \in (C - U)$.*

Lemma 16.31. *Let $h : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a topological embedding, U be a neighborhood that retracts to $h(\mathbb{S}^{n-1})$, let $f, g : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be simplicial maps such that for each point $x \in \mathbb{S}^{n-1}$, the straight line segments between $h(x)$ and $f(x)$ and $h(x)$ and $g(x)$ lie in U. Let A^n_f and C^n_f be $\mathbb{Z}_2$ n-chains (from the Two Chains Theorem) such that $\partial(A^n_f) = \partial(C^n_f) = f_{\#}(\mathbb{S}^{n-1})$ and $A^n_f \cup C^n_f = \mathbb{S}^n$ and let $a \in (A - U)$ and $c \in (C - U)$. Let A^n_g and C^n_g be $\mathbb{Z}_2$ n-chains such that $\partial(A^n_g) = \partial(C^n_g) = g_{\#}(\mathbb{S}^{n-1})$ and $A^n_g \cup C^n_g = \mathbb{S}^n$ where $a \in A_g$. Then $c \notin A_g$.*

Lemma 16.32. *Let $h : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a topological embedding, U be a neighborhood such that there is a retract $r : U \to h(\mathbb{S}^{n-1}$, and V be an open set in U such that for each $x \in V$, the straight line segment from x to $r(x)$ is in U. Let $a \notin U$ and let T be a standard triangulation of $\mathbb{S}^n$ with simplexes so small that for any simplex $\sigma \in T$, if $\sigma \cap (\mathbb{S}^n - V) \neq \varnothing$, then $\sigma \cap h(\mathbb{S}^{n-1} = \varnothing$. Let τ_0 be an n-simplex in T such that $a \in \tau_0$. Let A be the union of all n-simplexes τ_k in T such that there are n-simplexes $\{\tau_i\}_{i=0,\cdots,k}$ such that (1) each τ_i contains a point in $\mathbb{S}^n - V$, and (2) for each i, τ_i and τ_{i+1} share an $(n-1)$-face. Then $\partial A \subset V$ and $h^{-1}(r(\partial A))$ is the non-trivial element of $H_{n-1}(\mathbb{S}^{n-1})$. Also, $r(\partial A) = h(\mathbb{S}^{n-1})$.*

Lemma 16.33. *Let $h : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a topological embedding, let $\{U_i\}_{i \in \mathbb{N}}$ be open sets each containing $h(\mathbb{S}^{n-1})$ and each contained in the $\frac{1}{n}$-neighborhood of $h(\mathbb{S}^{n-1})$ with retract $r : U_1 \to h(\mathbb{S}^{n-1})$ and such that for every $i \in \mathbb{N}$ and every point $x \in U_{i+1}$, the straight line homotopy between x and $r(x)$ lies in U_i. Let $\{T_i\}_{i \in \mathbb{N}}$ be a sequence of triangulations of $\mathbb{S}^n$ where each triangulation T_{i+1} is a subdivision of T_i with simplices so small that any simplex of T_i that intersects $h(\mathbb{S}^{n-1})$ lies entirely in U_i. Let $a \in (\mathbb{S}^n - U_1)$. Let A_i be the component containing a of the union of all n-simplices of T_i that miss $h(\mathbb{S}^{n-1})$. Then $\partial(A_{i+1})$ and $\partial(A_{i+2})$ co-bound an n-chain in U_i, $\cup_{i \in \mathbb{N}} A_i \cap h(\mathbb{S}^{n-1}) =$, each point $x \in h(\mathbb{S}^{n-1})$ is a limit point of $\cup_{i \in \mathbb{N}} A_i$, there exists a point c in $(\mathbb{S}_n - \cup_{i \in \mathbb{N}} A_i - U_1)$, and if we do the same process that we did for a for c creating C_i's, then $\mathbb{S}_n = (\cup_{i \in \mathbb{N}} A_i) \cup h(\mathbb{S}^{n-1}) \cup (\cup_{i \in \mathbb{N}} C_i)$.*

Theorem 16.34 (Jordan-Brouwer Separation Theorem)**.** *Let $h : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a topological embedding. Then $h(\mathbb{S}^{n-1})$ separates $\mathbb{S}^n$ into precisely two components and is the boundary of each.*

Corollary 16.35 (Jordan-Brouwer Separation Theorem)**.** *Let $h : \mathbb{S}^{n-1} \to \mathbb{R}^n$ be a topological embedding. Then $h(\mathbb{S}^{n-1})$ separates $\mathbb{R}^n$ into precisely two components and is the boundary of each. The two components are distinguished topologically by the fact that one has a compact closure and the other does not.*

Theorem 16.36. *Every connected, compact topologically embedded $(n-1)$-manifold in $\mathbb{R}^n$ separates $\mathbb{R}^n$ into two components and is the topological boundary of each.*

Theorem 16.37. *The Klein Bottle cannot be embedded in $\mathbb{R}^3$.*

Chapter 17

Simplicial $\mathbb{Z}$-Homology: Getting Oriented

Exercise 17.1. *Check that this boundary map is well-defined: it does not depend on the oriented representative chosen for the definition.*

Exercise 17.2. *Find the boundary of the oriented 2-simplex $\tau = [v_0 v_1 v_2]$ and the boundary of the oriented 3-simplex $\sigma = [w_0 w_1 w_2 w_3]$. Repeat the procedure for $-\tau$ and $-\sigma$. What is the relationship between the boundary of τ and the boundary of $-\tau$? What is the relationship between the boundary of σ and the boundary of $-\sigma$?*

Theorem 17.3. *For any n-simplex σ*

$$\partial(-\sigma) = -\partial(\sigma).$$

Theorem 17.4. *For all $n \geq 0$,*

$$\partial_n \circ \partial_{n+1} = 0.$$

Theorem 17.5. *For any simplicial complex K, both $Z_n(K)$ and $B_n(K)$ are subgroups of $C_n(K)$, and $B_n(K) \subset Z_n(K)$.*

Exercise 17.6. *Create a triangulation of a Möbius band such that the central circle forms a 1-cycle γ. Show that the Möbius band's boundary 1-cycle α is equivalent to either 2γ or -2γ (depending on the orientation you give the two cycles).*

Theorem 17.7. *For a finite simplicial complex K, $H_n(K)$ is a finitely generated abelian group.*

Theorem 17.8. *If K is a connected simplicial complex, then $H_0(K)$ is isomorphic to $\mathbb{Z}$. If K has r connected components, then $H_0(K)$ is a free abelian group of rank r.*

Theorem 17.9. *If K is a one-point space, $\mathsf{H}_n(K) \cong 0$ for $n \geq 0$ and $\mathsf{H}_0(K) \cong \mathbb{Z}$.*

Theorem 17.10. *Let x see a complex K, and let $c \in C_n(K)$ be a chain. Then*

$$\partial \operatorname{\mathsf{Cone}}_x(c) + \operatorname{\mathsf{Cone}}_x(\partial c) = c.$$

Corollary 17.11. *For any complex K and x seeing K, the complex $x * K$ is acyclic.*

Theorem 17.12. *The complex K consisting of an n-simplex together with all its faces is acyclic.*

Theorem 17.13. *Let $f : K \to L$ be a simplicial map, and let $f_\#$ be the induced map $f_\# : C_n(K) \to C_n(L)$. Then for any chain $c \in C_n(K)$, $\partial(f_\#(c)) = f_\#(\partial(c))$. In other words, the diagram:*

$$\begin{array}{ccc} C_n(K) & \xrightarrow{f_\#} & C_n(L) \\ \partial \big\downarrow & & \big\downarrow \partial \\ C_{n-1}(K) & \xrightarrow{f_\#} & C_{n-1}(L) \end{array}$$

commutes.

Theorem 17.14. *Let $f : K \to L$ be a simplicial map. Then the induced homomorphism $f_* : H_n(K) \to H_n(L)$ is a well-defined homomorphism.*

Exercise 17.15. *Check that $C_n(K, K')$ is a free abelian group.*

Theorem 17.16. *There is a boundary map*

$$\partial_n : C_n(K, K') \to C_{n-1}(K, K')$$

such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \geq 0$.

Exercise 17.17. *Check that if $K' = \varnothing$, the empty set, then $H_n(K, K') = H_n(K)$ for all n, the usual homology groups.*

Exercise 17.18. *Show that $\tilde{H}_n(K) \cong H_n(K)$ for $n > 0$ and $H_0(K) \cong \tilde{H}_0(K) \oplus \mathbb{Z}$.*

Exercise 17.19. *Let K be the complex consisting of a triangle and all its faces. Determine $H_n(K, K')$ for all $n \geq 0$.*

Exercise 17.20. *Let K be a triangulation of an annulus, and let K' be the subcomplex consisting of the inner and outer edges of the annulus. Find a relative 1-cycle in $C_1(K, K')$ that is not a relative 1-boundary.*

Exercise 17.21. *Let K be a triangulation of a Möbius band, and let K' be its boundary. Determine $H_n(K, K')$ for $n \geq 0$.*

Theorem 17.22 (Excision). *Suppose K' is a subcomplex of K. Remove an open set U from K' such that what remains is a subcomplex L' of K', and remove U from K so that what remains is a subcomplex L of K. Then*

$$H_n(L, L') \cong H_n(K, K').$$

Theorem 17.23. *Given a simplicial map $f : (K, K') \to (L, L')$ there is an associated chain map $f_\# : C_n(K, K') \to C_n(L, L')$ and induced homomorphism $f_* : H_n(K, K') \to H_n(L, L')$.*

Exercise 17.24. *There are natural maps between chain groups:*

$$C_n(K') \xrightarrow{i} C_n(K) \xrightarrow{\pi} C_n(K, K')$$

What are the maps i and π, and what do you notice about them and their relationship with each other?

Theorem 17.25. *The boundary map $\partial : C_n(K) \to C_{n-1}(K')$ induces a well-defined map*

$$\partial_* : H_n(K, K') \to H_{n-1}(K').$$

Theorem 17.26 (Long Exact Sequence of a Pair). *If K' is a subcomplex of a simplicial complex K, then there is a long exact sequence:*

$$\cdots \xrightarrow{\partial_*} H_n(K') \xrightarrow{i_*} H_n(K) \xrightarrow{\pi_*} H_n(K, K') \xrightarrow{\partial_*} H_{n-1}(K') \xrightarrow{i_*} \cdots$$

Theorem 17.27 (Zig-Zag Lemma). *Suppose $\mathcal{C} = \{C_n, \partial_n^C\}, \mathcal{D} = \{D_n, \partial_n^D\}, \mathcal{E} = \{E_n, \partial_n^E\}$ are chain complexes, and $\phi : \mathcal{C} \to \mathcal{D}$ and $\psi : \mathcal{D} \to \mathcal{E}$ are chain maps such that*

$$0 \longrightarrow \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \longrightarrow 0$$

is a short exact sequence of chain complexes. Then there is a long exact sequence:

$$\cdots \xrightarrow{\partial_*} H_n(\mathcal{C}) \xrightarrow{\phi_*} H_n(\mathcal{D}) \xrightarrow{\psi_*} H_n(\mathcal{E}) \xrightarrow{\partial_*} H_{n-1}(\mathcal{C}) \xrightarrow{i_*} \cdots$$

where ∂_* is induced by ∂^D.

Theorem 17.28. *Given the commutative diagram of chain maps α, β, γ between the chain complexes of two short exact sequences:*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C} \arrow[r, "i"] \arrow[d, "\alpha"'] & \mathcal{D} \arrow[r, "\pi"] \arrow[d, "\beta"'] & \mathcal{E} \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & \mathcal{C}' \arrow[r, "i"] & \mathcal{D}' \arrow[r, "\pi"] & \mathcal{E}' \arrow[r] & 0
\end{tikzcd}$$

there are corresponding induced homomorphisms between the associated long exact sequences, such that the following diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_*"] & H_n(\mathscr{C}) \arrow[r, "\phi_*"] \arrow[d, "\alpha_*"'] & H_n(\mathscr{D}) \arrow[r, "\psi_*"] \arrow[d, "\beta_*"'] & H_n(\mathscr{E}) \arrow[r, "\partial_*"] \arrow[d, "\gamma_*"'] & H_{n-1}(\mathscr{C}) \arrow[r, "i_*"] \arrow[d, "\alpha_*"'] & \cdots \\
\cdots \arrow[r, "\partial_*"] & H_n(\mathscr{C}') \arrow[r, "\phi_*"] & H_n(\mathscr{D}') \arrow[r, "\psi_*"] & H_n(\mathscr{E}') \arrow[r, "\partial_*"] & H_{n-1}(\mathscr{C}') \arrow[r, "i_*"] & \cdots
\end{tikzcd}$$

Lemma 17.29 (The Five Lemma)**.** *Consider the following commutative diagram of groups and homomorphisms, where the rows are exact.*

$$\begin{tikzcd}
A \arrow[r, "q"] \arrow[d, "\alpha"'] & B \arrow[r, "r"] \arrow[d, "\beta"'] & C \arrow[r, "s"] \arrow[d, "\gamma"'] & D \arrow[r, "t"] \arrow[d, "\delta"'] & E \arrow[d, "\epsilon"'] \\
A' \arrow[r, "q'"] & B' \arrow[r, "r'"] & C' \arrow[r, "s'"] & D' \arrow[r, "t'"] & E'
\end{tikzcd}$$

If the rows are exact and $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is also an isomorphism.

Exercise 17.30. *In the proof of the Five Lemma, not all of $\alpha, \beta, \delta, \epsilon$ are required to be isomorphisms for the conclusion to still hold. Which isomorphisms can be relaxed?*

Exercise 17.31 (The Snake Lemma)**.** *Consider the following commutative diagram where the rows are short exact sequences.*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

Show there is an exact sequence

$$0 \to \operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma \to 0$$

where Coker *stands for the cokernel of a homomorphism: the quotient of its codomain by its image.*

Corollary 17.32 (Long Exact Sequence of a Pair)**.** *If K' is a subcomplex of a simplicial complex K, then there is a long exact sequence:*

$$\cdots \xrightarrow{\partial_*} H_n(K') \xrightarrow{i_*} H_n(K) \xrightarrow{\pi_*} H_n(K,K') \xrightarrow{\partial_*} H_{n-1}(K') \xrightarrow{i_*} \cdots$$

where the maps are induced by the inclusion maps $i : K' \to K$ and $\pi : (K, \varnothing) \to (K, K')$ and the boundary map $\partial : C_n(X) \to C_{n-1}(X)$.

Theorem 17.33. *Given a simplicial map $f : (K, K') \to (L, L')$, there is chain map between the long exact sequences:*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_*"] & H_n(K') \arrow[r, "i_*"] \arrow[d, "f_*"'] & H_n(K) \arrow[r, "\pi_*"] \arrow[d, "f_*"'] & H_n(K,K') \arrow[r, "\partial_*"] \arrow[d, "f_*"'] & H_{n-1}(K') \arrow[r, "i_*"] \arrow[d, "f_*"'] & \cdots \\
\cdots \arrow[r, "\partial_*"] & H_n(L') \arrow[r, "i_*"] & H_n(L) \arrow[r, "\pi_*"] & H_n(L,L') \arrow[r, "\partial_*"] & H_{n-1}(L') \arrow[r, "i_*"] & \cdots
\end{tikzcd}$$

Theorem 17.34 (Mayer-Vietoris). *Let K be a finite simplicial complex and A and B be subcomplexes such that $K = A \cup B$. Then there is a long exact sequence*

$$\cdots \xrightarrow{\delta} H_n(A \cap B) \xrightarrow{\phi_*} H_n(A) \oplus H_n(B) \xrightarrow{\psi_*} H_n(K) \xrightarrow{\partial_*} H_{n-1}(A \cap B) \xrightarrow{\phi} \cdots$$

Exercise 17.35. *Compute the $\mathbb{Z}$-homology of the Klein bottle, and compare it to the the $\mathbb{Z}_2$-homology of the Klein bottle.*

Exercise 17.36. *Compute the $\mathbb{Z}$-homology of every compact, triangulated 2-manifold.*

Theorem 17.37. *Let K be a finite, connected simplicial complex. Then*

$$H_1(K; \mathbb{Z}) \simeq (\pi_1(K))/[\pi_1(K), \pi_1(K)],$$

that is, the first homology group of K is isomorphic to the abelianization of the fundamental group of K.

Lemma 17.38. *If $f : \mathbb{S}^n \to \mathbb{S}^n$ is continuous, then* $\deg f$ *is well-defined. That is, it does not depend on the way in which we identify $H_n(\mathbb{S}^n)$ with $\mathbb{Z}$.*

Theorem 17.39. *Let $f, g : \mathbb{S}^n \to \mathbb{S}^n$ be continuous maps.*

1. *If f and g are homotopic, they have the same degree.*
2. $\deg(f \circ g) = (\deg f) \cdot (\deg g)$

Theorem 17.40. *The identity map on $\mathbb{S}^n$ has degree 1. The antipodal map has degree $(-1)^{n+1}$.*

Theorem 17.41 (Hairy Ball Theorem). *There exists a non-vanishing vector field on $\mathbb{S}^n$ if and only if n is odd.*

Exercise 17.42. *In the definition above and using a little linear algebra, show that* $\mathsf{Tr}(h_{free})$ *does not depend on the choice of basis for G_{free}.*

Exercise 17.43. *Construct a simple example of a map homotopic to the identity map on the triangulated circle whose induced chain map does not have the same trace as the identity chain map.*

Theorem 17.44. *Suppose $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated abelian groups and $f_A : A \to A$ and $f_B : B \to B$ are homomorphisms such that $i \circ f_A = f_B \circ i$. Then there is an induced homomorphism $f_C : C \to C$ that makes the following diagram commutative:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \\
 & & {\scriptstyle f_A}\downarrow & & {\scriptstyle f_B}\downarrow & & {\scriptstyle f_C}\downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{\pi} & C & \longrightarrow & 0
\end{array}$$

Moreover,

$$\mathsf{Tr}(f_B) = \mathsf{Tr}(f_A) + \mathsf{Tr}(f_C)$$

Theorem 17.45. *(The Hopf Trace Formula) Let K be a finite simplicial complex and let $f : K \to K$ be a simplicial map. Then*

$$\sum(-1)^i \,\mathsf{Tr}(f_{\#n}) = \sum(-1)^i \,\mathsf{Tr}(f_{*n}).$$

Theorem 17.46. *(Lefschetz Fixed Point Theorem) Let $f : |K| \to |K|$ be a continuous map on a simplicial complex K. If $\Lambda(f) \neq 0$, then f has a fixed point.*

Exercise 17.47. *Compute the Lefschetz number of the "mirror-reversing" self-map of a circle: $g : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $g(x, y) = (-x, y)$ viewing $\mathbb{S}^1$ as a subset of $\mathbb{R}^2$. Argue that any reversing map of a circle must have a fixed point.*

Chapter 18

Singular Homology: Abstracting Objects to Maps

Theorem 18.1 (Eilenberg-Steenrod)**.** *Fix a group G. Any two homology theories on compact triangulable pairs with coefficient group G are isomorphic.*

Exercise 18.2. *Let $X \subset \mathbb{E}^\infty$ be star-convex with respect to a point x. Verify that $\Phi_0^{n+1} \circ \mathrm{Cone}_x$ is the identity map on $S_n(X)$.*

Theorem 18.3. *For all $n \geq 0$,*

$$\partial_n \circ \partial_{n+1} = 0.$$

Hence $\mathrm{Im}\, \partial_{n+1} \subset \mathrm{Ker}\, \partial_n$.

Theorem 18.4 (Dimension Axiom)**.** *If P is a point, $H_n(P) \cong 0$ for all $n > 0$, and $H_0(P) \cong \mathbb{Z}$.*

Theorem 18.5. *Let $X \subset \mathbb{E}^\infty$ be star convex with respect to $x \in X$. For any singular n-simplex σ,*

$$\partial_{n+1}(\mathrm{Cone}_x\, \sigma) + \mathrm{Cone}_x(\partial_n \sigma) = \sigma.$$

Theorem 18.6. *Show that any star-convex space is acyclic.*

Theorem 18.7. *For a space X, show that $H_0(X)$ is a free abelian group with a generator for every path-connected component of X.*

Theorem 18.8. *Let $f : X \to Y$ be a continuous map. Then for any chain $c \in S_n(X)$, $\partial(f_\#(c)) = f_\#(\partial(c))$. In other words, the diagram:*

$$\begin{array}{ccc} S_n(X) & \xrightarrow{f_\#} & S_n(Y) \\ \partial \big\downarrow & & \big\downarrow \partial \\ S_{n-1}(X) & \xrightarrow{f_\#} & S_{n-1}(Y) \end{array}$$

commutes.

Exercise 18.9. *Check that the induced homomorphism is well-defined and a homomorphism.*

Theorem 18.10. *The identity map $i : X \to X$ induces the identity homomorphism on each homology group.*

Theorem 18.11. *If $f : X \to Y$ and $g : Y \to Z$ are continuous maps between topological spaces, then $(g \circ f)_* = g_* \circ f_*$*

Theorem 18.12. *If $f : X \to Y$ is a homeomorphism, then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism between singular homology groups.*

Theorem 18.13 (Homotopy Axiom). *If f and g are homotopic maps from X to Y, then they induce the same homomorphism in homology.*

Theorem 18.14. *There is a boundary map*

$$\partial : S_n(X, A) \to S_{n-1}(X, A)$$

such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \geq 0$.

Exercise 18.15. *Check that if $A = \varnothing$, the empty set, then $H_n(X, A) = H_n(X)$, the usual homology.*

Exercise 18.16. *Let X be the complex consisting of a triangle and all its faces. Determine $H_n(X, A)$ for all $n \geq 0$.*

Exercise 18.17. *Let K be a triangulation of an annulus, and let K' be the subcomplex consisting of the inner and outer edges of the annulus. Find a relative 1-cycle in $C_1(X, A)$ that is not a relative 1-boundary.*

Exercise 18.18. *Let K be a triangulation of a Möbius band, and let K' be its boundary. Determine $H_n(X, A)$ for $n \geq 0$.*

Theorem 18.19. *Given a continuous map $f : (X, A) \to (Y, B)$ there is an associated chain map $f_\# : S_n(X, A) \to S_n(Y, B)$ and induced homomorphism $f_* : H_n(X, A) \to H_n(Y, B)$.*

Theorem 18.20 (Long Exact Sequence of a Pair)**.** *Let A be a subspace of X. Then there is a long exact sequence:*

$$\cdots \xrightarrow{\partial_*} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{\pi_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} \cdots$$

where the maps are induced by the inclusion maps $i : A \to X$ and $\pi : (X, \varnothing) \to (X, A)$ and the boundary map $\partial : S_n(X) \to S_{n-1}(X)$.

Exercise 18.21. *Verify that* SD *is a chain map, commuting with* ∂*, and verify that it is natural, which means that for any continuous map* $f : X \to Y$*, the following diagram commutes.*

$$\begin{array}{ccc} S_n(X) & \xrightarrow{f_\#} & S_n(Y) \\ {\scriptstyle \mathsf{SD}}\downarrow & & \downarrow{\scriptstyle \mathsf{SD}} \\ S_n(X) & \xrightarrow{f_\#} & S_n(Y) \end{array}$$

.

Theorem 18.22. *Let* $\mathcal{U}$ *be an open cover of a space* X*. For any singular simplex* σ*, there exists an* m *such that each term of* $\mathsf{SD}^m(\sigma)$ *has an image that lies within one of the elements of* $\mathcal{U}$*.*

Theorem 18.23. *There is a chain homotopy between* SD^m *and the identity map on* $S_n(X)$*; in other words, there exists a homomorphism* $D_X : S_n(X) \to S_{n+1}(X)$ *such that*

$$\partial D_X \sigma + D_X \partial \sigma = \mathsf{SD}^m\, \sigma - \sigma$$

for every singular simplex σ *of* X*. Moreover, this chain homotopy is natural, meaning it commutes with maps of spaces: if* $f : X \to Y$*, then* $f_\# \circ D_X = D_Y \circ f_\#$*.*

Theorem 18.24. *For each* $n \geq 0$*, the induced homomorphism*

$$\mathsf{SD}^m_* : H_n(X) \to H_n(X)$$

is an isomorphism.

Theorem 18.25. *Suppose* U *and* A *are subspaces of* X *such that* $\overline{U} \subset \operatorname{Int} A$*. Then the inclusion map of* $(X - U, A - U)$ *in* (X, A) *induces an isomorphism*

$$H_n(X - U, A - U) \cong H_n(X, A).$$

Theorem 18.26. *For the* $n \geq 1$*, show that* $H_n(S^n) \cong \mathbb{Z}$ *and* $H_n(S^k) \cong 0$ *if* $n \neq k$*.*

Chapter 19

The End: A Beginning—Reflections on Topology and Learning

Appendix A

Appendix - Group Theory Background

Exercise A.1.

1. *Show that the set of all permutations on n elements forms a group with the group operation of function composition.*

2. *Show that any permutation can be written as a composition of disjoint cycles.*

3. *Show that any m-cycle can be written as a composition of transpositions.*

Exercise A.2. *What is the order of S_n?*

Exercise A.3.

1. *Show that an n-cycle can be written as the composition of $n-1$ transpositions. Thus a 3-cycle is an even permutation and a 4-cycle is an odd permutation!*

2. *Show that the group of even permutations is a subgroup of S_n.*

Exercise A.4. *What is the order of A_n?*

Exercise A.5. *Show that if we let a represent a reflection along a line passing through the polygon's center and a vertex, and b a rotation of $2\pi/n$ around its center, then*

$$D_n = \{1, b, \ldots, b^{n-1}, ab, \ldots, ab^{n-1}\}$$

Exercise A.6. *Show that in D_n as above, we have $ab = b^{n-1}a$, and thus D_n is not abelian for $n > 2$.*

Exercise A.7. *Show that D_n is isomorphic to a proper subgroup of S_n.*

Exercise A.8. *Under what conditions, if ever, is D_n is isomorphic to a subgroup of A_n?*

Exercise A.9. *Let $g, g' \in G$. Then either $gH = g'H$ or $gH \cap g'H = \varnothing$.*

Theorem A.10 (Lagrange's Theorem)**.** *Let G be a finite group, and H a subgroup. Then the cardinality $|H|$ of H divides the cardinality $|G|$of G and*

$$[G : H] = \frac{|G|}{|H|}$$

Theorem A.11. *Let $H \lhd G$ be a normal subgroup. Then its left and right cosets coincide for all $g \in G$, in other words $gH = Hg$ for all $g \in G$.*

Theorem A.12. *An onto homomorphism $f : G \to H$ is an isomorphism if and only if $\operatorname{Ker} f = \{1_G\}$.*

Theorem A.13. *Let $f : G \to H$ be a homomorphism from a group G to a group H, then $\operatorname{Ker} f \lhd G$ and $f(G) < H$.*

Theorem A.14 (First isomorphism theorem)**.** *Let $f : G \to H$ be a homomorphism with $\operatorname{Ker} f = N$. Then $f(H) \cong G/N$.*

Theorem A.15. *A cyclic group that is infinite is isomorphic to $\mathbb{Z}$.*

Theorem A.16. *A finite cyclic group of order n is isomorphic to $\mathbb{Z}_n$, the integers with addition $\mod n$.*

Exercise A.17.

1. *Verify that the dihedral group $D_n = \{1, b, \ldots, b^{n-1}, ab, \ldots, ab^{n-1}\}$ is generated by $\{a, b\}$.*
2. *Show that the symmetric group S_n, for $n \geq 2$, is generated by the set of 2-cycles: $\{(12), (23), \ldots, (n-1, n)\}$.*
3. *Show that the symmetric group S_n, for $n \geq 2$, is generated by the pair of cycles (12) and $(12 \ldots n)$.*

Theorem A.18 (Classification of Finitely Generated Abelian Groups)**.** *Let G be a finitely generated abelian group. Then G is isomorphic to:*

$$H_0 \oplus H_1 \oplus \ldots \oplus H_m$$

*where H_0 is a free abelian group, and $H_i \cong \mathbb{Z}_{p_i}$ ($i = 1, \ldots, n$) where p_i is a power of a prime. The rank of H_0 is unique and is called the **rank** of G. The orders $p_1, \ldots, p_m$ are also unique up to reordering.*

Theorem A.19. *$G' \lhd G$, and is the smallest subgroup for which G/G' is abelian. In other words, if there is a subgroup $N \lhd G$ such that G/N is abelian, then $G' \subset N$.*

Theorem A.20. *Isomorphic groups have isomorphic abelianizations.*

Exercise A.21. *Confirm that the lists of generators and relations given above completely determine the groups.*

Theorem A.22. *Suppose that L is a set and R is a collection of words in L. Then there is a group whose presentation is $\langle L|R\rangle$.*

Exercise A.23. *What is a group presentation for an arbitrary finitely generated abelian group? for the symmetric group?*

Theorem A.24. *For $m \neq n$, $F_m \neq F_n$.*

Lemma A.25. *We have a natural injections $G \to G * H$ and $H \to G * H$ so that G and H may be viewed as subgroups of $G * H$.*

Theorem A.26. *Let G and H be disjoint groups. Each element in $G * H$ has a unique expression of the form $g_1h_1 \cdots g_nh_n$ where $g_1, \ldots, g_n \in G$, $h_1, \ldots, h_n \in H$, and g_1 and h_n are allowed to be the identity, but no other letter is.*