MATH 172 (SU): HW NUMBER 11

1. Let $I = \langle x^2y^2, x^3y \rangle$ be an ideal in $K[x, y]$.

(a) In the (m, n)-plane, plot the set of exponent vectors (m, n) of monomials $x^m y^n$ that appear as elements of I.

(b) If we apply the division algorithm to an element $f \in K[x, y]$ using the generators of I as divisors, what terms can appear in the remainder?

2. Complete the last part of the proof of Dickson's lemma:

Let $I = \langle x^\alpha : \alpha \in A \rangle$ be a monomial ideal, and suppose there is a finite basis for I such that

$$I = \left\langle x^{\beta(1)}, x^{\beta(2)}, ..., x^{\beta(s)} \right\rangle.$$

Prove that there is a finite basis for I from among the original generating set:

$$I = \left\langle x^{\alpha(1)}, x^{\alpha(2)}, ..., x^{\alpha(s)} \right\rangle$$

where each $\alpha(i) \in A$.

3. Show that the ring $K[x_1, ..., x_n]$ is Noetherian.

(Recall that at the beginning of this class we defined a *Noetherian* ring to be a ring in which there is no infinite ascending chain of ideals: i.e., in any chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

there is an n beyond which $I_k = I_n$ for all $k \geq n$.)

Hint: use the Hilbert Basis Theorem that says that every ideal in this ring is finitely generated.

(Note: the book defines Noetherian differently: as a ring in which every ideal is finitely generated. As it turns out, the two definitions are equivalent, and in this problem you are showing one half of this equivalence: that the book's definition implies our definition of Noetherian.)