Inverse Semigroups Associated with Markov Subshifts

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The Question

Under what conditions is an inverse semigroup isomorphic to the inverse hull of a Markov subshift?

Definitions and Background Knowledge

A **semigroup** is a set S together with an associative binary operation. The semigroup we want to consider is a semigroup associated with a **Markov subshift**. Given the following Markov transition matrix:

$$T = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

we define a set of words L_T over the alphabet $A = \{a, b, c\}$. Define $L_T = \{d_1 d_2 \cdots d_n : d_i \in A \text{ and } T_{d_i, d_{i+1}} = 1 \text{ for all } i\} \cup A$. Then $L_T = \{a, b, c, aa, , ab, ac, ba, bc, cb, aaa, aab, \ldots\}$ and $L^0 = L_T \cup \{0\}$ is the semigroup associated with a Markov subshift.

Suppose $a_1 a_2 \cdots a_n$, $b_1 b_2 \cdots b_m \in L_T$. Then

$$(a_1 a_2 \cdots a_n) * (b_1 b_2 \cdots b_m) = \begin{cases} a_1 a_2 \cdots a_n b_1 b_2 \cdots b_m & \text{if } T_{a_n, b_1} = 1 \\ 0 & \text{otherwise} \end{cases}$$

An **inverse semigroup** is a semigroup H such that for all $s \in H$ there exists a unique $s^* \in H$ such that $ss^*s = s$ and $s^*ss^* = s^*$.

Let $s \in L^0$ and define $\boldsymbol{\theta_s}$ as a function such that its domain is $\{x \in L^0 : sx \neq 0\}$ and its range is $\{sx \in L^0 : sx \neq 0\}$.

The **inverse hull** of a Markov subshift, denoted $H(L^0) = \langle \theta_s : s \in L^0 \rangle$, is an example of an inverse semigroup [1]. We found that all nonzero elements of $H(L^0)$ are of the form $\theta_s \theta_{x_1}^{-1} \theta_{x_1} \ldots \theta_{x_n}^{-1} \theta_{x_n} \theta_w^{-1}$, where $s, w \in L^0 \cup \{1\}, x_i \in A$.

Consider an inverse semigroup, H where $L \subseteq H$, For all $\alpha \in H$, $D_\alpha = \{y \in L : yy* < \alpha^* \alpha\}$.$H$ is **right reductive** relative to L if for $\alpha, \beta \in H$ such that $D_\alpha = D_\beta$ and $\alpha x = \beta x \ \forall x \in D_\alpha$ implies that $\alpha = \beta$.

The following equivalence relations are defined on an inverse semigroup. Green's relations: For $s, t \in S$ we say that:

1. $s\mathcal{L}t$ if and only if $s^*s = t^*t$,
2. $s\mathcal{R}t$ if and only if $ss^* = tt^*$,
3. $s\mathcal{H}t$ if and only if $s^*s = t^*t$ and $ss^* = tt^*$.
4. $s\mathcal{D}t$ if and only if there exists an e such that $s\mathcal{L}e$ and $e\mathcal{R}t$.

An **idempotent** is an element $e \in H$ such that $e^2 = e$. The set of idempotents in H is a commutative inverse semigroup, denoted as

$$E(H) = \{e \in H : e^2 = e\} = \{s^*s : s \in H\}.$$

The $\mathcal{H}$-class of an idempotent e, denoted $\mathcal{H}_e$, is the maximum subgroup of S with identity e. An inverse semigroup S is said to be **combinatorial** if $H_e = \{e\}$ for every $e \in E(S)$.

References

[1] R. Exel and B. Steinberg. "Representations of the Inverse Hull of a 0-Left Cancellative Semigroup". In: *arXiv e-prints* (Feb. 2018), arXiv:1802.06281.

[2] Douglas Lind and Brian Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995, pp. xvi+495.

Idempotents of $H(L^0)$

For $H(L^0)$, we can write the idempotents in the following form: $E[H(L^0)] = \{\theta_s \theta_{a_1}^{-1} \theta_{a_1} \cdots \theta_{a_n}^{-1} \theta_{a_n} \theta_s^{-1} : s \in L_T^1 \text{ and } a_i \in A\}$.

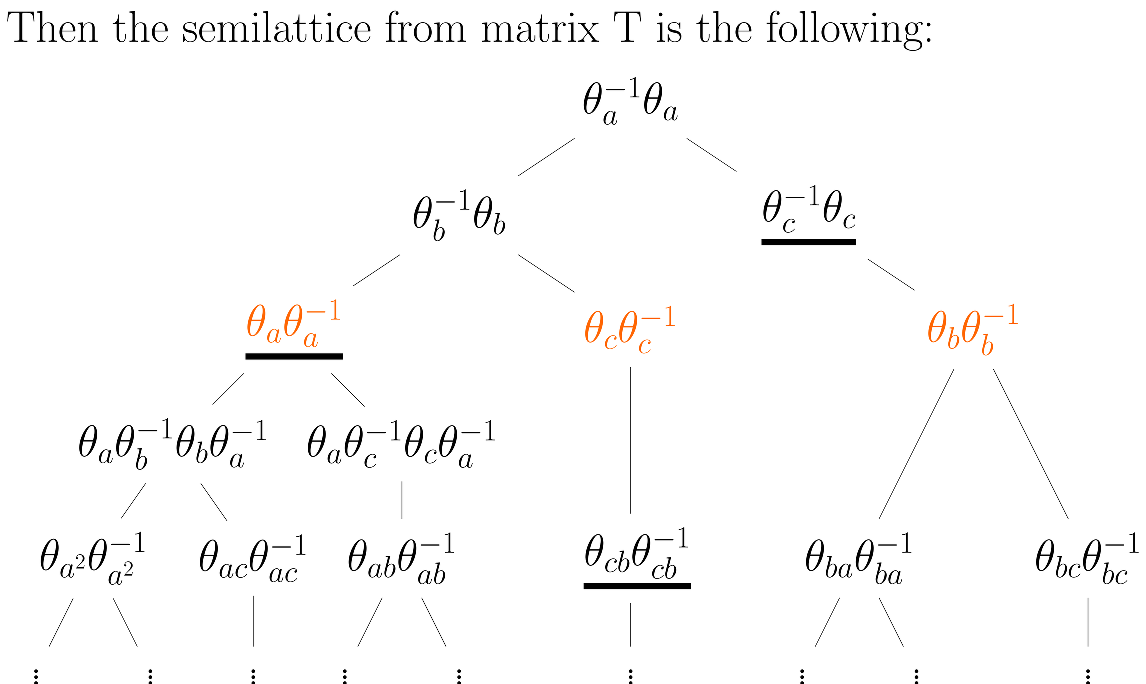
orange idempotents, are elements contained in the following set: $\mathcal{O} = \{\theta_a \theta_a^{-1} : a \in A\}$. We can write the set of all idempotents in the following form: $E[H(L^0)] = (\mathcal{O}^\uparrow - \mathcal{O}) \cup \mathcal{O}^\downarrow$ such that $\mathcal{O}^\uparrow - \mathcal{O} = \{\theta_{a_1}^{-1} \theta_{a_1} \cdots \theta_{a_n}^{-1} \theta_{a_n} : a_i \in A\}$ and $\mathcal{O}^\downarrow = \{\theta_s \theta_{a_1}^{-1} \theta_{a_1} \cdots \theta_{a_n}^{-1} \theta_{a_n} \theta_s^{-1} : s \in L_T \text{ and } a_i \in A\}$.

Example

We will construct the poset of idempotents of a Markov subshift. Consider the following matrix:

$$T = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Then the semilattice from matrix T is the following:



The picture above is a visual representation of the **natural partial order** of idempotent in the semigroup S, such that for $s, t \in S$, $s \leq t$ if and only if there exists an idempotent $e \in S$ such that $te = s$.

Shapes

L is a Markov subshift. Let L_n be the words in L of length n.

$$O(n) = \{\theta_w \theta_w^{-1} \in H(L^0) : w \in L_n\}.$$

In particular, $O(1) = O$. The semilattice can be reconstructed from translations of a finite collection of sets, one set for each letter in A. Given $a \in A$, let

$$S_a^\circ = a^*a \cap O(1).$$

We refer to S_a° as the **shape** of a. Similarly, given a word w of length $n \geq 1$ we let

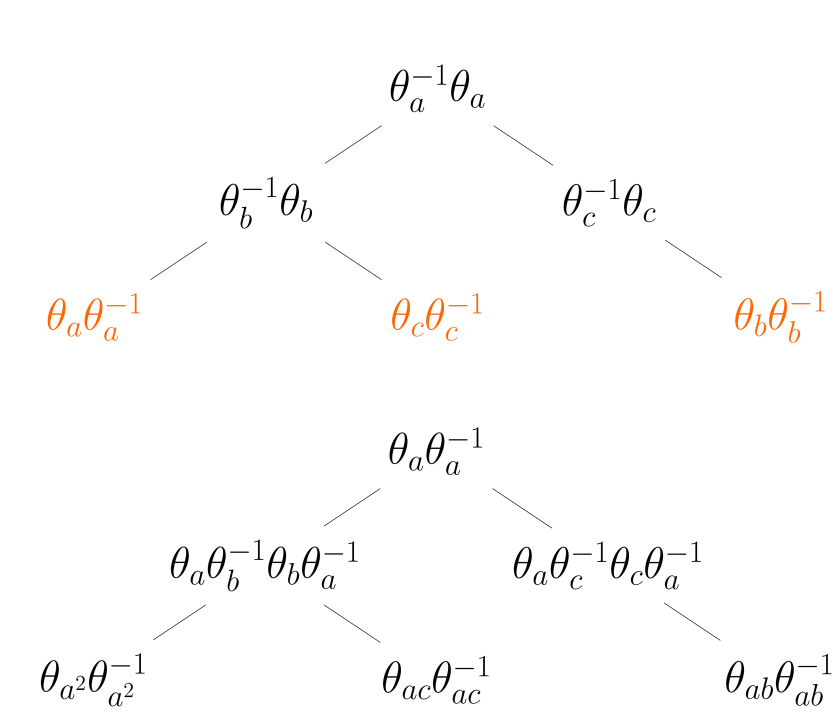
$$S_w = ww^* \cap O_{n+1}.$$

Theorem: Let w be a word in L that ends in the letter a in A. Then the map

$$f_w : S_a^\circ \to S_w$$

where $f_w(x) = wxw^*$ is an order isomorphism. Moreover, $x f_w(x)$ for each x in S_a°.

We can visually see these shapes in the above semilattice example. To illustrate this, bellow are the sub-semilattices of S_a° followed by S_a.



Properties

We want to know under what conditions is an inverse semigroup isomorphic to the inverse hull of a Markov subshift. We have the following three properties:

(1) H contains a set $\mathcal{O}$ of idempotents with the following properties:

1. $ef = 0$ for all $e \neq f \in \mathcal{O}$
2. For all $e \in E(H)$, $e \geq x$ or $e \leq x$ for some $x \in \mathcal{O}$
3. $\mathcal{O}^\uparrow - \mathcal{O} \cup \{0\}$ and $\mathcal{O}^\uparrow \cup \{0\}$ are closed under multiplication.
4. For $e \in \mathcal{O}$, $\mathcal{D}_e$ contains at most one element in $\mathcal{O}^\uparrow - \mathcal{O}$
5. $\mathcal{O}^\uparrow - \mathcal{O}$ is right reductive relative to $\mathcal{O}$

(2) H is combinatorial.

(3) L generates H.

We define the following sets given the above properties of H.
$A = \{a \in H : a^*a \in \mathcal{O}^\uparrow - \mathcal{O}, aa^* \in \mathcal{O}\}$
$L = \{a_1 a_2 ... a_n \neq 0 : a_i \in A\}$, where $L^0 = L \cup \{0\}$ and $L^1 = L \cup \{1\}$ such that for all $w \in L$, $w1 = w = 1w$.

Note that A acting like the alphabet of H, and L is acting like the language.

Main Theorem

Theorem: Let H be an inverse semigroup. Then H is isomorphic to the inverse hull of a Markov shift if and only if there exists a set $\mathcal{O}$ of idempotents such that $(H, \mathcal{O})$ satisfies properties **(1)**, **(2)**, and **(3)**

Applications

We can use our theorem to prove that we can have two Markov subshifts, L_1 and L_2, where $L_1 \ncong L_2$ but $H(L_1) \cong H(L_2)$.

Let us consider $H(L_1)$ generated by the matrix T. $\mathcal{O}_1$ is indicated the the semilattice by orange text. Let us pick a different set $\mathcal{O}_2$ underlined in black. $(H(L_1), \mathcal{O}_2)$ satisfy conditions (1) through (3).

Transition matrix for L_1:

$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Transition matrix for L_2:

$$\begin{array}{c} \\ x \\ y \\ z \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Despite $H(L_1^0) \cong H(L_2^0)$, $L_1 \ncong L_2$.

There are three common invariants that are considered when comparing languages like Markov subshifts: entropy, Bowen-Franks groups, and the Period [2].

We have found that for all three of these invariants there exists L_1 and L_2 that do not share the invariant, but $H(L_1^0) \cong H(L_2^0)$. Note in the above example, the entropy of L_1 is not equal to the entropy of L_2.

Theorem: If $H(L_1^0) \cong H(L_2^0)$ it must be the case that $|A_1| = |A_2|$.

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