Algebra univers. **48** (2002) 129–131
0002-5240/02/010129 – 03

Algebra Universalis

MAILBOX

The decidability of the affine completeness generation problem

KALLE KAARLI AND ALDEN PIXLEY

In our recent book [KP] we posed the problem (Problem 4.4.10) of characterizing those finite algebras of finite type which generate affine complete varieties. We had in mind an algebraic characterization and thus did not recognize and point out that the following theorem is a simple consequence of results appearing in the book. The purpose of the present note is to make the proof explicit.

Theorem. *There is an effective procedure for deciding whether or not a given finite algebra of finite type generates an affine complete variety.*

Background. All necessary background can be found in [KP] or, alternatively, in [K] and [KM]. Throughout, we shall make references to results in [KP]. We first recall that an algebra $\mathbf{A}$ is called affine complete if each congruence compatible operation on $\mathbf{A}$ is a polynomial of $\mathbf{A}$. A variety is affine complete if each of its members is affine complete. Also recall, for use below, that if a finite algebra $\mathbf{A}$ generates an affine complete variety then any subalgebra of $\mathbf{A}$ generates the same variety. Further, for some n, $\mathbf{A}$ must have an $(n+1)$-ary near unanimity term. This implies that the term functions of $\mathbf{A}$ are those functions on A which preserve the subuniverses of $\mathbf{A}^n$ and also that $\mathbf{A}$ generates a congruence distributive variety. Our proof essentially observes that we can effectively test for the existence of such a term. (For an arbitrary finite algebra of finite type there seems to be no known method for effectively testing for the existence of such a term. See e.g., [CD, page 296].)

In [G, Problem 6] G. Grätzer raised the question of characterizing affine complete algebras. Since non-congruence distributive affine complete algebras are common, while our present result solves the related decision problem for variety generation, it should be noted that it casts no new light on Grätzer's problem.

Presented by J. Berman.

Received May 10, 2001; accepted in final form November 14, 2001.

2000 *Mathematics Subject Classification*: 08A40, 08B05, 08B10.

Key words and phrases: Affine complete variety, near unanimity term, variety generation, decidability.

Work of the first author was supported by grant no. 4353 of the Estonian Science Foundation.

The proof. The following lemma summarizes well-known facts.

Lemma 1. *For any finite algebra* $\mathbf{A}$ *of finite type one can effectively construct* **Con** $\mathbf{A}$ *and more generally, construct the subalgebras of* $\mathbf{A}^n$ *for any* n. *Also one can effectively enumerate the term functions and the polynomial functions of* $\mathbf{A}$ *in order of increasing arity (first nullary, then unary, etc.). In addition one can also effectively enumerate the clone* $\mathrm{Comp}\mathbf{A}$, *of congruence compatible functions, in order of increasing arity.*

Lemma 2. *Let* C *be a clone on a finite set* A *and suppose that* C *contains an* $(n+1)$*-ary near unanimity function. Then* C *is generated by its members of arity not greater than* $|A|^n$.

Proof. This is just an observation about the familiar proof that C is finitely generated if it contains a near unanimity function (e.g., Theorem 3.2.4 of [KP]). Specifically, since C is the clone of term functions of the algebra $\mathbf{A} = \langle A; C \rangle$, to obtain a set of generators, for each of the finitely many non-subuniverses B of $\mathbf{A}^n$, we need only select an $f \in C$ such that f fails to preserve B. Since $|A^n| = |A|^n$, for any $f \in C$ failing to preserve B, by possible identification of arguments, there will also be an at most $|A|^n$-ary $f \in C$ which fails to preserve B. $\square$

Lemma 3. *If* $\mathbf{A}$ *is a finite algebra of finite type which generates an affine complete variety, then* $\mathbf{A}$ *must have an* $(n+1)$*-ary near unanimity term* u, *where*

$$n + 1 \leq m = |\mathrm{Con}\, \mathbf{F_A}(1)|$$

and hence u *can be effectively constructed.*

Proof. From the proofs of Theorems 2.3.9 and 4.4.1 of [KP] (originally appearing in [K]) the existence of a term u of arity no greater than m is guaranteed. (This is the critical observation of our proof.) If $(a_1, \ldots, a_k)$ is a list of the elements of A, then this k-tuple generates $\mathbf{F_A}(1)$ as a subalgebra of $\mathbf{A}^k$. Hence $\mathbf{F_A}(1)$, its congruences, and the integer m can be effectively constructed. Then we effectively enumerate all of the term functions of $\mathbf{A}$ up to arity m and test to find a u. $\square$

Proof of the Theorem. Assuming $\mathbf{A}$ is non-trivial, first test (a) to determine that it has no 1-element subalgebras and then (b) to determine that it possesses at least one near unanimity term (using Lemma 3). If $\mathbf{A}$ passes both of these tests then construct any subalgebra $\mathbf{B}$ of $\mathbf{A}$ having no proper subalgebras and test (c) to determine that $\mathbf{A}$ and $\mathbf{B}$ generate the same variety. By congruence distributivity, this can be effectively done by determining if each subdirectly irreducible member of $HS(\mathbf{A})$ is isomorphic to a member of $HS(\mathbf{B}) = H(\mathbf{B})$.

If all of (a)–(c) are passed then we may replace $\mathbf{A}$ by $\mathbf{B}$ and, using this new $\mathbf{A}$, construct the unique minimal subalgebra $\mathbf{M} = \mathbf{M}(\mathbf{A})$ of $\mathbf{F_A}(1)$ as described by

Theorem 4.4.8 of [KP]. Then, according to that theorem, **A** generates an affine complete variety just in case each of the quotients $\mathbf{M}/\theta$, $\theta \in \text{Con } \mathbf{M}$, is affine complete. Now Pol $\mathbf{M}/\theta \subseteq$ Comp $\mathbf{M}/\theta$, so Comp $\mathbf{M}/\theta$ contains a, say $(n+1)$-ary, near unanimity function. Hence by Lemma 2 it remains then to simply test to determine whether or not each function of arity not greater than $|M/\theta|^n$ in Comp $\mathbf{M}/\theta$ appears in Pol $\mathbf{M}/\theta$.

If we find a ternary near unanimity term of **A**, then we can also test to determine if **A** has a Pixley term; if so, then after passing test (c), by Theorem 4.4.11, we can conclude immediately that **A** generates an (arithmetical) affine complete variety and thus shorten the procedure above.

References

[CD] D. Clark and B. Davey, *Natural Dualities for the Working Algebraist*, Cambridge Univ. Press, Cambridge, New York, Melbourne, 1998.

[G] G. Grätzer, *Universal Algebra*, 2nd edition, Springer, New York, 1979.

[K] K. Kaarli, *Locally finite affine complete varieties*, J. Austral. Math. Soc. (Series A) **62** (1997), 141–159.

[KM] K. Kaarli and R. McKenzie, *Affine complete varieties are congruence distributive*, Algebra Universalis **38** (1997), 74–90.

[KP] K. Kaarli and A. F. Pixley, *Polynomial Completeness in Algebraic Systems*, Chapman & Hall/CRC, Boca Raton, 2001.

Kalle Kaarli
University of Tartu, 50090 Tartu, Estonia
e-mail: kaarli@math.ut.ee

Alden Pixley
Harvey Mudd College, Claremont, CA 91711
e-mail: alden_pixley@hmc.edu