Algebra univers. **55** (2006) 203–212
0002-5240/06/020203 – 10
DOI 10.1007/s00012-006-1948-2

Algebra Universalis

Weakly diagonal algebras and definable principal congruences

Kalle Kaarli and Alden Pixley

This paper is dedicated to Walter Taylor.

Abstract. An algebra is called weakly diagonal if every subuniverse of its square contains the graph of an automorphism. We show that every variety generated by a finite algebra with no proper subalgebras has a weakly diagonal generator. The result is applied in several ways and, in particular, to show that every arithmetical affine complete variety of finite type has equationally definable principal congruences.

1. Introduction

Throughout the paper an algebra is called *minimal* if it has no proper subalgebras. Finite minimal algebras naturally appear in the theory of affine complete varieties, i.e., varieties each of whose member algebras is affine complete in the sense that its polynomial functions are precisely its finitary congruence compatible functions. For general background on affine completeness and for most of the terminology of the present paper we refer to our book [5]. In particular from Section 4.4 of [5] we recall that every affine complete variety is congruence distributive, that an affine complete variety of finite type is always generated by a finite minimal algebra, and that an arithmetical variety of finite type is affine complete if and only if it is generated by a finite minimal algebra. Varieties generated by finite primal algebras, specifically the variety of Boolean algebras, are arithmetical and affine complete and have historically inspired the study of affine completeness. Weakly diagonal algebras were introduced in [4] with the aim of generalizing some results obtained in [1] for minimal functionally complete algebras. A special class of weakly diagonal algebras was studied in [2]. From [5], Section 3.1.4, recall that an algebra $\mathbf{A}$ is called *weakly diagonal* if every non-empty subuniverse of $\mathbf{A}^2$ contains the graph of some automorphism of $\mathbf{A}$. Clearly all weakly diagonal algebras are minimal but

Presented by J. Hyndman.
Received February 22, 2005; accepted in final form June 3, 2005.
2000 *Mathematics Subject Classification*: 08A40, 08A30, 08B10.
Key words and phrases: affine complete varieties, weakly diagonal algebras, minimal algebras, definable principal congruences.
Work of the first author was supported by grant No. 5368 from The Estonian Science Foundation.

the converse is not true. Our main observation in this paper is that in the case of finite algebras there is no difference between these properties if we are only concerned with the varieties the algebras generate; we prove that a variety generated by a finite minimal algebra also has a unique weakly diagonal generator so that the weakly diagonal algebras are canonical generators of the varieties generated by finite minimal algebras. We explain this claim now. In [3] it was observed that for any finite minimal algebra $\mathbf{A}$, the variety $\mathrm{Var}\,\mathbf{A}$ has a largest minimal member $\mathbf{M}$. In particular, in [5] (Prop. 4.4.7) the algebra $\mathbf{M}$ is characterized as any minimal subalgebra of the free algebra with one free generator $\mathbf{F}$ in $\mathrm{Var}\,\mathbf{A}$; from this it follows that $\mathbf{M}$ has the property that all minimal members of $\mathrm{Var}\,\mathbf{A}$ are its homomorphic images, hence $\mathbf{M}$ is unique up to isomorphism and if $\mathbf{A}$ is of finite type, it is constructable from $\mathbf{A}$.

The first and basic result of Section 2 of the present paper claims that finite weakly diagonal algebras are precisely the largest minimal algebras of the varieties they generate, i.e., are the algebras $\mathbf{M}$ described above. The importance of this result for the theory of varieties generated by finite minimal algebras, in particular affine complete varieties of finite type, is apparent. It reduces all the theory to the case of varieties with weakly diagonal generators. In the rest of the Section 2 we show that the structure of finite weakly diagonal algebras is relatively transparent, so there is a good reason to believe that our theorem really simplifies the study of varieties generated by a finite minimal algebra. We also show, roughly speaking, that the universe of such an algebra decomposes into the direct product of a group and a set, and that congruences respect this decomposition.

Section 3 is concerned with applications to the theory of affine complete varieties. We first show that all affine complete varieties of finite type have definable principal congruences, and then, more importantly, that arithmetical affine complete varieties of finite type have equationally definable principal congruences via the same formulas as in primal varieties and in Boolean algebras, and hence are *principal arithmetical varieties.* (See [5] for the definition and background.)

2. Weakly diagonal generators of varieties

We start with the proof of the basic theorem. Surprisingly, the proof is rather easy.

Theorem 2.1. *Let V be a variety generated by a finite minimal algebra $\mathbf{A}$. Then the following are equivalent:*

(1) $\mathbf{A}$ *is a largest minimal algebra in V;*
(2) $\mathbf{A}$ *is weakly diagonal.*

Proof. We first prove that $\mathbf{M}$, the largest minimal generator of V is weakly diagonal. Suppose it is not. Then $\mathbf{M}^2$ has a minimal subalgebra $\mathbf{B}$ which is not the graph of some automorphism of $\mathbf{M}$. Since $\mathbf{B}$ is subdirect in $\mathbf{M}^2$, we have $|B| \geq |M|$, and since equality would mean that B is the graph of an automorphism of $\mathbf{M}$, we must have $|B| > |M|$, which contradicts the definition of $\mathbf{M}$.

For the converse, assume that $\mathbf{A}$ is weakly diagonal. Let $\mathbf{F}$ be a free algebra in one generator of V and $\mathbf{M}$ its minimal subalgebra. Then $\mathbf{F}$ and consequently also $\mathbf{M}$ can be embedded into some finite power $\mathbf{A}^n$ of $\mathbf{A}$, $n \leq |A|$. Suppose that n is the minimal integer such that $\mathbf{M}$ can be embedded into $\mathbf{A}^n$. Since $\mathbf{A}$ is minimal, $\mathbf{M}$ is actually subdirect in $\mathbf{A}^n$. We want to show that $n = 1$. Suppose that, in the contrary, $n > 1$, and let $M_{12} = \{\langle a_1, a_2\rangle \mid \langle a_1, \ldots, a_n\rangle \in M\}$. Obviously M_{12} is a subuniverse of $\mathbf{A}^2$ and the algebra $\mathbf{M}_{12}$ being a homomorphic image of $\mathbf{M}$, must be minimal. Since $\mathbf{A}$ is weakly diagonal, M_{12} contains the graph of an automorphism of $\mathbf{A}$; let it be D. Since $\mathbf{M}_{12}$ is minimal, its universe must coincide with D, hence $\mathbf{M}_{12}$ is isomorphic to $\mathbf{A}$. This implies that $\mathbf{M}$ can be embedded into $\mathbf{A}^{n-1}$, a contradiction with the minimality of n. □

Theorem 2.1 allows us to reformulate one of the main results of [3] as follows.

Corollary 2.2. *A finitely generated congruence distributive variety is affine complete iff it has a weakly diagonal generator all of whose quotient algebras are affine complete.*

In order to continue, we need the following lemma which was proved in [4].

Lemma 2.3. *Let* $\mathbf{A}$ *be a finite minimal algebra and* $\mathbf{G} = \mathbf{Aut}\,\mathbf{A}$*. Then the following are equivalent:*

(1) $\mathbf{A}$ *is weakly diagonal;*
(2) *given* $a \in A$*, there is a unary term* u *such that* $u(A) \subseteq a^G$*.*

Remarks 2.4. 1. Here a^G denotes the $\mathbf{G}$-orbit containing a. We will denote the image of $a \in A$ under the automorphism g by a^g. Hence $a^G = \{a^g \mid g \in G\}$.

2. Clearly, if $\mathbf{A}$ is any algebra, $u(x)$ a unary term function and $\mathbf{G} = \mathbf{Aut}\,\mathbf{A}$, then $u(A)$ is a union of $\mathbf{G}$-orbits. Hence the inclusion of the second statement can be strengthened to the equality $u(A) = a^G$.

The following lemma establishes an important property of weakly diagonal algebras which we shall use in Section 3.

Lemma 2.5. *Let* $\mathbf{A}$ *be a finite weakly diagonal algebra and let* $t(x, \ldots, z, a)$ *and* $u(x, \ldots, z, a)$ *be polynomials obtained by replacing all occurances of the variable* w *in the terms* $t(x, \ldots, z, w)$ *and* $u(x, \ldots, z, w)$ *by the element* $a \in A$*. If* $t(x, \ldots, z, a) = u(x, \ldots, z, a)$ *for all* $x, \ldots, z \in A$*, then there is a unary term* $s(x)$ *such that* $t(x, \ldots, z, s(x)) = u(x, \ldots, z, s(x))$ *for all* $x, \ldots, z \in A$*.*

Remarks 2.6. 1. We do not claim that the term functions of the lemma equal the corresponding polynomial functions (which would mean that the polynomial functions were term functions) but only that the same identity is satisfied.

2. As a consequence of the lemma it follows that for any finite minimal algebra $\mathbf{A}$ the term identities defining the variety $\mathrm{Var}\,(\mathbf{A})$ are determined in the manner indicated in the lemma by the polynomial identities of the unique weakly diagonal algebra $\mathbf{M}$ which also generates $\mathrm{Var}\,(\mathbf{A})$.

Proof. Let $s(x)$ be a unary term chosen by Lemma 2.3, that is, $s(A) \subseteq a^G$ where $\mathbf{G} = \mathbf{Aut}\,\mathbf{A}$. We now show (by the same method as in the proof of Lemma 3.1.16 of [5]) that $t(x, \ldots, z, s(x)) = u(x, \ldots, z, s(x))$ is a term identity of $\mathbf{A}$.

For any given $x, \ldots, z \in A$ let g be an automorphism of $\mathbf{A}$ such that $s(x) = a^g$. Denoting the inverse of g by h we then have

$$\begin{aligned} t(x, \ldots, z, s(x)) &= t(x, \ldots, z, a^g) \\ &= t(x^{hg}, \ldots, z^{hg}, a^g) \\ &= (t(x^h, \ldots, z^h, a))^g \\ &= (u(x^h, \ldots, z^h, a))^g \\ &= u(x, \ldots, z, a^g) \\ &= u(x, \ldots, z, s(x)) \end{aligned}$$

which proves the lemma. □

We conclude this section with the decomposition results referred to in the Introduction. Let $\mathbf{A}$ be a finite weakly diagonal algebra and $\mathbf{G} = \mathbf{Aut}\,\mathbf{A}$. Also let u be a unary term of $\mathbf{A}$ from Lemma 2.3 and

$$B = u^{-1}(a) = \{x \in A \mid u(x) = a\}\,.$$

Then every $\mathbf{G}$-orbit contains precisely one element of B. Indeed, if $b, c \in B$, $g \in G$ and $b^g = c$, then $a = u(c) = u(b^g) = u(b)^g = a^g$. Since the set of all fixed-points of g is a subuniverse of $\mathbf{A}$, we conclude $g = 1$ which implies $b = c$. Because of the finiteness the term function u can be chosen idempotent. Hence we have the following lemma.

Lemma 2.7. *Every finite weakly diagonal algebra* $\mathbf{A}$ *with* $\mathbf{G} = \mathbf{Aut}\,\mathbf{A}$ *contains a subset* B *such that*

(1) *every* $a \in A$ *has a unique representation in form* $a = b^g$ *where* $b \in B$, $g \in G$;
(2) *there is a unary idempotent term function* u *such that* $|u(B)| = 1$.

Lemma 2.7 implies, in particular, that $\mathbf{G}$-orbits of $\mathbf{A}$ are of the same size, the size of $\mathbf{G}$, and hence the order of $\mathbf{G}$ is a divisor of $|\mathbf{A}|$. (Actually this simple fact holds in case of all finite minimal algebras.) So there are two extreme situations isolated by the conditions $|G| = 1$ and $|G| = |A|$. The following corollary characterizes these situations in terms of unary term functions.

Corollary 2.8. *If* $\mathbf{A}$ *is a finite weakly diagonal algebra with* $\mathbf{Aut}\,\mathbf{A} = \mathbf{G}$*, then*

(1) $|G| = 1$ *iff each element of* A *is the value of some constant unary term function.*
(2) $|G| = |A|$ *iff each unary term function of* $\mathbf{A}$ *is a bijection on* A*.*

Proof. 1. If $|G| = 1$ and $a \in A$ then the second claim of Lemma 2.7 implies that a is a value of a suitable constant term function. The converse is true for any algebra.

2. If $|G| = |A|$ and $a \in A$ then $A = a^G$ and each unary term function u is determined by the image of a: if $u(a) = a^h$, then $u(a^g) = a^{hg}$ for every $g \in G$. Clearly such a function is bijective. The converse follows from the proof of Lemma 2.7. Indeed, if $|G| < |A|$ then the unary term u such that $|u(B)| = 1$ is not a bijection. □

If $L = \operatorname{Con}\mathbf{A}$, then obviously $\mathbf{G} = \mathbf{Aut}\,\mathbf{A}$ acts on L via $\langle x, y\rangle \in l$ iff $\langle x^g, y^g\rangle \in l^g$. Also, if $l \in L$ and $a \in A$ we put $H_l = \{g \in G \mid \langle a^g, a\rangle \in l\}$. Obviously every H_l is a subuniverse of the group $\mathbf{G}$. Since $\mathbf{A}$ is minimal, the subgroups $\mathbf{H}_l$ do not depend on the choice of a. For the rest of this section, $\mathbf{A}$ is a finite weakly diagonal algebra and $\mathbf{G}$, $\mathbf{L}$, $\mathbf{H}_l$ ($l \in L$) have the meaning just defined. The symbols u and B will have always the meaning given in Lemma 2.7. The next lemma shows that the family of subgroups $\mathbf{H}_l$, $l \in L$, determines the restrictions of congruences of $\mathbf{A}$ to all $\mathbf{G}$-orbits.

Lemma 2.9. *For any finite weakly diagonal algebra* $\mathbf{A}$*, given any* $a \in A$*,* $g, h \in G$*, and* $l \in L$*, we have*

$$\langle a^g, a^h\rangle \in l \quad\Leftrightarrow\quad g^{-1}h \in H_l\,.$$

Proof. The relation $\langle a^g, a^h\rangle \in l$ can be written as $\langle a^g, (a^g)^{g^{-1}h}\rangle \in l$ which is equivalent to $g^{-1}h \in l$. □

The main result of this section clarifies the relationship between the lattice $\mathbf{L}$ and the system of subgroups $\mathbf{H}_l$, $l \in L$, and shows that the congruences of $\mathbf{A}$ are completely determined by their restrictions to B and by the subgroups $\mathbf{H}_l$, $l \in L$.

Theorem 2.10. *The following are true for any finite weakly diagonal algebra* $\mathbf{A}$*.*

(1) *the mapping* $l \mapsto \mathbf{H}_l$ *is a lattice homomorphism from* $\mathbf{Con}\,\mathbf{A}$ *to the lattice of subgroups of* $\mathbf{G}$*;*
(2) $H_{l^g} = g^{-1}H_l g$ *for every* $g \in G$ *and* $l \in L$*;*
(3) *for every* $b, c \in B$*,* $g, h \in G$*,* $l \in L$*, we have*

$$\langle b^g, c^h\rangle \in l \quad\Leftrightarrow\quad \langle b, c\rangle \in l^{g^{-1}} \text{ and } g^{-1}h \in H_l\,.$$

Proof. 1. If $l, m \in L$ then obviously $l \leq m$ implies $H_l \subseteq H_m$, hence for arbitrary $l, m \in L$ we have $H_{l \wedge m} \subseteq H_l \cap H_m$ and $H_l \vee H_m \subseteq H_{l \vee m}$. Let $g \in H_l \cap H_m$ and $a \in A$. Then $\langle a^g, a \rangle$ is both in l and m, implying $\langle a^g, a \rangle \in l \wedge m$, that is, $g \in H_{l \wedge m}$. Hence $H_{l \wedge m} = H_l \cap H_m$. Now take $g \in H_{l \vee m}$ and $a \in A$. Then $\langle a, a^g \rangle \in l \vee m$, thus there exist $a = c_0, c_1, \ldots, c_s = a^g \in A$ such that $\langle c_{i-1}, c_i \rangle \in \{l, m\}$, $i = 1, \ldots, s$. Using the term function u we can replace the elements c_i by $d_i = u(c_i)$, $i = 1, \ldots, s$. Then every d_i can be written in form a^{g_i} for suitable $g_i \in G$ ($g_0 = 1, g_s = g$), and $\langle d_{i-1}, d_i \rangle \in \{l, m\}$ implies $g_{i-1}^{-1} g_i \in H_l \cup H_m$ for every $i = 1, \ldots, s$. Clearly then $g = (g_0^{-1} g_1)(g_1^{-1} g_2) \cdots (g_{s-1}^{-1} g_s)$ belongs to the subgroup of G generated by H_l and H_m.

2. The equality follows from the next sequence of equivalences where a is an arbitrary element of A:

$$\begin{aligned} h \in H_{l^g} &\Leftrightarrow \langle a^g, a^{gh} \rangle \in l^g \Leftrightarrow \langle a, a^{ghg^{-1}} \rangle \in l \\ &\Leftrightarrow ghg^{-1} \in H_l \Leftrightarrow h \in g^{-1} H_l g \,. \end{aligned}$$

3. Let u be the unary idempotent term function such that $u(B) = a$. Then $\langle b^g, c^h \rangle \in l$ implies $\langle a^g, a^h \rangle \in l$, hence $h^{-1} g \in H_l$ and $\langle c^h, c^g \rangle \in l$ by Lemma 2.9. Now transitivity of l yields $\langle b^g, c^g \rangle \in l$ which implies $\langle b, c \rangle \in l^{g^{-1}}$. For the converse, suppose that $\langle b, c \rangle \in l^{g^{-1}}$ and $g^{-1} h \in H_l$. The first of these conditions implies $\langle b^g, c^g \rangle \in l$ and the second one together with Lemma 2.9 yields $\langle c^g, c^h \rangle \in l$. Thus by transitivity $\langle b^g, c^h \rangle \in l$. □

The final result of this section shows that in the congruence permutable case, the relationship between congruences of $\mathbf{A}$ and their restrictions to B and to a $\mathbf{G}$-orbit is even nicer.

Theorem 2.11. *Let the finite weakly diagonal algebra* $\mathbf{A}$ *be congruence permutable. Then*

(1) $H_l H_m = H_m H_l$ *and consequently* $H_l \vee H_m = H_l H_m$ *for every* $l, m \in L$*;*
(2) *the restrictions of every* $l, m \in L$ *to* B *permute;*
(3) *the restriction map* $l \mapsto l|_B$ *is a lattice homomorphism from* $\mathbf{L}$ *to the lattice of equivalence relations on* B*,*

Proof. 1. If $g \in H_l$ and $h \in H_m$ are arbitrary then $gh \in H_l \vee H_m = H_{l \vee m}$, hence $\langle a, a^{gh} \rangle \in l \vee m$ for any $a \in A$. We choose $a = u(B)$. Since l and m permute, we have $\langle a, a^{gh} \rangle \in m \circ l$, so there exists $x \in A$ such that $\langle a, x \rangle \in m$ and $\langle x, a^{gh} \rangle \in l$. This element x can be replaced by $y = u(x) \in a^G$. Now, if $y = a^f$ where $f \in G$, we have $f \in H_m$ and $f^{-1} gh \in H_l$. This proves $gh \in H_m H_l$.

2. Let $b, c \in B$ and $\langle b, c \rangle \in l \circ m$. Then there exists $x \in A$ such that $\langle b, x \rangle \in m$ and $\langle x, c \rangle \in l$. Let $x = d^g$ where $d \in B$ and $g \in G$. Then $\langle b, x \rangle \in m$ implies

$g \in H_m$, hence $\langle d^g, d\rangle \in m$ and by transitivity $\langle b, d\rangle \in m$. Similar arguments show that $\langle d, c\rangle \in l$.

3. Obviously the restriction map preserves the meets and $l|_B \vee m|_B \le (l \vee m)|_B$ for every $l, m \in L$. For the reverse inequality, let $b, c \in B$ be such that $\langle b, c\rangle \in (l \vee m)|_B$. Since l and m permute as congruences of $\mathbf{A}$ there exists $x \in A$ such that $\langle b, x\rangle \in l$ and $\langle x, c\rangle \in m$. Now, exactly as it was done in the proof of claim 2, we can replace x by $d \in B$. □

3. Definable principal congruences

Recall that a variety V has *definable principal congruences* (DPC for short) if there is a first order formula $\phi(x, y, u, v)$ such that for every $\mathbf{A} \in V$ and $a, b, c, d \in A$, we have

$$\langle a, b\rangle \in \mathrm{Cg}(c, d) \quad \Leftrightarrow \quad \mathbf{A} \vDash \phi(a, b, c, d)\,.$$

The formula $\phi(x, y, u, v)$ is the disjunction of *principal congruence formulas*, which are the existentially quantified conjunctions of equations which occur in Mal'cev's Lemma. (See [5] and references there for more details.)

If, moreover, the formula $\phi(x, y, u, v)$ is a conjunction of finitely many equations $t_i(a, b, c, d) \approx s_i(a, b, c, d)$ where $t_i(x, y, u, v)$ and $s_i(x, y, u, v)$, $i = 1, \ldots, n$, are terms, then we say that V has *equationally definable principal congruences* (EDPC for short) with respect to these equations.

We begin with a simple observation.

Theorem 3.1. *Every affine complete variety of finite type has DPC.*

Proof. This is a direct corollary of Lemma 6.1 from [7] which generalizes the fact, proved by R. McKenzie [6], that every directly representable variety has DPC. One just has to combine this lemma with the fact that every affine complete variety of finite type is generated by a finite minimal algebra $\mathbf{A}$ and hence also contains the unique largest minimal algebra $\mathbf{M}$ discussed in the Introduction. Let V be an affine complete variety of finite type. Then by Lemma 6.1 of [7] it is easy to see that the theorem is proved if one verifies the following three conditions:

(1) V is locally finite;
(2) V has only finitely many subdirectly irreducibles;
(3) for each subdirectly irreducible $\mathbf{A} \in V$ there is a positive integer $N = N(\mathbf{A})$ such that for each finite subdirect power $\mathbf{B}$ of $\mathbf{A}$ there is a subset $B_0 \subseteq B$ of size $|B_0| \le N$ such that B_0 projects onto each of the given subdirect factors.

Now let $\mathbf{M}$ be a largest (finite) minimal generator of V. Since all subdirectly irreducibles of V are minimal, they are homomorphic images of $\mathbf{M}$, so there are only finitely many of them. Hence the conditions (1) and (2) are fulfilled. Finally, in order to prove that (3) is fulfilled, we observe that $\mathbf{B}$ has a minimal subalgebra $\mathbf{B}_0$ whose universe B_0 projects onto each subdirect factor, because the latter are also minimal. Now obviously $|M|$ can be taken as the bound N. □

The remainder of the paper deals with arithmetical case. It is well known that a variety V is arithmetical iff it has a Pixley term, that is, a ternary term $t(x, y, z)$ such that the identities

$$t(x, y, y) \approx t(x, y, x) \approx t(y, y, x) \approx x \tag{1}$$

are satisfied in V. We say that the Pixley term $t(x, y, z)$ is a *principal Pixley term for* V if V has EDPC with respect to the equation $t(u, v, x) \approx t(u, v, y)$. Specifically this means that for every algebra $\mathbf{A}$ of V,

$$\langle x, y\rangle \in \mathrm{Cg}(u, v) \;\Leftrightarrow\; t(u, v, x) = t(u, v, y)\ .$$

If a variety has a principal Pixley term, it is called a *principal arithmetical variety.*

We recall that every ternary function f on any set A which satisfies the identities (1) is called a *Pixley function.* Thus, a Pixley term for V is a ternary term that induces Pixley functions on all members of V.

It is perhaps not so well known that the analogs of these results hold for arithmetical equivalence lattices on finite sets [5]. (Note that an equivalence lattice $\mathbf{L}$ on a set A is called arithmetical if it is distributive and every two equivalences in L permute.) In particular by Theorem 2.2.6 of [5] every arithmetical equivalence lattice on a finite set A supports a principal Pixley function $f(x, y, z)$. The latter means that the equality $f(u, v, x) = f(u, v, y)$ with $x, y, u, v \in A$ holds iff $\langle x, y\rangle$ belongs to the equivalence relation $l \in L$ generated in $\mathbf{L}$ by the pair $\langle u, v\rangle$. Since the equivalence relations of $\mathbf{L}$ are joins of principal equivalence relations, any principal Pixley function—there may be many—uniquely determines $\mathbf{L}$.

Our main theorem is the following.

Theorem 3.2. *Every arithmetical affine complete variety* V *of finite type is a principal arithmetical variety, that is, it has EDPC with respect to the equation* $t(u, v, x) \approx t(u, v, y)$ *where* $t(u, v, z)$ *is a suitable Pixley term for* V*.*

The larger significance of Theorem 3.2 lies in the fact that it describes principal congruences in arithmetical affine complete varieties of finite type precisely as they are known to be described in discriminator varieties where the principal Pixley term is any discriminator term for the variety. In particular, for the variety of Boolean algebras, the simplest affine complete variety, when $t(x, y, z)$ is defined by Boolean operations, one of the usual formulas for describing principal congruences results from Theorem 3.2.

Proof of Theorem 3.2. Let V be an arithmetical affine complete variety of finite type. Then it has a largest minimal generator $\mathbf{M}$ which is weakly diagonal by Theorem 2.1. By Theorem 2.2.6 of [5] the lattice $\mathbf{L} = \mathbf{Con}\,\mathbf{M}$ supports a principal Pixley function on A. Since $\mathbf{M}$ is affine complete, this function is a polynomial, and since $\mathbf{M}$ is minimal, it can be written as $t(x, y, z, a)$ for any given $a \in A$ and a suitable 4-ary term $t(x, y, z, w)$. Now Lemma 2.5 implies that there is a unary term $s(x)$ such that $t(x, y, z, s(x))$ is a Pixley term for V.

Next we show, applying Theorem 1.2.6 of [5], that $t(x, y, z, s(x))$ is a principal Pixley term for V. We first observe that though in [5] that theorem is stated and proved for varieties, its proof literally carries over to the single algebra case. Hence, given any m-ary basic operation symbol f of V,

$$t(u, v, f(x_1, \dots, x_m), a) \approx t(u, v, f(t(u, v, x_1, a), \dots, t(u, v, x_m, a)), a)$$

is an identity of $\mathbf{M}$. Then by Lemma 2.5 also

$$t(u, v, f(x_1, \dots, x_m), s(x)) \approx t(u, v, f(t(u, v, x_1, s(x)), \dots, t(u, v, x_m, s(x))), s(x))$$

is an identity of $\mathbf{M}$, hence also an identity for V because $\mathbf{M}$ generates V. It remains to again apply Theorem 1.2.6 of [5] and conclude that V is a principal arithmetical variety. □

In conclusion we remark that an open problem ([5], Problem 1.2.7) is whether every arithmetical variety having EDPC is necessarily a principal arithmetical variety. Theorem 3.2 answers this question affirmatively for affine complete varieties of finite type.

References

[1] K. Kaarli, *On varieties generated by functionally complete algebras*, Algebra Universalis **29** (1992), 495–502.

[2] K. Kaarli, *On certain classes of algebras generating arithmetical affine complete varieties*, General Algebra and Applications, Heldermann Verlag, Research and Exposition in Mathematics, Vol. **20**, Berlin, 1992, pp. 152–161.

[3] K. Kaarli, *Locally finite affine complete varieties*, J. Austral. Math. Soc. (Series A) **62** (1997), 141–159.

[4] K. Kaarli and A. Pixley, *Finite arithmetical affine complete algebras having no proper subalgebras*, Algebra Universalis **31** (1994), 557–579.

[5] K. Kaarli and A. Pixley, *Polynomial Completeness in Algebraic Systems*, Chapman & Hall/CRC, Boca Raton, 2001.

[6] R. McKenzie, *Para-primal varieties: a study of finite axiomatizability and definable principal congruences in locally finite varieties*, Algebra Universalis **8** (1978), 336–348.

[7] A. Pixley, *Principal congruence formulas in arithmetical varieties*, in: Universal Algebra and Lattice Theory, Lecture Notes in Math. **1149** (1984), Springer-Verlag, New York, pp. 238–254.

Kalle Kaarli
Institute of Pure Mathematics, University of Tartu, 50090 Tartu, Estonia

Alden Pixley
Department of Mathemetics, Harvey Mudd College, Claremont, California 91711, USA