What's So Special About Boolean Algebras

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For Kalle Kaarli, for over three decades my friend and most valued colleague, in this year of his seventy fifth and my ninety fifth birthdays

Since their conception in the mid-nineteenth century Boolean algebras have played a special role in mathematics, primarily as the "algebra of logic" and as the "algebra of classes" and have subsequently inspired alternative algebras of logic and also the the study of infinitary Boolean operations and topological properties. In the present essay our purpose is to examine a well known purely algebraic property of Boolean algebras, namely their dual roles as both algebras based on lattice properties and as special types of rings, i.e.; Boolean rings. We show how, taken together, properties of these two features enable us to locate Boolean algebras/rings as the simplest example of a much broader general class of algebraic structures, "arithmetical polynomial complete varieties" which naturally inherit many of the special features of the Boolean case.

There have been many variations of Boolean algebras in the past, usually obtained by strengthening or weakening the properties of the basic operations or by adding new operations, usually retaining some of the underlying lattice structure. Thus this might be called a "bottom- up" approach. Our program is to take what we might call a "top-down" approach: thus we start with the basic variety setting for Boolean theory, arithmetical varieties, and study how adding appropriate properties, recaptures Boolean properties. The program is at an early stage and what is presented here may be considered a progress report.

This essay is expository and it's intended audience is graduate students with a little background in general algebraic systems. Detail background through the year 2000 may be found in [1], and in [2]-[4] for later developments. While these works are published as joint work of Alden Pixley and Kalle Kaarli, the former author freely acknowledges that they could never have appeared without the generous participation of the latter.

1. Varieties and the variety of Boolean algebras

Model theory and varieties In mathematical logic, *model theory* is the study of the relationship between formal theories (a collection of sentences in a formal language expressing statements about a mathematical structure), and their models (those structures in which the statements of the theory hold). Since our interest is in algebras, and since the most common algebraic systems, groups, rings, lattices, and Boolean algebras, can be defined by equations, we restrict ourselves to certain specialized kinds of formal theories, *equational theories*, in which the sentences are just universally quantified equations, and their models are algebras satisfying these equations. The class of all algebras modeling an equational theory is called a *variety* (or *equational class*).

To make our definition more precise we define an *algebraic language* or *type* to be a set of finitary operation symbols denoted by

$$f(x_1, \ldots, x_n), g(x_1, \ldots, x_m), \cdots, m, n \geq 0, etc.$$

Terms are then formal compositions of operation symbols, and *equations* are formal equalities $t(x_1, \ldots, x_n) \approx s(x_1, \ldots, x_m)$ (briefly $t \approx s$). An *algebra* $\mathbf{A} = \langle A, F \rangle$ of a particular type is understood to be a non-empty set A, the universe of $\mathbf{A}$, together with a set F of finitary operations which are interpretations of the operation symbols of the type, an operation symbol $f(x_1, \ldots, x_n)$ interpreting as an operation in $A^n \to A$, and each term interpreting as a term function. Homomorphisms and direct products are defined as usual and a subalgebra is an algebra of the same type with universe a nonempty subset of A which is closed under the operations of $\mathbf{A}$.

The following is an important characterization of varieties, due to Garrett Birkhoff (1931).

> **Birkhoff**1 *A class of algebras of a given type is definable by a set of equations of the type iff the class is closed under the successive formation of homomorphic images of subalgebras of direct products.*

In symbols this means that V is a variety ifff $V = HSPV$. If V is a variety, i.e.: defined by a set of equations, satisfaction of this formula is rather obvious; proof of the converse, that satisfaction of the formula implies the existence of a set of defining equations for V, entails showing that satisfaction of the formula enables construction of a *free algebra* in $HSPV$.

If K is a class of algebras of the same type, then $V(K)$ is the least variety containing K, the variety defined by all of the identities common to all of the algebras of K. From **Birkhoff**1 it follows that $V(K) = HSPK$. While these formulas look simple, the complexity of the algebras resulting from forming homomorphic images of subalgebras of direct products can be formidable. More of this later.

Consider the familiar domain of groups. The choice of operations is important. For example, groups may be defined as systems $\mathbf{G} = \langle G, \cdot \rangle$ where $\cdot$ satisfies the associative identity and it is stipulated that G contains an identity element and that each member of G has an inverse with respect to this identity. In this formulation of the definition if we consider only the identities, ie.: associativity for $\cdot$, we have the variety of semigroups, which properly contains the class of groups. The problem here is that in this formulation of groups a subalgebra (i.e.: a subset closed under the operation) is not necessarily a subgroup. (For example the positive integers are not a subgroup of the group of integers under addition.) On the other hand groups defined as systems $\mathbf{G} = \langle G, \cdot, e, ^{-1} \rangle$, ($e$ the nullary identity operation and $^{-1}$ the unary inverse), satisfying the usual identities, are a variety.

In the case of groups the distinction is not really very important, since in either formulation the classes of structures described are essentially the same. This is because in the first description the subalgebras of groups which are groups are the same as the subalgebras in the second formulation. (This in turn is because in groups the identity and inverses "descend" to each subgroup.) Thus the variety of groups is unambiguously described either way.

Properly describing Boolean algebras is a bit more delicate. For distributive lattices, complements, when they exist, are unique and Boolean lattices are defined as complemented distributive lattices, i.e.: as systems $\mathbf{L} = \langle L, \vee, \wedge \rangle$ in which the operations $\vee$ and $\wedge$ satisfy the (dual) identities defining distributive lattices and in which there are distinct largest and least elements, 1 and 0 relative to the the lattice order, and such that each element $x \in L$ has a complement $x' \in L$ relative to 1 and 0. In contrast to groups, which contain exactly one 1-element subgroup, each element of L is a 1-element subalgebra and a subalgebra of a Boolean lattice need not be a Boolean lattice. For example Boolean lattices always contain sublattices which are chains which, of course are not complemented unless they have only two elements. Moreover subalgebras of Boolean lattices, when they are Boolean, generally have only "local" 0 and 1 elements and complements. Clearly Boolean lattices do not form a variety.

On the other hand Boolean algebras are defined as systems $\mathbf{B} = \langle B, \vee, \wedge, 0, 1, ' \rangle$ in which $\vee$ and $\wedge$ satisfy the dual identities defining distributive lattices and additionally, for complementation $'$, the identities $0' = 1$, $1' = 0$ and the dual DeMorgan identities. As a result every subalgebra of a Boolean algebra is also a Boolean algebra and always contains both the 0 and 1 of the parent algebra; in contrast to the usual pattern in groups, rings, and lattices, no non-trivial Boolean algebra contains a 1-element subalgeba; as we shall see later, this is a very special feature of Boolean algebras. Boolean algebras clearly do form a variety.

2. Basic features of Boolean algebras

Boolean algebras vs Boolean rings An alternative approach to Boolean algebras is via ring theory: i.e.: Boolean rings and their inter-definability with Boolean algebras, which we now review. A Boolean ring is a system $\mathbf{B} = \langle B, +, \cdot, 0, 1\rangle$ which is a ring in which every element is idempotent, i.e.: $x^2 = x$ for all $x \in B$. Thus for $x, y \in B$,

$$\begin{aligned} (x+y)(x+y) &= x+y, \\ \text{so } x^2 + yx + xy + y^2 &= x + yx + xy + y = x + y. \end{aligned}$$

Canceling and setting $x = y$ yields both $x + x = 0$, characteristic 2, and $yx = xy$, commutativity. Therefore Boolean rings, defined by the usual ring identities for $+, \cdot, 0, 1$ together with idempotence, form a variety. The important fact is that these two varieties are inter-definable: given a Boolean ring, define

$$\begin{aligned} x \wedge y &= xy, \\ x \vee y &= x + y + xy, \\ \text{and } x' &= 1 + x, \end{aligned}$$

to obtain a Boolean algebra as a special kind of lattice. The other way around, given a Boolean algebra, define

$$\begin{aligned} xy &= x \wedge y, \\ \text{and } x + y &= (x \vee y) \wedge (x \wedge y)', \end{aligned}$$

to obtain a Boolean ring. (Either way the 0 and 1 are the same.) Thus the varieties of Boolean algebras and of Boolean rings are inter-definable. For our purposes the significance of this inter–definability is that it shows how Boolean algebras/rings bridge two major areas of mathematics: order and lattice theory on the one hand, and group/ring theory on the other. No wonder their joint character has intrigued mathematicians for so long!

This has been recognized for many years but it's deeper significance is only more recently understood and is our major purpose here.

The two element Boolean algebra Almost from the historical beginning of Boolean algebras particular attention has been paid to the two element Boolean algebra

$$\mathbf{B}_2 = \langle \{0,1\}, \vee, \wedge, ', 0, 1 \rangle,$$

because of it's role in logic. Specifically, this is because every finitary operation $f \in \{0,1\}^n \to \{0,1\}$, n a non-negative integer, is a term function. In logic this means that every truth function can be composed from the Boolean operations using a familiar algorithm; in electrical engineering this fact it is sometimes called the "fundamental theorem of switching circuit theory".

A more recently recognized feature of this variety (Marshall Stone 1936) is that the algebra $\mathbf{B}_2$ generates the entire variety B of all Boolean algebras in the sense that every Boolean algebra is isomorphic to a subdirect power of $\mathbf{B}_2$, i.e.: $B = IP_S\mathbf{B_2}$. An algebra is a *subdirect product* of factor algebras if it is a subalgebra of a product and each of the projections into a factor of the product is onto that factor. The operation P_S represents the formation of subdirect products. The representation $B = IP_S\mathbf{B_2}$ obviously implies that the defining identities of the variety of Boolean algebras are just the identities of the algebra $\mathbf{B}_2$. The formula also implies that every Boolean algebra is isomorphic to a field of sets: a collection of subsets of some universe U containing the empty set and U and which is closed under the operations of finite set unions and intersections and complementation. The "points" of the set U are the maximal filters (dual ideals) of the algebra.

3. THE STRUCTURE OF VARIETIES AND MALCEV CONDITIONS The representation $B = IP_S\mathbf{B_2}$ of the variety B of Boolean algebras is intuitively so appealing: we think of each Boolean algebra $\mathbf{B}$ as consisting of a class of vectors $b = (b_1, \ldots, b_k, \ldots)$, over some index set K, each $b_k \in \{0,1\}$ and for each $k \in K$, the projection $proj_kB$ of B into the set of all kth coordinates, is equal to $\{0,1\}$. (Generally, K is possibly uncountable and B is a subset of the functions $\{0,1\}^K \to \{0,1\}$.) To understand this representation as compared with the rather opaque general $HSPK$ representation, we need a few

more basics about varieties.

First, more about subdirect products (Most of these facts are due to Garrett Birkhoff; their proofs can be found in any introductory text on universal algebra.)

> **Birkhoff**2 *Suppose an algebra* $\mathbf{A}$ *is a subalgebra of the direct product of algebras* $\mathbf{A}_k, k \in K$ *for some index set* K*, and suppose the projection homomorpsms* $p_k : \mathbf{A} \to \mathbf{A}_k$ *are onto the* A_k*, so that for their kernels* $\phi_k \in Con\mathbf{A}$*,* $\mathbf{A}/\phi_k \cong A_k$*, then*
>
> $$\bigwedge\{\phi_k : k \in K\} = 0$$
>
> *if and only if* $\mathbf{A}$ *is a subdirect product of the* $\mathbf{A}_k$*.*

Notice that when $\mathbf{A}$ is a subdirecrt product as above, then $\mathbf{A}$ isomorphic to a product in $\prod\{\mathbf{A}/\phi_k : k \in K\}$. This gives an internal description of when $\mathbf{A}$ is isomorphic to some subdirect product of algebras: If $\{\phi_k : k \in K\}$ is a family of congruences in $Con\mathbf{A}$ then the mapping $a \mapsto \prod\{a/\phi_k : k \in K\}$ is an isomorphism of $\mathbf{A}$ onto a subdirect product in $\prod\{\mathbf{A}/\phi_k : k \in K\}$ iff $\bigwedge\{\phi_k : k \in K\} = 0$.

A non-trivial algebra $\mathbf{A}$ is *simple* if 0 and 1 are its only congruences. More generally, $\mathbf{A}$ is *subdirectly irreducible* if it has at least one congruence larger than 0 and if the intersection of all congruences larger than 0 is a congruence μ which is also larger than 0. μ is called the monolith of $Con\mathbf{A}$. Thus if $\mathbf{A}$ is subdirectly irreducible then in any subdirect representation of $\mathbf{A}$, one of the subdirect factors is isomorphic with $\mathbf{A}$.

> **Birkhoff**3 *Every non-trivial algebra is isomorphic with a subdirect product of subdirectly irreducible algebras.*

This appealing fact has the equally appealing consequence: Each variety V is generated by it's subdirectly irreducible members, V_{SI}, so V can be represented as $V = IP_S(V_{SI})$, like the appealing "vector" representation of Boolean algebras. In fact it is easy to show that $\mathbf{B}_2$ is the only subdirectly irreducible member of the variety of Boolean algebras, which explains the representation $\mathcal{B} = IP_S\mathbf{B_2}$ noted above.

On the other hand, in many other varieties the description of the subdirectly irreducible members is not so simple. For example the class of subdirectly irreducible groups is as large as the class of all groups themselves.

In these varieties the subdirect representation **Birkhoff**3 is not so appealing.

Malcev conditions The HSP characterization of varieties of **Birkhoff1** makes a vital connection between the abstract "syntactical" description of a variety, i.e: a set of equations, and a "semantical" description, i.e.: as a class of actual algebras determined by closure under H, S, and P. (In fact this description provides a proof of the completeness of the rules for equational logic.) This suggests that there might be more specific connections between the equations defining a variety and some more specific algebraic properties.

In fact this is true and is the significance of Malcev conditions, the first of which was discovered by A. I. Malcev. In general the join of two congruences in the lattice of congruence relations of an algebra is the union of all of their finite relation products:

$$\theta \vee \phi = \theta \cup \phi \cup \theta \circ \phi \cup \theta \circ \phi \circ \theta \cup \cdots .$$

In the special case where $\theta \vee \phi = \theta \circ \phi$ the congruences are *permutable*. In certain varieties, for example any algebra with a group reduct, the congruences always permute. Congruence permutability for varieties was characterized by Malcev as follows:

> **Malcev** *A variety V is congruence permutable iff there is a ternary term $p(x, y, z)$ in the language of V, such that the identities*
>
> $$p(x, x, y) \approx y \approx p(y, x, x)$$
>
> *are satisfied by all of the algebras of V.*

For example in any variety of algebras with group reducts, the term $xy^{-1}z$ is a Malcev term. This is thus a direct link between a specific property of the equational theory of a variety and a specific algebraic property shared by all of the algebras of the variety. Inspired by this early result an important and large literature of Malcev conditions has arisen, each linking some algebraic property of the variety with the formal equations defining the variety.

For our purposes in examining Boolean algebras we take a more focused approach. Our key is to observe that since Boolean algebras can be construed as both rings and as special lattices, and that as rings they have a Malcev term witnessing congruence permutability. Indeed, the term $p(x, y, z) = x + y + z$ where $x + x = 0$ is such. So we seek a central property of lattices which might be correspondingly represented. Such a property is *congruence distributivity*: for congruence lattices each of the (equivalent) dual distributive laws holds. This has long been known, and if one checks the proof it is clear that it only depends upon the fact that lattices have a ternary majority term, i.e.: a term $m(x, y, z)$ satisfying the identities

$$m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x.$$

Such a term in the language of lattice theory is

$$m(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z).$$

Thus the existence of a majority term for a variety is sufficient for congruence distributivity. Conditions both necessary and sufficient for congruence distributivity are known and are more complex. However it is not hard to show that the existence of both Malcev and majority terms is both necessary and sufficient for a variety to be both congruence permutable and distributive, and this is what we have for Boolean algebras.Such a variety is called an *arithmetical* variety.

In fact we can say more:

> **Arithmeticity**1 *A variety is both congruence permutable and congruence distributive iff there is a term $t(x, y, z)$ in the language of the variety satisfying the identities*
>
> $$t(x, y, y) \approx t(x, y, x) \approx t(y, y, x) \approx x.$$

Such a term is called an *arithmetical term* or a *Pixley term.* Notice that if m, p, t are respectively majority, Malcev, and arithmetical terms, then clearly $t(x, t(x, y, z), z)$ is a majority term and $m(x, p(x, y, z), z)$ is an arithmetical term. An arithmetical term in the language of Boolean algebras is

$$t(x, y, z) = (x \vee y') \wedge (x \vee z) \wedge (y' \vee z).$$

where $'$ is Boolean complementation.

Arithmetical varieties This then is a big part of what is so special about Boolean algebras: the variety of Boolean algebras is arithmetical.

We want to emphasize that while it was well known years earlier that Boolean algebras, via their dual nature as lattices and rings, enjoyed arithmeticity, little was made of this on the varietal level until the discovery and exploitation of Malcev conditions.

What does it mean algebraically for a variety to be arithmetical? What is gained by combining permutability of congruences with distributivity? In the variety of groups, the most fundamental exemplar of congruence permutability, the congruence lattices are determined by the normal subgroups and the join of two normal subgroups N and M is just NM. This somewhat obscures our earlier description, which is to observe that for the congruences θ_1, θ_2 determined by these normal subgroups,

$$(a, b) \in \theta_1 \vee \theta_2 \leftrightarrow (a, x) \in \theta_1 \text{ and } (b, x) \in \theta_2$$

for some element x of the group. Alternatively, congruence permutability asserts that the pair of congruences

$$\begin{array}{rcl} a & \equiv & x\ (\theta_1) \\ b & \equiv & x\ (\theta_2) \end{array}$$

is solvable iff $a \equiv b\ \ (\theta_1 \vee \theta_2)$.

Another suggestive way of thinking of congruence permutability is geometrically. Let $\mathbf{A}$, $\mathbf{B}$ be algebras in a congruence permutable variety and let S be a subuniverse in their direct product $\mathbf{A} \times \mathbf{B}$. Let $(x, y), (x, v), (u, v)$ be three vertices of a rectangle in $A \times B$ and suppose they also lie is S. Then applying the Malcev term $p(, x, y, z)$ we have

$$p(x, y), (x, v), (u, v)) = (p(x, x, u), p(y, v, v)) = (u, y),$$

so the fourth vertex of the rectangle must also lie in S. Thus the possibilities for subuniverses of direct products in congruence permutable varieties is quite restricted.

There are many other consequences of congruence permutabilityy; for example it implies that the lattice of congruences is modular which for groups translates into the Jordan-Hölder Theorem.

What about congruence distributivity? For congruence distributive varieties a most fundamental fact is that there are no skew congruences for subdiret decompositions of an algebra into finitely many factors This means that whenever $\rho, \theta_1, \ldots, \theta_n$ are congruence relations of an algebra $\mathbf{A}$ in such a variety and $\theta_1 \wedge \cdots \wedge \theta_n = 0$, then $(\theta_1 \vee \rho) \wedge \cdots \wedge (\theta_n \vee \rho) = \rho$. So the congruences of $\mathbf{A}$ are specifically determined by those of the subirect factors, a major simplification! Significantly, this condition, the absence of skew congruences in finite subdiret decompositions, is actually equivalent to congruence distributivity for varieties.

A further critical fact about lattices is that the existence of a majority term endows them with not only congruence distributivity but in addition an important interpolation property not shared by all congruence distributive varieties. This property asserts that if $\mathbf{L}$ is a lattice and $f : L^m \to L$ is a partial function with domain D a finite subset of L^m and all subuniverses of $\mathbf{L}^2$ are closed under f, applied coordinate-wise where it is defined, then there is a term function of $\mathbf{L}$ which agrees with f on D, i.e.: f can be interpolated by a term function. This property leads to an important characterization of arithmetical varieties, the following so called Chinese Remainder Theorem, which generalizes the solvability criterion for two congruences as described above:

> **Chineses Remainder Theorem** *A variety V is arithmetical if and only if for every algebra $\mathbf{A}$ in V, elements $a_1, \ldots, a_n$ in A, and congruence relations $\theta_1, \ldots, \theta_n$ of $\mathbf{A}$, the system of congruences*
>
> $$\begin{aligned} a_1 &\equiv x \qquad (\theta_1) \\ &\ldots \\ a_n &\equiv x \qquad (\theta_n) \end{aligned}$$
>
> *is solvable in A iff the congruences are pairwise compatible, meaning that for each $1 \leq i < j \leq n$, $a_i \equiv a_j(\theta_i \vee \theta_j)$.*

4. Polynomial completeness We noted earlier two special properties of the two element Boolean algebra $\mathbf{B}_2$ which we return to now:

> **Boolean** *1. Each operation on $\{0,1\}$, i.e..: in $\{0,1\}^n \to \{0,1\}$, $n = 0, 1, \ldots$ is a term function of $\mathbf{B}_2$ (the fundamental theorem of switching circuit theory).*
>
> *2. $\mathbf{B}_2$ generates the variety of Boolean algebras.*

The first of these properties is called primality: a finite algebra $\mathbf{A} = \langle A, F\rangle$ is *primal* if each operation on A is a term function of $\mathbf{A}$. Until the discovery of Malcev conditions this was regarded as an oddity and not really an algebraic property of algebras. Now with our characterization of arithmetical varieties (as having an arithmetical term) we can characterize primality algebraically as follows:

> **Primal** *A non-trivial finite algebra is primal iff i) it is simple (has only the congruences 0 and 1),ii) it is rigid (has only the identity automorphism), iii) it is minimal (has no non-trivial subalgebras), and iv) it has an arithmetical term.*

Since primality entails conditions i)-iii) the notion does not apply to varieties. However it has long been known that the variety of Boolean algebras is *polynomial complete*, .i.e: every congruence compatible operation on B is a Boolean polynomial. While this property of varieties until recently was pretty much ignored the following algebraic characterization of arithmetical varieties was suggestive:

> **Arihmeticity**2 *A variety V is arithmetical iff for each algebra $\mathbf{A} \in V$, each finite partial operation on A which is congruence compatible (where defined) agrees with some polynomial of $\mathbf{A}$.*

This and other related results inaugurated research into polynomial complete varieties (originally called affine complete varieties). Some sample results of this research include:

Polynomial Completeness

- *Every polynomial complete variety V is residually finite (all subdirectly irreducible members are finite),*
- *and is also congruence distributive. Additionally,*
- *if $\mathbf{A} \in V$ and $\mathbf{B} < \mathbf{A}$ then any (congruence) compatible operation on $\mathbf{B}$ has exactly one compatible extension on $\mathbf{A}$. From this it immediately follows that*

  i) if $\mathbf{B} < \mathbf{A} \in V$, then $V(B) = V(A)$, and

  ii) V is semidegenerate (no non-trivial algebra in V has a one element subalgebra) This condition is equivalent to: For each $\mathbf{A} \in V$ the unit congruence $1_A \in Con\mathbf{A}$ is compact, i.e.: is the join of finitely many principal congruences).

- *For a polynomial complete variety V the following are equivalent: i) V is generated by a finite minimal algebra; ii) is locally finite (every finitely generated algebra is finite; iii) is term equivalent to a variety of finite type; iv) has only finitely many siubvarieties.*

5. CONCLUSION Because of these strong properties we ask if polynomial completeness for a variety can be characterized algebraically. For arithmetical varieties, (remembering that the variety of Boolean algebras is such), we have the following important result:

> **Arithmetical Completeness** *An arithmetical variety V of finite type is polynomial complete iff it is generated by a finite minimal algebra. In this case $V = IP_SH\mathbf{A}$, where $\mathbf{A}$ is the generating algebra. V is called an arithmetical complete variety*

From this representation it follows that for polynomial complete arithmetical varieties of finite type the minimal generating algebra is the unique largest finite minimal algebra in the variety.

Thus Boolean algebras are just the simplest example of this characterization: Suppose an arithmetical complete variety contains a two element minimal algebra $\mathbf{A}$. Obviously $\mathbf{A}$ must be simple. Unless the exchange the two elements is an automorphism $\mathbf{A}$ is rigid and it follows, from the characterization **Primal** above, that $\mathbf{A}$ is a primal and hence is term equivalent to the two element Boolean algebra.This is "what is so special about Boolean algebras".

Characterizing general (necessarily congruence distributive) polynomial complete varieties of finite type is more complicated. Whereas for the arithmetical case all finite algebras are polynomial complete, in the general congruence distributive case we must check that all minimal algebras are polynomial complete. The most we can say in this case is that there is a constructive process for doing this, but which might be quite complex.

Finally, it should be emphasized that the above characterization **Arithmetical Completeness** gives us a simple tool kit for constructing such varieties, in fact, all such varieties: For any finite set A and arithmetical partition lattice L on A containing both the equality and diversity relations, choose a compatible arithmetical function $t(x, y, z)$ as an operation. (Such an operation necessarily exists and can be constructed.) Add a finite set $F = \{g, h, \ldots\}$ of other operations to guarantee that the algebra

$\mathbf{A} = \langle A, \{t\} \cup F \rangle$ is minimal and that L is the congruence lattice of $\mathbf{A}$. (It is well known that this is possible because L is distributive.) Therefore $\mathbf{A}$ generates an arithmetical polynomial complete variety of finite type in the same way that $\mathbf{B}_2$ generates the variety of Boolean algebras.

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