

Solution to CMJ Problem 952

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Prove that for $m, n \geq 0$,

$$\sum_{k=0}^n \binom{n-k}{k} \binom{k}{m} 2^{n-2k} (-1)^{k-m} = \binom{n+1}{2m+1}.$$

Proof. If we ignore the sign term, the sum on the left counts the ways to tile a strip of length n with squares and dominoes, where each tile can be red or white, and we are required to have exactly m dominoes. To see this, note that to create such a tiling with k dominoes (and therefore $n - 2k$ squares) we can arrange the $n - k$ uncolored tiles in $\binom{n-k}{k}$ ways, color the m dominoes in $\binom{k}{m}$ ways, then color the squares 2^{n-2k} ways. Since the number of white dominoes is $k - m$, the signed sum enumerates the number of tilings with an even number of white dominoes minus those with an odd number of white dominoes.

Next we pair up these tilings as best as we can, so that in each pair, exactly one tiling has an even number of white dominoes. For a given tiling T , we find its mate by looking for the first occurrence of a white domino or an rw (a red square followed by a white square). Assuming that one of these two objects can be found, we replace the first such object with the other, producing a new tiling T' , which still has length n and has m red dominoes, but the number of white dominoes is of opposite parity. Note that applying the same rule to T' produces T .

It remains to describe and count those exceptional tilings that contain no white dominoes and no rw . Such tilings have $k - m = 0$ and are therefore counted positively in the sum. Such tilings can be represented uniquely in the form

$$w^{x_1} r^{y_1} D w^{x_2} r^{y_2} D w^{x_3} r^{y_3} D \dots w^{x_m} r^{y_m} D w^{x_{m+1}} r^{y_{m+1}},$$

where D denotes a red domino, and the total number of squares is $n - 2m$. Hence the number of exceptional tilings is the number of nonnegative integer solutions to

$$x_1 + y_1 + x_2 + y_2 + \dots + x_{m+1} + y_{m+1} = n - 2m.$$

Since the number of nonnegative integer solutions to the equation $z_1 + \dots + z_n = k$ is $\binom{n+k-1}{k}$, the number of exceptional tilings is $\binom{(2m+2)+(n-2m)-1}{n-2m} = \binom{n+1}{n-2m} = \binom{n+1}{2m+1}$, as desired. \square

We note that another way to create an exceptional tiling is to choose $\{x_1, x_2, \dots, x_{2m+1}\}$, a subset of $\{1 \dots, 2n + 1\}$, which can be done $\binom{n+1}{2m+1}$ ways. For $i = 1 \dots, m$, the i th domino starts on cell $x_{2i} - 1$. Before the first domino, we have $x_1 - 1$ white squares followed by $x_2 - 1 - x_1$ red squares. To the immediate right of the domino starting on cell $x_{2i} - 1$, we have $x_{2i+1} - x_{2i} - 1$ white squares followed by $x_{2i+2} - x_{2i+1} - 1$ red squares (except, when $i = m$, $n - x_{2m+1}$ red squares).