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It is clear that $L_{(P)}$ is a linear operator from $\mathcal{H}_k$ to $\mathcal{H}_k$. Moreover, we claim that $L_{(P)}$ is self-adjoint and positive. In fact, for any u in $\mathcal{H}_{m+k}$ and v in $\mathcal{H}_k$ we infer from (2) that

$$\langle P(\partial)u, v\rangle_k = \langle u, \overline{P}v\rangle_{k+m}.$$

It follows that for Q and R in $\mathcal{H}_k$

$$\begin{aligned}\langle L_{(P)}[Q], R\rangle_k &= \langle P(\partial)(PQ), R\rangle_k = \langle \overline{P}Q, \overline{P}R\rangle_{k+m} \\ &= \langle Q, P(\partial)(\overline{P}R)\rangle_k = \langle Q, L_{(P)}[R]\rangle_k,\end{aligned}$$

which means $L_{(P)}$ is self-adjoint.

Analogously, by (4) we have

$$\langle L_{(P)}[Q], Q\rangle_k = \langle \overline{P}Q, \bar{P}Q\rangle_{k+m} = \|\overline{P}Q\|^2_{k+m} \geq \|P\|^2_m \|Q\|^2_k, \tag{5}$$

showing that $L_{(P)}$ is positive.

Therefore, we see that all eigenvalues of $L_{(P)}$ are positive and, on the basis of (5), that $\|P\|^2_m$ furnishes a lower bound for them. Furthermore, equality holds in (4) if and only if either $P \equiv 0$ or $\|P\|^2_m$ is the smallest eigenvalue of $L_{(P)}$ and Q is an eigenvector corresponding to it (unless $Q \equiv 0$). A particular case in which equality holds in (4) occurs when $P = P(y)$ belongs to $\mathcal{H}_m$ and $Q = Q(z)$ to $\mathcal{H}_k$, where $y \in \mathbb{R}^p$, $z \in \mathbb{R}^q$, and $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$.

Added in proof. Professor Luo Xuebo, who was one of his coauthor's Ph.D. supervisors, died in March 2004. Zhu-Jun Zheng expresses his deep respect for and everlasting memory of his deceased colleague and mentor.

Institute of Applied Mathematics, Northwestern Polytechnical University, Xi'an, 710072, P. R. China.

Institue of Mathematics, Henan University, Kaifeng, 475001, P. R. China.
zhengzj@henu.edu.cn

Combinatorial Proofs of Fermat's, Lucas's, and Wilson's Theorems

Peter G. Anderson, Arthur T. Benjamin, and Jeremy A. Rouse

In this note, we observe that many classical theorems from number theory are simple consequences of the following combinatorial lemma:

Lemma 1. *Let S be a finite set, let p be prime, and suppose $f : S \to S$ has the property that $f^p(x) = x$ for any x in S, where f^p is the p-fold composition of f. Then $|S| \equiv |F| \pmod p$, where F is the set of fixed points of f.*

Proof. The set S is the disjoint union of sets of the form $\{x, f(x), \ldots, f^{p-1}(x)\}$. Since p is prime, each set has either size one or size p. ∎

The *Lucas numbers* 2, 1, 3, 4, 7, 11, 18, 29, 47, ..., named in honor of Edouard Lucas (1842–1891), are defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

It is easy to show that, for $n \geq 1$, L_n counts the ways to create a bracelet of length n using beads of length one or two, where bracelets that differ by a rotation or a reflection are still considered distinct. For example, there are four bracelets of length three. (Such a bracelet can have three beads of length one, or it can have a bead of length two and a bead of length one, where the bead of length one can be in position one, two, or three.) Let f act on bracelets of prime length p by rotating each bead clockwise one unit. Clearly f^p leaves any bracelet unchanged. Since f has just one fixed point (when all beads have length one), we conclude that $L_p \equiv 1 \pmod{p}$ for each prime p.

More generally, for nonnegative integers a and b the *Lucas sequence (of the second kind)* is defined, as in [**4**], by $V_0 = 2$, $V_1 = a$, and $V_n = aV_{n-1} + bV_{n-2}$ for $n \geq 2$. Again, it is easy to show [**1**] that V_n with $n \geq 1$ counts colored bracelets of length n, where there are a color choices for beads of length one and b color choices for beads of length two. By the same argument as earlier, with the exception of those bracelets consisting of length one beads all of the same color, when p is prime every bracelet can be rotated to create p distinct bracelets. Thus

$$V_p \equiv a \pmod{p}$$

for each prime p. In the special case where $b = 0$, it is clear that $V_p = a^p$. Consequently, we have *Fermat's Theorem:* if p is a prime, then

$$a^p \equiv a \pmod{p}.$$

This combinatorial proof of Fermat's theorem was originally given in [**2**].

Next, consider colored bracelets of length pk, where p is prime. If we rotate the beads k units at a time, then there are exactly V_k fixed points, obtained by taking any colored bracelet of length k and "replicating" it p times. Our lemma concludes that for p prime

$$V_{pk} \equiv V_k \pmod{p}.$$

In particular, $V_{p^e} \equiv V_{p^{e-1}}$ when $e \geq 1$. Consequently, for p prime and e nonnegative,

$$V_{p^e} \equiv a \pmod{p}.$$

Now consider the set S of permutations of $\{0, 1, \ldots, p-1\}$ with exactly one cycle; thus, $|S| = (p-1)!$. Define $f : S \to S$ by

$$f\big((a_0, a_1, \ldots, a_{p-1})\big) = (1 + a_0, 1 + a_1, \ldots, 1 + a_{p-1}),$$

where addition is done modulo p. For each π in S, $f^p(\pi) = \pi$. For a satisfying $1 \leq a \leq p-1$ those permutations of the form $\pi_a = (0, a, 2a, 3a, \ldots, (p-1)a)$ (with multiplication done modulo p) are fixed points of f since $f(\pi_a)$ remains an "arithmetic progression." Conversely, if π is a fixed point of f and $\pi(0) = a$, then $\pi = f^a(\pi)$ must send a to $2a$ and, in general, $\pi = f^{ka}(\pi)$ sends ka to $(k+1)a$. Thus $\pi = \pi_a$, and f has exactly $p-1$ fixed points. This establishes *Wilson's Theorem:* if p is a prime, then

$$(p-1)! \equiv (p-1) \pmod{p}.$$

The same approach can be applied to the set S of k-element subsets of

$$\{0, 1, \ldots, p-1\}.$$

Define $f : S \to S$ by $f(\{a_1, a_2, \ldots, a_k\}) = \{1 + a_1, 1 + a_2, \ldots, 1 + a_k\}$, where again addition is done modulo p. When $1 \le k \le p - 1$ there are no fixed points of f. Consequently, for p prime and k satisfying $1 \le k \le p - 1$,

$$\binom{p}{k} \equiv 0 \pmod{p}.$$

We conclude with *Lucas's Theorem:* For p prime, let n and k have base p notation $n = \sum_{i \ge 0} b_i p^i$ and $k = \sum_{i \ge 0} c_i p^i$, where $0 \le b_i, c_i < p$. Then

$$\binom{n}{k} \equiv \prod_{i \ge 0} \binom{b_i}{c_i} \pmod{p}.$$

Proof. It suffices to show that $\binom{pn+r}{pk+s} \equiv \binom{n}{k}\binom{r}{s} \pmod{p}$ when $0 \le r, s < p$, and then proceed inductively. Let S denote the set of ordered pairs (A, v), where A is a binary $p \times n$ matrix and v is a binary $r \times 1$ vector such that among the $pn + r$ entries of A and v exactly $pk + s$ are equal to one. Hence $|S| = \binom{pn+r}{pk+s}$. Let Q denote the $p \times p$ permutation matrix with nonzero entries $q_{1p} = 1$ and $q_{i,i-1} = 1$ for $i = 2, 3, \ldots, p$. Thus QA has the same rows as A, each shifted "down" by one row.

Define $f : S \to S$ by $f((A, v)) = (QA, v)$. Then $f^p((A, v)) = (Q^p A, v) = (A, v)$. There are $\binom{n}{k}\binom{r}{s}$ fixed points of f, consisting of those pairs (A, v) such that the first row of A contains exactly k ones, the other rows of A are the same as the first row, and v contains exactly s ones in its r positions. Note that if $s > r$, then $\binom{r}{s} = 0$. Thus, by our lemma, $\binom{pn+r}{pk+s} \equiv \binom{n}{k}\binom{r}{s} \pmod{p}$, as desired. ∎

For another fine combinatorial proof of Lucas's theorem, see [3].

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Department of Computer Science, Rochester Institute of Technology, Rochester, NY 14623-5608
anderson@cs.rit.edu

Department of Mathematics, Harvey Mudd College, Claremont, CA 91711
benjamin@hmc.edu

Department of Mathematics, University of Wisconsin, Madison, WI 53706
rouse@math.wisc.edu