NOTE

Composite Fermions and Integer Partitions

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We utilize the KOH theorem to prove the unimodality of integer partitions with at most a parts, all parts less than or equal to b, that are required to contain either repeated or consecutive parts. We connect this result to an open question in quantum physics relating the number of distinct total angular momentum multiplets of a system of N fermions, each with angular momentum ℓ, to those of a system in which each Fermion has angular momentum $\ell^* = \ell - N + 1$. © 2001 Academic Press
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1. INTRODUCTION

Mathematics and physics have a symbiotic relationship. Insights in one field often lead to advances in the other. For example, the proof of a purely mathematical construct, the alternating sign matrix theorem, was only made possible with the tools developed by physicists studying the statistical mechanics of square ice [2]. In this paper, tools from partition theory (specifically, recent combinatorial proofs due to O'Hara and Zeilberger

[3, 4, 12, 13]) make it possible to answer an open question of quantum physics involving the composite Fermion model of the fractional quantum Hall effect.

The study of electronic properties of quasi-two-dimensional systems under high magnetic fields and low temperatures has been a fruitful area of investigation garnering the 1985 and 1998 Nobel Prizes in physics for the discovery and explanation of the integral [10] and fractional [6, 9] quantum Hall effects. For completeness, we provide a brief description of both of these effects, though a true physical understanding is not required to appreciate the partition results. Mathematicians may skip the physics presented in the next two paragraphs.

The integral quantum Hall effect results from the highly degenerate nature of the single particle energy levels $\varepsilon_n = (n + \frac{1}{2}) \hbar \omega_c$, where n is a non-negative integer and the cyclotron energy $\hbar \omega_c$ is proportional to the applied magnetic field B_0. Each level contains $N_\phi = A B_0 / \phi_0$ single particle states, where A is the area of the sample and $\phi_0 = \hbar c / e$ is the quantum of magnetic flux. Each particle state can be occupied by a single electron or can be empty. When the ratio ν of the number of electrons N to N_ϕ is an integer, an energy gap $\hbar \omega_c$ separates the ground state from the nearest excited state. This gap causes the integral quantum Hall effect. When ν takes on a value corresponding to an odd denominator fraction (e.g., $\nu = \frac{1}{3}, \frac{1}{5}, \frac{2}{3}, \frac{2}{5}, ...$) the fractional quantum Hall (FQH) effect is found. The energy gap causing the FQH effect results from the Coulomb interaction between the electrons in the highly degenerate single particle states of the partially filled energy level.

A simple intuitive understanding of the FQH effect has been obtained from the composite Fermion (CF) model [5]. An electron constrained to move on the surface of a sphere in the presence of a radial magnetic field of constant magnitude can be described by an eigenvector $|\ell, m\rangle$, where ℓ is the angular momentum of an individual electron and m is its projection onto a given direction. According to elementary quantum mechanics the allowed values of m belong to the set $\mathcal{M}_\ell = \{ -\ell, -\ell + 1, ..., \ell - 1, \ell \}$ where for the present case ℓ is either an integer or half integer, i.e., $2\ell \in \mathbb{Z}$. Many electron eigenfunctions can be constructed from antisymmetrized products of single particle eigenfunctions $\prod_{i=1}^{N} |\ell, m_i\rangle$ in which every m_i in a given product must be a different member of the set $\mathcal{M}_\ell$. We represent such a state by $\mathbf{m} = (m_1, m_2, ..., m_N)$ where $\ell \geqslant m_1 > m_2 > \cdots > m_N \geqslant -\ell$ and refer to it as an M-state if $\sum_{i=1}^{N} m_i = M$. From the $\binom{2\ell+1}{N}$ distinct products obtained by choosing N different values of m_i from the $2\ell + 1$ members of $\mathcal{M}_\ell$, we can form linear combinations $|L, M, \alpha\rangle$ which are eigenfunctions of the total angular momentum operator $\hat{L} = \sum_{i=1}^{N} \hat{\ell}_i$ (where $\hat{\ell}_i$ is the angular momentum operator of the ith particle of angular momentum ℓ) and of $M = \sum_{i=1}^{N} m_i$. The label α is used to distinguish

distinct angular momentum multiplets (the $2L+1$ states $|L, M\rangle$ with M belonging to $\mathcal{M}_L$ are referred to as a *multiplet of angular momentum L*) with the same value of L. Let $f_\ell(N, M)$ count the number of M-states using N Fermions of angular momentum ℓ. The *number of distinct multiplets of angular momentum L* is defined as

$$g_\ell(N, L) = \begin{cases} f_\ell(N, L) - f_\ell(N, L+1) & \text{for } L \geqslant 0 \\ f_\ell(N, L+1) - f_\ell(N, L) & \text{for } L < 0. \end{cases} \quad (1)$$

For the non-physicist reader, we now summarize the essential ideas. A *fermion of angular momentum* ℓ is a particle that takes on a value from the set $\mathcal{M}_\ell = \{-\ell, -\ell+1, ..., \ell\}$, where the number 2ℓ is an integer. In an N-fermion system of angular momentum ℓ, all fermions will have different values from $\mathcal{M}_\ell$. We represent such a system by $\mathbf{m} = (m_1, m_2, ..., m_N)$ where $\ell \geqslant m_1 > m_2 > \cdots > m_N \geqslant -\ell$ and call it an *M-state* if $\sum_{i=1}^{N} m_i = M$. We let $f_\ell(N, M)$ count the number of M-states using N fermions of angular momentum ℓ. Notice $f_\ell(N, M) = f_\ell(N, -M)$. In an N-fermion system where each particle has angular momentum ℓ, the *number of distinct multiplets of (total) angular momentum L*, denoted $g_\ell(N, L)$, is the difference between the number of M states when M equals L and $L+1$, as defined in Eq. (1).

In this paper we show $g_\ell(N, L) \geqslant 0$ and establish the conjecture of Quinn and Wójs [8] that the number of distinct multiplets for N Fermions of angular momentum ℓ is greater than or equal to the number of distinct multiplets for N Fermions of angular momentum $\ell - N + 1$. In other words, for all L,

$$g_\ell(N, L) \geqslant g_{\ell-N+1}(N, L).$$

This conjecture was useful in understanding why the mean field composite Fermion picture correctly predicted the lowest energy band of multiplets for any value of the applied magnetic field, because the transformation from N electrons to mean field composite Fermions [7, 8] involves changing the angular momentum from ℓ to $\ell^* = \ell - N + 1$.

We illustrate these ideas through a small example. For a given ℓ and N, define the function $G_\ell(N) = \sum g_\ell(N, M) q^M$. Table I displays $G_\ell(3)$ for different angular momenta ℓ in 3 particle systems. The term $2q^3$ in $G_4(3)$ means that there are 2 more ways to express 3 as a sum of three distinct integers from -4 to 4 than there are to express 4. Notice that simply decreasing the angular momentum ℓ does not guarantee containment of distinct multiplets. Comparing $G_5(3)$ to $G_4(3)$, containment violations occur for 3-states and 1-states. While 3-states occur at both angular momenta, the number of distinct multiplets when $\ell = 4$ exceeds the number of distinct multiplets when $\ell = 5$. The 1-state is allowed only when $\ell = 4$ and

TABLE I

Calculations of the Number of Distinct Multiplets for a 3 Fermion System
Using Angular Momenta of $\ell = 5, 4, 3, 2, 1$

$$G_\ell(3) = \sum g_\ell(N, M)\, q^M$$

ℓ												
5	q^{12} $+q^{-12}$	$+q^{10}$ $+q^{-10}$	$+q^9$ $+q^{-9}$	$+q^8$ $+q^{-8}$	$+q^7$ $+q^{-7}$	$+2q^6$ $+2q^{-6}$	$+q^5$ $+q^{-5}$	$+2q^4$ $+2q^{-4}$	$+q^3$ $+q^{-3}$	$+q^2$ $+q^{-2}$		$+q^0$
4			q^9 $+q^{-9}$		$+q^7$ $+q^{-7}$	$+q^6$ $+q^{-6}$	$+q^5$ $+q^{-5}$		$+2q^3$ $+2q^{-3}$		$+q^1$ $+q^{-1}$	
3						q^6 $+q^{-6}$		$+q^4$ $+q^{-4}$	$+q^3$ $+q^{-3}$	$+q^2$ $+q^{-2}$		$+q^0$
2									q^3 $+q^{-3}$		$+q^1$ $+q^{-1}$	
1												q^0

not when $\ell = 5$. However, by reducing the angular momentum by $N - 1 = 2$ we see that term-by-term, $G_5(3) \geqslant G_3(3) \geqslant G_1(3)$ and $G_4(3) \geqslant G_2(3)$.

Studying the relationships between distinct total angular multiplets effectively reduces to a problem of restricted integer partitions. Let $\mathscr{P}_a(b, c)$ denote the set of partitions of c into at most b positive parts, with all parts less than or equal to a, and let $p_a(b, c) = |\mathscr{P}_a(b, c)|$. For $a, b > 0$, the generating function $\sum p_a(b, c)\, q^c$ is precisely the Gaussian polynomial [1]

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q = \frac{(1 - q^{a+b})(1 - q^{a+b-1}) \cdots (1 - q^{b+1})}{(1 - q^a)(1 - q^{a-1}) \cdots (1 - q)}.$$

The coefficients of this polynomial are unimodal and symmetric about $ab/2$. Many elegant proofs of this fact have been discovered (see [12]), but it was not until 1990 that a direct combinatorial proof was established by Kathy O'Hara [4]. Subsequent work by Zeilberger [12, 13] distilled O'Hara's combinatorics into the powerful KOH theorem. We utilize the KOH theorem to prove the unimodality of integer partitions with at most a parts, all parts less than or equal to b, that are required to contain either repeated or consecutive parts. This further restriction will be the key to unlocking the Fermion conjecture.

2. RESTRICTED INTEGER PARTITIONS AND UNIMODALITY

We represent a partition in $\mathscr{P}_a(b, c)$ by a b-tuple $\mathbf{y} = (y_1, y_2, ..., y_b)$ where $a \geqslant y_1 \geqslant y_2 \geqslant \cdots \geqslant y_b \geqslant 0$ and $\sum_{i=1}^b y_i = c$. We say $\mathbf{y}$ has *repeated or*

consecutive parts if $y_{i+1} = y_i$ or $y_i - 1$ for some $1 \leqslant i < b$. Notice that a partition y with fewer than $b - 1$ positive parts falls in this category since $y_{b-1} = y_b = 0$. Let $r_a(b, c)$ count the number of partitions in $\mathscr{P}_a(b, c)$ that have repeated or consecutive parts. We wish to show that for $a, b > 0$, the coefficients of the generating function $R(a, b; q) = \sum r_a(b, c) q^c$ are unimodal and symmetric about $ab/2$.

Observe that the subset of partitions in $\mathscr{P}_a(b, c)$ *without* repeated or consecutive parts can be put into one-to-one correspondence with $\mathscr{P}_{a-2b+2}(b, c - b(b-1))$ via the bijection

$$(y_1, y_2, ..., y_{b-1}, y_b) \leftrightarrow (y_1 - 2(b-1), y_2 - 2(b-2), ..., y_{b-1} - 2, y_b).$$

Thus

$$R(a, b; q) = \sum (p_a(b, c) - p_{a-2b+2}(b, c - b^2 + b)) q^c$$

$$= \begin{bmatrix} a+b \\ b \end{bmatrix}_q - q^{b^2 - b} \begin{bmatrix} a-b+2 \\ b \end{bmatrix}_q.$$

The fact that $\begin{bmatrix} a+b \\ b \end{bmatrix}_q$ and $q^{b^2-b} \begin{bmatrix} a-b+2 \\ b \end{bmatrix}_q$ are both unimodal and symmetric about $ab/2$ does not guarantee that their difference has the same feature. Fortunately, we can appeal to the KOH Theorem, which we state here using the formulation from Bressoud [3].

THEOREM 2.1 [KOH].

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q = \sum_{y \in \mathscr{P}_b(b, b)} q^{y_1^2 + \cdots + y_b^2 - b} \prod_{i=1}^{b} \begin{bmatrix} (a+2) i - Y_{i-1} - Y_{i+1} \\ y_i - y_{i+1} \end{bmatrix}_q,$$

where $y_{b+1} = 0$, $Y_0 = 0$, *and* $Y_i = y_1 + \cdots + y_i$ *for* $i = 1, ..., b + 1$.

As shown in [13], all summands of the KOH identity are unimodal and symmetric about $ab/2$ and consequently so is their sum. The partition $(b, 0, ..., 0) \in \mathscr{P}_b(b, b)$ contributes the term $q^{b^2-b} \begin{bmatrix} (a+2)-b \\ b \end{bmatrix}_q$ to the sum. Consequently $R(a, b; q)$ is the sum of the remaining terms and hence is also unimodal and symmetric about $ab/2$, as desired.

Interestingly, the generating function whose coefficients enumerate restricted partitions with repeated parts need not be unimodal, nor will the generating function that enumerates restricted partitions with consecutive parts. For example when $a = b = 2$, the generating functions are $1 + q^2 + q^4$ and $q + q^3$, respectively.

3. CONNECTIONS TO QUANTUM PHYSICS

Returning to our original problem, let $\mathscr{F}_\ell(N, M)$ denote the set of M-states using N Fermions of angular momentum ℓ. For $\mathbf{m} \in \mathscr{F}_\ell(N, M)$ the bijection

$$(m_1, m_2, ..., m_{N-1}, m_N)$$
$$\leftrightarrow (m_1 + \ell - N + 1, m_2 + \ell - N + 2, ..., m_{N-1} + \ell - 1, m_N + \ell)$$

establishes a one-to-one correspondence between $\mathscr{F}_\ell(N, M)$ and $\mathscr{P}_{2\ell-N+1}(N, M + N\ell - N(N-1)/2)$. This correspondence is valid even when ℓ is a half-integer. Hence we have the generating function

$$\sum_{M=N(N-1)/2-N\ell}^{N\ell-N(N-1)/2} f_\ell(N, M) \, q^M = q^{N(N-1)/2 - N\ell} \begin{bmatrix} 2\ell+1 \\ N \end{bmatrix}_q. \tag{2}$$

Since $\begin{bmatrix} a+b \\ b \end{bmatrix}_q$ is unimodal and symmetric about $ab/2$, it follows that $\sum f_\ell(N, M) q^M$ is unimodal and symmetric about 0. Thus for $M < 0$, $f_\ell(N, M) \leqslant f_\ell(N, M+1)$ and for $M \geqslant 0$, $f_\ell(N, M) \geqslant f_\ell(N, M+1)$; so $g_\ell(N, M) \geqslant 0$.

We are now ready to establish the Quinn and Wójs conjecture.

THEOREM 3.1. *The number of distinct multiplets for N Fermions of total angular momentum ℓ is greater than or equal to the number of distinct multiplets for N Fermions of angular momentum $\ell - N + 1$. In other words, for all M,*

$$g_\ell(N, M) \geqslant g_{\ell-N+1}(N, M). \tag{3}$$

Proof. This is equivalent to proving for $N, \ell > 0$, and for $M \geqslant 0$, that

$$f_\ell(N, M) - f_\ell(N, M+1) \geqslant f_{\ell-N+1}(N, M) - f_{\ell-N+1}(N, M+1),$$

or equivalently,

$$f_\ell(N, M) - f_{\ell-N+1}(N, M) \geqslant f_\ell(N, M+1) - f_{\ell-N+1}(N, M+1).$$

Hence it suffices to show that for fixed N, ℓ, $S(\ell, N; q) = \sum (f_\ell(N, M) - f_{\ell-N+1}(N, M)) q^M$ is unimodal and symmetric about $M = 0$. From Eq. (2), we have

$$S(\ell, N; q) = q^{N(N-1)/2 - N\ell} \begin{bmatrix} 2\ell+1 \\ N \end{bmatrix}_q - q^{N(N-1)/2 - N(\ell-N+1)} \begin{bmatrix} 2\ell-2N+3 \\ N \end{bmatrix}_q$$

$$= q^{N(N-1)/2 - N\ell} \left\{ \begin{bmatrix} 2\ell+1 \\ N \end{bmatrix}_q - q^{N^2-N} \begin{bmatrix} 2\ell-2N+3 \\ N \end{bmatrix}_q \right\}.$$

Substituting $a = 2\ell - N + 1$, $b = N$ reduces the braced factor to $R(a, b; q)$, which from Section 2 is unimodal and symmetric about $ab/2 = N\ell - N(N-1)/2$. Thus, $S(\ell, N; q)$ is unimodal and symmetric about 0, as desired. ∎

Repeatedly reducing the angular momentum by $N - 1$ gives the immediate corollary:

COROLLARY 3.1. *For a positive integer k, the number of distinct multiplets for N Fermions of total angular momentum ℓ is greater than or equal to the number of distinct multiplets for N Fermions of total angular momentum $\ell - k(N - 1)$. In other words, for all M,*

$$g_\ell(N, M) \geqslant g_{\ell - k(N-1)}(N, M).$$

The preceding analysis can also be applied to quasi-particles called *bosons*. A *boson of angular momentum ℓ* takes on a value from the set $\mathcal{M}_\ell$ where ℓ is again an integer or half-integer. An N-boson system of angular momentum ℓ differs from a N-Fermion system of angular momentum ℓ in that bosons may take on repeated values from $\mathcal{M}_\ell$. We represent such a system by $\mathbf{z} = (z_1, z_2, ..., z_N)$ where $\ell \geqslant z_1 \geqslant z_2 \geqslant \cdots \geqslant z_N \geqslant -\ell$. Here M-states and the number of distinct multiplets of a given total angular momentum are defined as before. There is a one-to-one correspondence between N-Fermion systems with angular momentum ℓ and N-boson systems with angular momentum $\ell - \frac{1}{2}(N - 1)$, via the bijection

$$(m_1, m_2, ..., m_{N-1}, m_N)$$

$$\updownarrow$$

$$(m_1 - \tfrac{1}{2}(N - 1), m_2 - \tfrac{1}{2}(N - 1) + 1, ..., m_{N-1}$$

$$- \tfrac{1}{2}(N - 1) + (N - 2), m_N - \tfrac{1}{2}(N - 1) + (N - 1)).$$

So the number of distinct multiplets for N Fermions each with angular momentum ℓ exactly equals the number of distinct multiplets for N bosons each with angular momentum $\ell - \frac{1}{2}(N - 1)$.[1] Therefore, analogous results for bosons hold.

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[1] The same result was arrived at independently by B. Wybourne [11] using group decompositions.

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REFERENCES

1. G. E. Andrews, "The Theory of Partitions," Addison–Wesley, Reading, MA, 1976.
2. D. M. Bressoud, "Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture," Math. Assoc. of America, Washington, DC, 1999.
3. D. M. Bressoud, In the land of OZ, in "q-Series and Partitions" (D. Stanton, Ed.), IMA Volumes in Mathematics and Its Applications, Springer-Verlag, New York, 1989.
4. K. O'Hara, Unimodality of the Gaussian coefficient: A constructive proof, *J. Combin. Theory Ser. A* **53** (1990), 29–52.
5. J. K. Jain, Composite Fermion approach for the fractional quantum Hall effect, *Phys. Rev. Lett.* **63** (1989), 199–202.
6. R. B. Laughlin, Anomalous quantum Hall effect: An incompressible quantum fluid with fractionally charged excitations, *Phys. Rev. Lett.* **50** (1983), 1395–1398.
7. J. J. Quinn and A. Wójs, Composite fermions in fractional quantum Hall effect: Essential role of the pseudopotential, *Phys. E* **6** (2000), 1–4.
8. J. J. Quinn and A. Wójs, Composite fermions in fractional quantum Hall systems, *J. Phys.* **12** (2000), R265–R298.
9. D. C. Tsui, H. L. Störmer, and A. C. Gossard, Two dimensional magnetotransport in the extreme quantum limit, *Phys. Rev. Lett.* **48** (1982), 1559–1562.
10. K. von Klitzing, G. Dorda, and M. Pepper, New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance, *Phys. Rev. Lett.* **45** (1980), 494–497.
11. B. Wybourne, private communication, June 5, 2000.
12. D. Zeilberger, A one-line high school proof of the unimodality of the Gaussian polynomials $\begin{bmatrix} n \\ k \end{bmatrix}$ for $k < 20$, in "q-Series and Partitions" (D. Stanton, Ed.), IMA Volumes in Mathematics and Its Applications, Springer-Verlag, New York, 1989.
13. D. Zeilberger, Kathy O'Hara's constructive proof of the unimodality of the Gaussian polynomials, *Amer. Math. Monthly* **96** (1989), 590–602.