One

What is a Partial Differential Equation?

1.1 What is a Partial Differential Equation?

You've probably seen an ordinary differential equation (ODE) before. For example, the pendulum equation

$$\frac{d^2\Theta}{dt^2} + \frac{g}{L}\sin\Theta = 0 \tag{1.1}$$

describes the angle $\Theta(t)$ a pendulum makes with the vertical as a function of time t. Here g (the acceleration due to gravity) and L (the length of the pendulum) are constants, t is the *independent variable*, and Θ is the *dependent variable*. This is an ODE because there is only one independent variable, t.

If there is more than one independent variable in a problem, we are likely to encounter a partial differential equation (PDE). A PDE relates the partial derivatives of a function of two or more independent variables. For example, Laplace's equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0, \tag{1.2}$$

relating the partial derivatives of a function $\Phi(x, y)$, arises in many places in mathematics and physics. For simplicity, we will use subscript notation for partial derivatives, so this equation can also be written $\Phi_{xx} + \Phi_{yy} = 0$.

A function is a *solution* to a PDE if it satisfies the equation in addition to any side conditions. Side conditions often consist of prescribed values the solution must take at an initial time, or at certain points in its domain.

A given PDE (with side conditions) could have no solution or many solutions, so the question of whether a solution *exists* and when it is *unique* is important to consider.

Exercise 1.1. Show that $\Phi_1 = x$ and $\Phi_2 = x^2 - y^2$ are solutions to Laplace's equation (1.2). Can you combine them to create a new solutions?

Exercise 1.2. Let c be a constant and f be any differentiable function. Show that $u(x,t) = f(x - ct)$ is a solution to the *transport equation*

$$u_t + cu_x = 0. \tag{1.3}$$

Find a solution $u(x,t)$ to the transport equation that satisfies the initial condition $u(x,0) = \sin x$.

Exercise 1.3. Show that

$$Z(x,y) = \ln\left(\frac{\sin(y)}{\sin(x)}\right)$$

is a solution to the *minimal surface equation*

$$(1 + Z_y^2)Z_{xx} - 2Z_xZ_yZ_{xy} + (1 + Z_x^2)Z_{yy} = 0 \tag{1.4}$$

in the region $0 < x < \pi$, $0 < y < \pi$. What happens on the boundary of this region? Suppose we consider a constant multiple of $Z(x,y)$ – is it still a solution of the PDE?

Exercise 1.4. Show that the function

$$h(x,t) = 2\alpha^2 \text{sech}\left(\alpha(x - 4\alpha^2 t)\right)$$

is a solution to the Korteweg-deVries (KdV) equation,

$$h_t + 6hh_x = h_{xxx}. \tag{1.5}$$

This solution is known as a *soliton* and can be thought of a model of a tsunami propagating in the ocean. A computer algebra system such as MAPLE or MATHEMATICA may be helpful with the algebra.

1.2 Classifying PDEs: Order, Linear vs. Nonlinear

When studying ODEs we classify them in an attempt to group similar equations which might share certain properties, such as methods of solution. We classify PDEs in a similar way. Two important properties for classification are order and linearity. The *order* of a differential equation is the highest derivative that appears in the equation. So, the transport equation (1.3) is first order, and Laplace's Equation (1.2) and the minimal surface equation (1.4) are both second order. Note that mixed derivatives also count, so for example $u_{xxy} = 0$ would be third order.

Much of this book will concentrate on *linear* PDE. These are the most commonly encountered PDE in mathematics and physics and enjoy the greatest availability of solution techniques. Roughly, a PDE is linear if all terms involving the dependent variables only appear to degree one. There is no restriction on how the independent variables can appear. To give a more precise definition of linear, we first define a *linear operator*.

Definition 1.6. A *linear operator* L is a mapping that takes functions u to functions $L[u]$ in such a way that for any two functions u_1 and u_2

$$L[c_1u_1 + c_2u_2] = c_1L[u_1] + c_2L[u_2]$$

for any constants c_1 and c_2. In words, *a linear operator respects linear combinations.*

For example, the Laplacian operator in two dimensions is defined as

$$L = \partial_{xx} + \partial_{yy}.$$

L is written using *operator notation*, which allows us to avoid naming the function L is operating on. We could also just write

$$L[\Phi] \equiv \Phi_{xx} + \Phi_{yy}.$$

A quick computation shows that $L[c_1\Phi + c_2\Psi] = c_1L[\Phi] + c_2L[\Psi]$, so L is a linear operator.

We can now define a *linear partial differential equation* as a PDE of the form $L[\Phi] = f$ where f is a known function of the independent variables. A *nonlinear* PDE as any equation that cannot be written in this form (i.e. a PDE that is not linear).

We should also define the *domain* and *codomain* (or *range*) of the operator. The domain is the space of functions which an operator acts on. So, for example, we might consider the domain of the Laplace operator as twice differentiable functions of x and y. The codomain is a space of functions that contains $L[u]$ where L is a linear operator and u is in the domain of L. For the Laplace operator we might consider the codomain to be just the set of functions of two variables, x and y. Note that for the purposes of this text, unless stated otherwise, we will assume that functions can be differentiate as many times as needed to do a particular calculation.

Exercise 1.5. Convince yourself that any linear combination of linear operators (with the same domain and codomain) is also a linear operator.

Example 1.1. Find the most general first-order linear PDE for $u(x,t)$.
Solution: Because the PDE is first-order, it can depend only on the independent variables (x,t), the dependent variable u and its first derivatives u_t and u_x. Also, for an operator $L[u]$ to be linear it must be linear in u, u_t and u_x. Therefore the most general first-order linear operator is

$$L[u] \equiv a(x,t)u_t + b(x,t)u_x + c(x,t)u$$

or in operator notation

$$L \equiv a(x,t)\partial_t + b(x,t)\partial_x + c(x,t)$$

where a, b and c are arbitrary functions. The most general first-order linear PDE for $u(x,t)$ is

$$a(x,t)u_t + b(x,t)u_x + c(x,t)u = f(x,t), \tag{1.7}$$

or in operator notation

$$L[u] = f(x,t).$$

Strictly speaking, we should also specify that a, b are not both identically zero for the equation to be first-order. ■

Exercise 1.6. Which of Laplace's equation (1.2), the convection equation (1.3), the minimal surface equation (1.4) and the Korteweg-deVries equation (1.5) are linear?

Exercise 1.7. Write down the most general constant coefficient linear second-order equation for $\Phi(x,y)$.

1.3 Homogeneous PDEs and Vector Spaces of Solutions

Linear equations can further be classified as *homogeneous*, which take the form $L[u] = 0$ and *inhomogeneous* which take the form $L[u] = f$ for some non-zero function of the independent variables, f. So the transport equation

$$u_t + cu_x = 0$$

is homogeneous, but its cousin, the general first-order linear PDE for $u(x,t)$, is inhomogeneous

$$a(x,t)u_t + b(x,t)u_x + c(x,t)u = f(x,t),$$

unless $f(x,t) = 0$.

By the definition if two solutions, say u_1 and u_2, satisfy a linear homogeneous PDE, that any linear combination of them

$$u = c_1u_1 + c_2u_2 \tag{1.8}$$

is also a solution because

$$L[u] = L[c_1u_1 + c_2u_2] = c_1L[u_1] + c_2L[u_2] = 0. \tag{1.9}$$

So, for example, since

$$\Phi_1 = x^2 - y^2 \qquad \Phi_2 = x$$

both satisfy Laplace's equation, $\Phi_{xx} + \Phi_{yy} = 0$, so does any linear combination of them

$$\Phi = c_1\Phi_1 + c_2\Phi_2 = c_1(x^2 - y^2) + c_2x.$$

This property is extremely useful for constructing solutions which satisfy certain initial conditions and boundary conditions. Mathematicians refer to a set of functions closed under addition and scalar multiplication as a *vector space*. The set of solutions to a linear homogeneous PDE form a vector space, a fact that is incredibly useful for constructing solutions that satisfy particular initial conditions or boundary conditions. Physicists, mathematicians and engineers often refer to the idea *superposition* of solutions, whereby a solution can be represented as a sum of simpler solutions; an example of this is a vibrating string that oscillates with some superposition of its fundamental frequency and its harmonics or overtones. The most common origin of superposed solutions in physics is the underlying linearity of the governing equations.

1.4 Challenge Problems for Lecture 1

Problem 1.1. Classify the follow differential equations as ODEs or PDEs, linear or nonlinear, and determine their order. For the linear equations, determine whether or not they are homogeneous.

(a) The *diffusion equation* for $h(x,t)$:

$$h_t = Dh_{xx}$$

(b) The *wave equation* for $w(x,t)$:

$$w_{tt} = c^2 w_{xx}$$

(c) The *thin film equation* for $h(x,t)$:

$$h_t = -(hh_{xxx})_x$$

(d) The *forced harmonic oscillator* for $y(t)$:

$$y_{tt} + \omega^2 y = F\cos(\Omega t)$$

(e) The *Poisson Equation* for the electric potential $\Phi(x,y,z)$:

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 4\pi\rho(x,y,z)$$

where $\rho(x,y,z)$ is a known charge density.

(f) *Burger's equation* for $h(x,t)$:

$$h_t + hh_x = \nu h_{xx}$$

Problem 1.2. Show that the helicoid

$$Z(x,y) = \tan^{-1}(y/x)$$

satisfies the minimal surface equation,

$$(1+Z_y^2)Z_{xx} - 2Z_xZ_yZ_{xy} + (1+Z_x^2)Z_{yy}$$

MAPLE or MATHEMATICA may be helpful with the algebra.