Ten

The Wave Equation

Parts of this chapter are based on notes by M. Vajiac & J. Tolosa

We are all familiar with the oscillation of a string under tension. A violin, a guitar and a piano all work in basically the same way. A string of length L oscillates with a small vertical displacement, which we call $u(x,t)$ where x which is between 0 and L represents the distance along the undisplaced string.

The oscillations can be modeled by a one-dimensional wave equation has the form:

$$\mathbb{DE} : u_{tt} = c^2 u_{xx} \qquad 0 < x < L, \quad t > 0 \tag{10.1}$$

where u_{tt} can be thought of as the (non-dimensional) acceleration and u_{xx} is the (non-dimensional) restoring force proportional to the curvature of the string. The speed $c = \sqrt{T/\rho}$ where T is the tension (with units of Force) and ρ is the (line) density of the string (with units of Mass/Length). Intuitively, if you increase the tension, the speed of the string and oscillation frequency increases, whereas if you increase the density, the oscillation frequency decreases.

We need to specify the position of the ends of the string. Most commonly one fixes both ends at zero displacement,

$$\mathbb{BC} : u(0,t) = u(L,t) = 0 \qquad t > 0,$$

but one can imagine more exotic boundary conditions where these positions are specified as a function of time. We also need to specify the strings initial velocity and acceleration,

$$\mathbb{IC} : u(x,0) = f(x), \quad u_t(x,0) = g(x) \qquad 0 < x < L.$$

Intuitively, since the wave equation gives the acceleration at each point we need to specify the initial position and velocity. Another way to think of this is by analogy with ordinary differential equations; this equation is second-order in time, so we need to specify the initial position and its derivative.

We can also consider the case where the string is *pushed* with an external force $h(x,t)$, which correspond to plucking or strumming.

$$u_{tt} = c^2 u_{xx} + h(x,t) \qquad 0 < x < L, \quad t > 0,$$

which yields the inhomogeneous *forced* wave equation.

Another important case is the *damped* wave equation; a plucked guitar string does not oscillate forever. this is due to air resistance and a simple model of this system is to include a linear damping term,

$$u_{tt} + k u_t = c^2 u_{xx} \qquad 0 < x < L, \quad t > 0,$$

where k is the damping constant.

10.1 DIRICHLET PROBLEM AND SEPARATION OF VARIABLES

The simplest model of the plucked string is the *Dirichlet Problem* for the wave equation:

THE DIRICHLET PROBLEM FOR THE WAVE EQUATION

$$\begin{array}{rccl} \mathbb{DE}: & \multicolumn{2}{c}{u_{tt} = c^2 u_{xx},} & 0 < x < L, t > 0 \\ \mathbb{BC}: & u(0,t) = 0, & u(L,t) = 0, & t > 0 \\ \mathbb{IC}: & u(x,0) = f(x), & u_t(x,0) = g(x) & 0 < x < L \end{array}$$

As you have seen in Chapter ** for the diffusion equation, the method of separation of variables yields a set of solutions for PDEs that (hopefully) form a basis for an arbitrary initial condition.

First, let us present the solution:

Theorem 10.1. *The Dirichlet problem for the wave equation:*

$$\begin{array}{rccl} \mathbb{DE}: & \multicolumn{2}{c}{u_{tt} = c^2 u_{xx},} & 0 < x < L, t > 0 \\ \mathbb{BC}: & u(0,t) = 0, & u(L,t) = 0, & t > 0 \\ \mathbb{IC}: & u(x,0) = f(x), & u_t(x,0) = g(x) & 0 < x < L \end{array}$$

has a solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(\frac{n\pi ct}{L}) + B_n \frac{L}{n\pi c} \sin(\frac{n\pi ct}{L}) \right] \sin(\frac{n\pi x}{L}),$$

where:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L}),$$

and

$$g(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L}).$$

where the coefficients A_n *and* B_n *are the Fourier coefficients of the initial velocity and displacement respectively:*

$$A_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L})\, dx \qquad B_n = \frac{2}{L} \int_0^L g(x) \sin(\frac{n\pi x}{L})\, dx.$$

If $f(x)$ *and* $g(x)$ *are continuous, piecewise differentiable and vanish at* $x = 0$ *and* $x = L$*, this solution converges uniformly.*

We will derive this solution via the method of separation of variables, we look for solutions $u(x,t)$ that are a product of two function, one that depends only on the variable t and a second function that depends only on the variable x.

Let $u(x,t) = X(x)T(t)$ and substitute in the equation $u_{tt} = c^2 u_{xx}$, to obtain:

$$X(x)T''(t) = c^2 X''(x)T(t),$$

or dividing by $c^2 X(x)T(t)$, we obtain:

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)},$$

thus the equality is one of functions of different variables, so both quotients have to be constant,

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

Here λ is the *separation constant*. If we also separate the boundary conditions,

$$u(0,t) = X(0)T(t) = 0 \qquad u(0,t) = X(L)T(t) = 0$$

we see that either $T(t) = 0$ which yields the trivial solution $u(x,t) = 0$ or $X(0) = X(L) = 0$.

This yields the boundary value problem of Sturm-Liouville type

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = X(L) = 0$$

which we have examined previously. This eigenvalue problem has a countable number of positive eigenfuctions,

$$X(x) = X_n(x) \equiv \sin(\frac{n\pi x}{L}), \qquad \lambda_n = \mu_n^2 \equiv \left(\frac{n\pi}{L}\right)^2$$

where X_n is arbitrary up to a constant multiple.

We can now solve the T equation for every eigenvalue $\lambda_n = \mu_n^2$,

$$T''(t) + c^2\mu_n^2 T(t) = 0,$$

which has solution

$$T(t) = T_n(t) = a_n\cos(\mu_n ct) + b_n\sin(\mu_n ct) = a_n\cos(\frac{n\pi ct}{L}) + b_n\sin(\frac{n\pi ct}{L})\ .$$

we can define now a solution

$$u(x,t) = u_n(x,t) \equiv X_n(x)T_n(t) = \left[a_n\cos(\frac{n\pi ct}{L}) + b_n\sin(\frac{n\pi ct}{L})\right]\sin(\frac{n\pi x}{L})$$

Since the equation is homogeneous, the most general solution is a linear combination of these solutions, namely:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos(\frac{n\pi ct}{L}) + b_n\sin(\frac{n\pi ct}{L})\right] c_n\sin(\frac{n\pi x}{L}).}$$

The only conditions left to check are the initial conditions. Note that for our solution

$$u(x,0) = \sum_{n=1}^{\infty} a_n\sin(\frac{\pi n}{L}x).$$

and comparing this to

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n\sin(\frac{n\pi x}{L})$$

we conclude $\boxed{a_n = A_n}$. Similarly, we note that

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin(\frac{n\pi}{L}x).$$

and comparing this to the initial velocity

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L}),$$

we see that $\boxed{b_n = \frac{L}{n\pi c} B_n}$. This yields our solution

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(\frac{n\pi ct}{L}) + B_n \frac{L}{n\pi c} \sin(\frac{n\pi ct}{L}) \right] \sin(\frac{n\pi x}{L}).}$$

Note that if $f(x)$ and $g(x)$ are continuous and piecewise differentiable this guarantees that the Fourier sine series converges uniformly.

10.2 THE EXAMPLE OF THE PLUCKED STRING

The *plucked string* refers to the initial condition for the Dirichlet problem, where the initial displacement $f(x)$ is a piecewise linear function, the equilibrium position assumed when a string is plucked at a point and that the velocity initially is zero.

$$\begin{aligned} \mathbb{DE}: &\quad u_{tt} = c^2 u_{xx}, && 0 < x < L, t > 0 \\ \mathbb{BC}: &\quad u(0,t) = 0, u(L,t) = 0, && t > 0 \\ \mathbb{IC}: &\quad u(x,0) = f(x), u_t(x,0) = 0 && 0 < x < L \end{aligned}$$

where

$$f(x) = \begin{cases} U\frac{x}{\bar{x}}, & 0 \le x \le \bar{x} \\ U\frac{x-L}{\bar{x}-L}, & \bar{x} \le x \le L \end{cases}$$

Let's find a formal solution to the "plucked string" equation. Clearly $A_n = 0$ and the B_n's are the Fourier sine coefficients of $f(x)$.

$$\begin{aligned} B_n &= \int_0^L f(x) \sin(\frac{n\pi x}{L}) dx \\ &= \frac{2L^2 U}{\pi^2 \bar{x}(L-\bar{x})} \frac{1}{n^2} \sin(\frac{n\pi \bar{x}}{L}). \end{aligned}$$

Now we can write the formal solution to the plucked string equation:

$$u(x,t) = \frac{2L^2U}{\pi^2\bar{x}(L-\bar{x})}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(\frac{n\pi\bar{x}}{L})\cos(\frac{n\pi ct}{L})\sin(\frac{n\pi x}{L}).$$

An amazing fact is that this solution is actually a piecewise linear function, a fact we will prove using the d'Alembert solution to the wave equation.

10.2.1 Musical instruments

Many instruments produce sound by making strings vibrate; such are the harp, the piano, the harpsichord, the guitar, the violin, and others. Strings are kept fixed at the endpoints, but they way the instruments are played create different initial conditions. In instruments like the guitar, the string is plucked; this produces an initial perturbation with no initial velocity. In the piano, on the other hand, the string is hit, which creates an initial velocity but no initial perturbation from the initial position.

The oscillations of the string are described by

$$u(x,t) = \sum_{n=1}^{\infty}[A_n\cos\omega_n t + B_n\sin\omega_n t]\sin\lambda_n x,$$

where

$$\lambda_n = n\frac{\pi}{L} \qquad \text{and} \qquad \omega_n = c\lambda_n = cn\frac{\pi}{L}.$$

The sound we hear is thus a combination of the main harmonic sounds (eigenfunctions)

$$u_n(x,t) = (A_n\cos\omega_n t + B_n\sin\omega_n t)\sin\lambda_n t.$$

The contribution of each particular harmonic is measured by its *energy*, which turns out to be equal to:

$$E_n = \frac{\omega_n^2 M}{4}(A_n^2 + B_n^2),$$

where $M = DL$ is the total mass of the string (recall that D was the density).

For the plucked string, the energy is given by:

$$E_n = \frac{MU^2L^2c^2}{n^2\pi^2\bar{x}^2(L-\bar{x})^2}\sin^2\frac{\pi n\bar{x}}{L}.$$

The energy decreases as n^{-2}, so only the main tone u_1 and a few other harmonics are audible.

On the other hand, if we hit the string with a flat hammer of length 2δ with center at $\bar{x}$ and producing an initial velocity v_0, the energy of the n^{th} harmonic is:

$$E_n = \frac{4MV_0^2}{n^2}\pi^2 \sin^2 \frac{\pi n \bar{x}}{L} \sin^2 \frac{\pi n \delta}{L},$$

and the energy again decreases as n^{-2}. However, if the hammer is sufficiently narrow, letting δ tend to zero (the blade of a knife), we get the model of a string getting an impulse concentrated at a point $\bar{x}$. The corresponding energy is:

$$E_n = \frac{v_0^2}{L} \sin^2 \frac{\pi n \bar{x}}{L}.$$

Thus, for a very narrow hammer the energies of all harmonics are of the same order and the generated sound will be saturated with harmonics. This can be checked experimentally, by hitting a string with the blade of a knife. The sound will have a metallic quality.

Not all harmonics are desirable. The first ones, u_2 up to u_6, sound well together with the main harmonic u_1. However, the 7^{th} and the first harmonics sounding together produce a sense of dissonance.

There are several ways to try to "kill" those harmonics by percussion (as in the piano).

- **The position of the hammer.** The presence of the factor $\sin \frac{\pi n \bar{x}}{L}$ shows that by choosing the center $\bar{x}$ of the hammer at the node of the undesired harmonic we may make it disappear (make the corresponding A_n and B_n be equal to zero). In modern pianos the position of the hammer is chosen near the nodes of the 7^{th} and the 7^{th} harmonics, to "kill" them.

- **The shape of the hammer**. In modern pianos the hammers are not flat, but rather round. One can model this situation by choosing the initial velocity to be, say, a parabola on the interval $[\bar{x} - \delta, \bar{x} + \delta]$, instead of a horizontal line. Older pianos, which had flatter and narrower hammers, produced a more piercing, shrilled sound.

- **The rigidity of the hammer**. If instead of being rigid the hammer is softer the motion is not described by its initial position and velocity but rather by a short-time acting force,

Exercise 10.1. Can you model the percussion strategies above with our solution to the wave equation?

10.3 CHALLENGE PROBLEMS

Problem 10.1. A string of length π is held fixed at both endpoints. Its initial position is $f(x) = \sin(x)$ and its initial velocity is $g(x) = \sin 2x$. Assuming that $c = 1$, find the position of the string $u(x,t)$ for every $x \in [0,\pi]$ and for every $t > 0$. Animate the approximation and draw a $3D$ plot.

Problem 10.2. Solve the following problem for the wave equation:

$$u_{tt} = u_{xx},$$

with boundary conditions

$$u(0,t) = 0, \quad u_x(\pi,t) = 0 \quad \text{for every } t > 0;$$

and initial conditions

$$u(x,0) = \sin(x), \quad u_t(x,0) = 0, \quad \text{for every } x \in [0,\pi]\,.$$

Notice the change in the boundary conditions. This will lead to different eigenvalues and eigenfunctions. Use Maple to animate the solution you found, to draw a $3D$ plot, and to check that the solution satisfies the conditions of the problem.

Problem 10.3. A damped string of length 1 has equation

$$u_{tt} = c^2 u_{xx} - \gamma u_t,$$

where γ is a small damping coefficient. Find the solution $u(x,t)$ assuming that both endpoints are fixed, the initial condition is $x(1-x)$ and the initial velocity is zero. Plot and animate the solution for the case when $c = \frac{1}{4}$ and $\gamma = \frac{1}{5}$.

Problem 10.4. Solve the string equation $u_{tt} = c^2 u_{xx}$ for $L = 1$, with the boundary conditions $u(0,t) = 0$ and $u(1,t) = 1$, with zero initial velocity, assuming that the initial position is

(a) $u(x,0) = 0$,

(b) $u(x,0) = x^2$.

Hint: You cannot use the superposition principle, since the boundary condition at $x = 1$ is not homogeneous. Try a change of coordinates first, $v(x,t) = u(x,t) + h(x)$, where $h(x)$ is a suitable (easy) function that would guarantee that v also satisfies the string equation, now with homogeneous boundary conditions.

Problem 10.5. Solve the equation

$$u_{tt} = c^2 u_{xx} + \sin x \qquad 0 \le x \le \pi, \quad t > 0.$$

With the boundary conditions

$$u(0,t) = u_t(\pi,t) = 0,$$

and the initial conditions

$$u(x,0) = 0. \qquad u_t(x,0) = 0.$$

Hint: Make the change of coordinates $u(x,t) = y(x) + v(x,t)$, where $y(x)$ is a steady solution that satisfies

$$c^2 y'' + \sin x = 0,$$

with $y(0) = y(\pi) = 0$. Find y and then show $v(x,t)$ satisfies a homogeneous wave equation.

Problem 10.6. Solve the wave equation

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < 1, t > 0$$

with the boundary conditions

$$u(0,t) = 0, \qquad u_x(1,t) + u(1,t) = 0 \qquad t > 0$$

which corresponds a string with the left end fixed and the right end being attached to an elastic hinge. Use the initial conditions are

$$u(x,0) = x - \frac{2}{3}x^2, \qquad u_t(x,0) = x.$$

Note. This exercise is hard! The eigenvalues λ_n will be solutions of a transcendental equation.