Eleven

Laplace's Equation in a Disk

If we seek the steady-state distribution of temperature in a two-dimensional region, this leads from the heat equation $u_t = \Delta u$ to the Laplace equation $\Delta u = 0$, since $u_t = 0$ for the steady-state distribution. In Electrostatics, according to Maxwell's equations the electrostatic potential ϕ satisfies the equation $\Delta\phi = -4\pi\rho$, where ρ is the density of the charges. Thus, if there are no electric charges inside the region, the potential will satisfy Laplace's equation $\Delta\phi = 0$. Let's examine this problem in some more detail.

Suppose we have a region, Ω, in $\mathbb{R}^n$ and we specify the temperature $u(\vec{x})$ (where $\vec{x} \in \mathbb{R}^n$) on the boundary of the region, $\partial\Omega$. The equilibrium temperature satisfies Laplace's Equation

$$\mathbb{DE} : \Delta u = 0 \qquad \vec{x} \in \Omega,$$

$$\mathbb{BC} : u = f \qquad \vec{x} \in \partial\Omega.$$

Below we will solve this problem for a disk in two-dimensions using separation of variables. Functions $u(x, y)$ that satisfy Laplace's Equation are called *harmonic* and play a central role in the study of functions of a complex variable.

11.1 The Laplacian in Polar Coordinates

When a problem has rotational symmetry, it is often convenient to change from Cartesian to polar coordinates. It is then useful to know the expression for the Laplacian acting on $u(x, y)$,

$$\Delta u = u_{xx} + u_{yy}$$

in polar coordinates. Recall that

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Using the chain rule,

$$u_x = u_r r_x + u_\theta \theta_x.$$

Let us find r_x and θ_x. Implicit functions help simplify the computations a little bit. Differentiating both sides of

$$r^2 = x^2 + y^2$$

with respect to x, we get

$$2rr_x = 2x,$$

whence $r_x = \frac{x}{r}$. You can find $r_y = \frac{y}{r}$ in a similar way.

To compute θ_x, we can start from the relation

$$y = r\sin\theta.$$

Differentiating both sides with respect to x, we get

$$0 = r_x \sin\theta + r\cos\theta \cdot \theta_x,$$

whence

$$\theta_x = -\frac{r_x \sin\theta}{r\cos\theta} = -\frac{r_x}{r}\tan\theta = -\frac{r_x}{r}\cdot\frac{y}{x}.$$

Substituting here the value of r_x we have just found, we get

$$\theta_x = -\frac{y}{r^2}.$$

You can find $\theta_y = \frac{x}{r^2}$ in a similar way, starting from the equation $x = r\cos\theta$.

Differentiating once again, we can show that

$$r_{xx} = \frac{y^2}{r^3}, \qquad \theta_{xx} = \frac{2xy}{r^4},$$

and find similar values for r_{yy}, θ_{xx}, and θ_{yy}.

This will yield the following expressions for u_{xx} and u_{yy}:

$$u_{xx} = \frac{x^2}{r^2}u_{rr} - \frac{2xy}{r^3}u_{r\theta} + \frac{y^2}{r^4}u_{\theta\theta} + \frac{y^2}{r^3}u_r + \frac{2xy}{r^4}u_\theta;$$
$$u_{yy} = \frac{y^2}{r^2}u_{rr} + \frac{2xy}{r^3}u_{r\theta} + \frac{x^2}{r^4}u_{\theta\theta} + \frac{x^2}{r^3}u_r - \frac{2xy}{r^4}u_\theta.$$

Exercise 11.1. Compute r_{xx}, r_{yy}, θ_{xx}, θ_{yy}, and obtain the above expressions for u_{xx} and u_{yy} using the chain rule.

Adding up both expressions, doing a couple of cancellations and regrouping, we obtain

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

This is the desired expression of the Laplacian in polar coordinates. Sometimes it is convenient to write it in a slightly different way:

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}; \tag{11.1}$$

for the second expression we combined the first two terms, using the product rule.

11.2 Separation of Variables

We will solve the Dirichlet problem for the Laplace equation on a circle, that is, the problem of finding a function that is harmonic inside a circle and has a prescribed value on the boundary; let us call a the radius of the circle which we will assume is centered at the origin,

$$\begin{aligned} \mathbb{DE}: &\quad \Delta u = 0 &\qquad x^2 + y^2 \le a^2 \\ \mathbb{BC}: &\quad u = f &\qquad x^2 + y^2 = a^2, \end{aligned}$$

where f is a known function on the boundary of the region. In polar coordinates

$$\begin{aligned} \mathbb{DE}: &\quad \Delta u(r,\theta) = 0 &\qquad \text{for every } \theta \text{ and for } r < a; \\ \mathbb{BC}: &\quad u(a,\theta) = f(\theta) &\qquad \text{for every } \theta, \end{aligned}$$

where $f(\theta)$ is now a specified periodic function with period 2π, (Periodicity is required because θ represents the polar angle, so $\theta+2\pi$ and θ are measures of the same angle.)

Using the expression(11.1) of the Laplacian in polar coordinates, we can rewrite the problem as

$$\begin{aligned} \mathbb{DE}: &\quad u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} = 0 &\qquad \text{for every } \theta \text{ and for } r < a; \\ \mathbb{BC}: &\quad u(a,\theta) = f(\theta) &\qquad \text{for every } \theta, \end{aligned}$$

Using the method of separation of variables, we will first forget for a moment all about the boundary condition and seek nontrivial solutions (eigenfunctions) of the Laplace equation on the circle as a product:

$$u(r, \theta) = R(r)\Theta(\theta).$$

Substituting into the Laplace equation, we get

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Multiplying both sides by r^2 we can rewrite this as

$$(r^2R'' + rR')\Theta = -\Theta''R.$$

Dividing both sides by $R\Theta$, which is assumed to be nonzero, this becomes

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta}.$$

Now we use an argument you have already heard many times: since the left-hand side depends only on r, and the right-hand side depends only on θ, both sides must be constant, call it λ:

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda. \tag{11.2}$$

The equation for Θ reads

$$\Theta'' + \lambda\Theta = 0,$$

with the additional conditions that

$$\Theta(2\pi) = \Theta(0), \quad \Theta'(2\pi) = \Theta'(0); \tag{11.3}$$

these conditions arise, again, from the fact that θ represents the polar angle.

This is the Fourier Eigenvalue Problem for periodic boundary conditions (5.13) with a period of 2π; the eigenvalues and eigenfunctions are given by (5.14)

$$\begin{aligned} \Theta_0(\theta) &= \frac{1}{2} & \lambda_0 &= 0 & & \\ \Theta_n^c(\theta) &= \cos(n\theta) & \lambda_n &= n^2 & \text{for} \quad & n = 1, 2, 3, \ldots. \\ \Theta_n^s(\theta) &= \sin(n\theta) & \lambda_n &= n^2 & \text{for} \quad & n = 1, 2, 3, \ldots. \end{aligned} \tag{11.4}$$

We can now solve for $R(r)$. Substituting $\lambda = \lambda_n \equiv n^2$ into (11.2) produces

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = n^2; \quad n = 0, 1, 2, \dots \tag{11.5}$$

The equation for R is now $r^2R'' + rR' = n^2R$, or

$$r^2R'' + rR' - n^2R = 0.$$

This is an ordinary differential equation which you probably have seen in your ODE course; it is called an *Euler equation*. The main feature of an Euler equation is that each term contains a power of r that coincides with the order of the derivative of R.

Euler equations always admit particular solutions of the form $R(r) = r^\alpha$, where α is a suitable (possibly fractional) power, which we will now find. Differentiating twice and substituting into the equation, we get

$$\alpha(\alpha - 1)r^\alpha + \alpha r^\alpha - n^2r^\alpha = 0,$$

or, cancelling the common (positive) factor r^α,

$$\alpha(\alpha - 1) + \alpha - n^2 = 0,$$

or just

$$\alpha^2 - n^2 = 0.$$

This is the *characteristic equation* for the Euler equation. In this case, the solutions are particularly easy to find: $\alpha = \pm n$.

Accordingly, we have two particular solutions for our equation: $R_1(r) = r^n$ and $R_2(r) = r^{-n}$. The general solution of the factor $R(r)$ is a linear combination of these:

$$R(r) = P_nr^n + Q_nr^{-n}, n = 0, 1, 2, \dots,$$

where P_n and Q_n are arbitrary constants.

For $n = 0$ the two solutions coincide and are equal to $r^0 = 1 = const$. In this case a second solution of the corresponding equation $r^2R'' + rR' = 0$ is $R(r) = \ln r$.

Exercise 11.2. Find the general solution of the Euler equation $r^2R''+rR' = 0$, corresponding to the case $n = 0$. We already know one particular solution, $R(r) = 1 = const$. To find the second solution, do the change of variables $R' = z$ and solve a first-order separable equation for z.

We will ignore this second solution, $R(r) = \ln r$, because it is not bounded at the center of the circle, when $r = 0$.

If Q_n is nonzero for some positive n, then $R(r)$ will contain the term r^{-n}, which blows up at the center of the circle. We don't want this kind of behavior, so we ask that $Q_n = 0$ for all nonzero n's.

Exercise 11.3. Can you imagine a problem involving $\Delta u = 0$ somewhere and $u = f$ on the boundary of a circle, for which Q_n nonzero could be useful? (Think outside the box!)

We have thus obtained the radial factor:

$$R_n(r) = P_n r^n, \quad n = 0, 1, 2, \dots$$

Multiplying the solutions for $R_n(r)$ and $\Theta_n(\theta)$ (and dropping the arbitrary constants) yields

$$u_0(\theta) = \frac{1}{2}, \quad u_n^c(\theta) = r^n \cos(n\theta), \quad u_n^s(\theta) = r^n \sin(n\theta) \tag{11.6}$$

All these functions satisfy the Laplace equation, that is, all these functions are *harmonic*. In particular, we get the following interesting result: the functions

$$r^n \cos n\theta \qquad \text{and} \qquad r^n \sin n\theta$$

are harmonic for every $n = 1, 2, 3 \dots$

Now we can get back to the original problem (if we still remember what it was!). Each u_n satisfies the Laplace equation. Since this equation is linear and homogeneous, any linear combination of the u_n's will also satisfy the same equation. This is even true for an infinite linear combination, provided the series converges nicely enough. So,

$$\begin{aligned} u(r,\theta) &= A_0 u_0 + \sum_{n=1}^{\infty} A_n u_n^c(r,\theta) + B_n u_n^s(r,\theta) && (11.7) \\ &= \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) && (11.8) \end{aligned}$$

will be a solution of $\Delta u = 0$. Our hope is to be able to pick the coefficients A_n and B_n so as to also satisfy the boundary condition $u(a,\theta) = f(\theta)$.

We have

$$u(a,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta). \tag{11.9}$$

To compare this with $f(\theta)$ it helps to expand this function in Fourier series:

$$\mathbb{FSS}[f(\theta)] = \frac{a_0}{2} + \sum_{n=0}^{\infty}(a_n \cos n\theta + b_n \sin n\theta), \tag{11.10}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\cos n\phi \, d\phi, \quad n = 0, 1, 2 \ldots \tag{11.11}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\sin n\phi \, d\phi, \quad n = 1, 2, 3 \ldots \tag{11.12}$$

The Fourier Series (11.10) is valid under mild assumptions on $f(\phi)$; for example, it holds if f is continuous and piecewise differentiable. For the moment we will assume that everything converges.

We want $u(a, \theta)$ to coincide with $f(\theta)$. Comparing (11.9) and (11.10), we get

$$A_0 = a_0; \qquad a^n A_n = a_n; \qquad a^n B_n = b_n, \quad n = 1, 2, \ldots$$

In turn, using (11.11) and (11.12), this implies that

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\, d\phi;$$

$$A_n = \frac{1}{a^n \pi}\int_{-\pi}^{\pi} f(\phi)\cos n\phi \, d\phi, \quad n = 1, 2, \ldots;$$

$$B_n = \frac{1}{a^n \pi}\int_{-\pi}^{\pi} f(\phi)\sin n\phi \, d\phi, \quad n = 1, 2, \ldots.$$

By substituting all these expressions into (11.8) we obtain a formula for the solution of our problem:

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{r}{a}\right)^n \cos n\theta + b_n \left(\frac{r}{a}\right)^n \sin n\theta. \tag{11.13}$$

Let's use this solution to solve some problems.

Example 11.1. As a concrete example, let us consider the problem of finding the steady-state distribution of temperature of a circular membrane, if the temperature is kept fixed and equal to 1 on half the boundary, and -1 on the other half. Namely, we will consider the problem of finding $u(r,\theta)$ on the circle $r \leq a$ such that

$$\begin{aligned} \mathbb{DE}: \quad & \Delta u(r,\theta) = 0 \qquad r < a \\ \mathbb{BC}: \quad & u(a,\theta) = \begin{cases} -1, & -\pi < \theta < 0, \\ 1, & 0 < \theta < \pi. \end{cases} \end{aligned}$$

Solution: we can read our solution off directly from (11.13) we can say that the solution will look like:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n \cos n\theta + b_n \sin n\theta),$$

where:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \quad n = 0, 1, 2, \ldots \tag{11.14}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta, \quad n = 1, 2, \ldots \tag{11.15}$$

In our case $f(\theta)$ is odd; therefore, all the coefficients a_n will vanish.

As regards the b_n, using the symmetry of the integrand and the interval of integration, we can write

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(\theta) \sin n\theta \, d\theta \\ &= -\frac{2}{n\pi}(\cos n\pi - \cos 0) = \frac{2}{n\pi}\left(1 - (-1)^n\right) \\ &= \begin{cases} 0, & \text{if } n \text{ is even}, \\ \frac{4}{n\pi}, & \text{if } n \text{ is odd}. \end{cases} \end{aligned}$$

Therefore, the solution is

$$\begin{aligned} u(r,\theta) &= \frac{4}{\pi}\left(\frac{r\sin\theta}{a} + \frac{r^3 \sin 3\theta}{3a^3} + \frac{r^5 \sin 5\theta}{5a^5} + \cdots\right) \\ &= \frac{4}{\pi} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^{2n+1} \frac{\sin(2n+1)\theta}{2n+1}. \end{aligned}$$

We leave it as an exercise for the reader to examine the convergence property of the solutions; not surprisingly there is Gibb's phenomena near the boundary at $\theta = 0, \pi$ where the boundary data is discontinuous. ■

Example 11.2. We seek a function harmonic inside the unit circle and equal to (the restriction of) the polynomial $x^3 - y^3$ on the boundary of the unit circle. Let us write the problem in polar coordinates. Since on the unit circle we have $x = \cos\theta$ and $y = \sin\theta$, we must solve the problem

$$\begin{aligned} \mathbb{DE}: \quad & \Delta u(r,\theta) = 0 \quad && r < 1, && (11.16)\\ \mathbb{BC}: \quad & u(1,\theta) = \cos^3\theta - \sin^3\theta \quad && 0 \le \theta \le 2\pi && (11.17) \end{aligned}$$

Solution: To expand the boundary condition, $\cos^3\theta - \sin^3\theta$, in Fourier series, it is perhaps faster to use trigonometric identities. We begin with the identities

$$\begin{aligned} \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta && (11.18)\\ \sin 3\theta &= -\sin 3\theta + 3\cos^2\theta\sin\theta. && (11.19) \end{aligned}$$

which in turn imply that

$$\begin{aligned} \cos^3\theta &= \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta), && (11.20)\\ \sin^3\theta &= \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta). && (11.21) \end{aligned}$$

Identities (11.20,11.21) yield the desired Fourier expansion of the boundary condition:

$$\cos^3\theta - \sin^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta) - \frac{3}{4}\sin\theta + \frac{1}{4}\sin(3\theta). \quad (11.22)$$

By using (11.13) with $r = 1$ we see the solution will look like

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n\cos n\theta + b_n\sin n\theta).$$

At $r = 1$, the solution will be

$$u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta).$$

Comparing this with (11.22) we conclude that

$$a_1 = \frac{3}{4}, \qquad b_1 = -\frac{3}{4}, \qquad a_3 = \frac{1}{4}, \qquad b_3 = \frac{1}{4},$$

and all the other coefficients will be equal to zero. In conclusion, the solution is

$$u(r,\theta) = \frac{3}{4}r(\cos\theta - \sin\theta) + \frac{1}{4}(\cos 3\theta + \sin 3\theta).$$

This solution can be written back in Cartesian coordinates, recalling that $r\cos\theta = x$ and $r\sin\theta = y$. Thus, the first two terms are just $\frac{3}{4}(x-y)$. To get the other two terms in Cartesians, we use again the identities (11.24) and (11.25):

$$r^3\cos 3\theta = r^3(\cos^3\theta - 3\cos\theta\sin^2\theta) = x^3 - 3xy^3;$$

$$r^3\sin 3\theta = r^3(-\sin^3\theta + 3\cos^2\theta\sin\theta) = -y^3 + 3x^2y.$$

Therefore, in Cartesian coordinates the solution is

$$u(x,y) = \frac{3}{4}(x-y) + \frac{1}{4}(x^3 - 3xy^2 - y^3 + 3x^2y). \tag{11.23}$$

This is an example of a *harmonic polynomial*. ■

Exercise 11.4. Use de Moivre's formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

with $n = 3$ to prove the formulas

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \tag{11.24}$$

$$\sin 3\theta = -\sin 3\theta + 3\cos^2\theta\sin\theta. \tag{11.25}$$

Exercise 11.5. Check directly that (11.23) is harmonic inside the unit circle (in fact, it is harmonic *everywhere*), and that it is equal to $x^3 - y^3$ on the boundary of the circle.

The previous example is a special case of a theorem for *harmonic polynomials*.

Definition 11.26. A *harmonic polynomial*, $H(x,y)$ is a polynomial that satisfies Laplace's equation, $H_{xx} + H_{yy} = 0$.

Theorem 11.1. *Suppose $H(x,y)$ is harmonic in a disk, Ω, that is $H_{xx} + H_{yy} = 0$. Moreover, suppose that $H(x,y)$ satisfies the boundary condition $H(x,y) = P(x,y)$ on $\partial\Omega$ where $P(x,y)$ is a polynomial in x and y of degree n. Then the unique solution for $H(x,y)$ is a harmonic polynomial of degree n.*

Remark. In the example above $P(x, y) = x^3 - y^3$ and

$$H(x, y) = \frac{3}{4}(x - y) + \frac{1}{4}(x^3 - 3xy^2 - y^3 + 3x^2y)$$

which demonstrates that the two polynomials are not necessarily equal.

11.3 The Poisson Kernel

Formula (11.13) is usually as far as one can go, using the method of separation of variables. Sometimes, though, one gets lucky and one can obtain a more compact expression. That's what we will do now. By substituting all the coefficients into (11.13) we obtain a formula for the solution of our problem:

$$u(r, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\, d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \int_{-\pi}^{\pi}(\cos n\theta \cos n\phi + \sin n\theta \sin n\phi) f(\phi)\, d\phi.$$

Using the addition formula for the cosine, we can simplify a little this expression:

$$u(r, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\, d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \int_{-\pi}^{\pi}\cos n(\theta - \phi) f(\phi)\, d\phi. \tag{11.27}$$

Let us work a little more with (11.27). To begin with, let us assume that it is legal to interchange summation and integration. Again, you need some assumptions on $f(\phi)$ for this to hold, but we will again ignore this fact for the time being. Then we can rewrite (11.27) as

$$u(r, \theta) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta - \phi)\right] f(\phi)\, d\phi. \tag{11.28}$$

The surprising fact is that you can actually compute the sum of this series. We don't want to spoil you the fun of actually doing it yourself.

Exercise 11.6. Compute the sum:

$$\frac{1}{2} + \sum_{n=1}^{\infty} t^n \cos n\alpha, \qquad |t| < 1. \tag{11.29}$$

by using complex variables to relate this to a geometric series. The idea is that this series is the real part of the series

$$\frac{1}{2} + \sum_{n=1}^{\infty} t^n e^{in\alpha}, \tag{11.30}$$

which converges for every t less than 1 in absolute value. Find the sum of (11.30) using geometric series. Then the real part of your answer will be the sum of (11.29).

The answer you should get:

$$\frac{1}{2} + \sum_{n=1}^{\infty} t^n \cos n\alpha = \frac{1 - t^2}{2(1 - 2t\cos\alpha + t^2)}, \qquad |t| < 1. \tag{11.31}$$

Continuing with our computations, let us apply (11.31) to find the sum of the series in (11.28):

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - \left(\frac{r}{a}\right)^2}{1 - 2\left(\frac{r}{a}\right)\cos(\theta - \phi) + r^2} f(\phi)\,d\phi,$$

or, multiplying top and bottom by a^2,

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2} f(\phi)\,d\phi. \tag{11.32}$$

This formula makes sense for every θ and every $r < a$. When $r = a$, expression (11.32) no longer makes sense.

The function

$$K(r,\theta,a,\phi) = \frac{1}{2\pi}\frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2} \tag{11.33}$$

is called the *Poisson kernel*. Using it, we can write the solution to the problem of finding u such that $\Delta u = 0$ inside the circle, and $u = f$ on the boundary, in a very compact form:

$$u(r,\theta) = \int_{-\pi}^{\pi} K(r,\theta,a,\phi) f(\phi)\,d\phi. \tag{11.34}$$

11.4 Validity of the Poisson Kernel Solution

Formula (11.32) or, equivalently, (11.34), looks very nice. However, we obtained it under unspecified assumptions for f, lighthearted assumptions that the infinite sum of solutions is still a solution, and a careless swapping of integration.

What one can try to do now is look at (11.32), forget how we obtained this formula, and see if it satisfies the given problem. One immediate problem we have already noticed is that (11.32) no longer makes sense when $r = a$, so there is little hope that (11.32) will directly satisfy the boundary condition. However, one can prove the following surprising result.

Theorem 11.2. *If $f(\theta)$ is continuous and periodic with period 2π then the function $u(r,\theta)$ given by (11.32) or (11.34) satisfies $\Delta u = 0$ for $r < a$ and:*

$$\lim_{r\to a^-,\theta\to\theta_0} = f(\theta_0), \tag{11.35}$$

for every θ_0.

Remark. In other words, if we define a function $u(r,\theta)$ for $r \le a$ and every θ as:

$$u(r,\theta) = \begin{cases} \int_{-\pi}^{\pi} K(r,\theta,a,\phi) f(\phi)\, d\phi, & \text{if } r < a, \\ f(\theta), & \text{if } r = a, \end{cases} \tag{11.36}$$

then we obtain a function that is continuous on the closed circle $r \le a$ and harmonic inside it. One of the reasons why this result is surprising is the fact that we know that the Fourier expansion for f, which was one of our key assumptions along the way, in general is not valid when f is just continuous. And important addition to this result is that *the solution is unique*.

What is more, one can prove that, in fact, if f is only *piecewise continuous* and has only jump discontinuities, then (11.32) *is still a harmonic function* inside the circle and satisfies the limit condition (11.35) at every point θ_0 at which $f(\theta)$ is continuous.

The proof of the theorem is very beautiful and uses several results of analysis and harmonic functions:

Proof. (a) If $f(\theta)$ is continuous or, even weaker, if $f(\theta)$ is bounded an integrable, the function $u(r,\theta)$ given by:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n \cos n\theta + b_n \sin n\theta), \tag{11.37}$$

where a_n and b_n are given by (11.11) and (11.12), is infinitely differentiable inside the circle. Indeed, the Fourier coefficients a_n and b_n are bounded, so any derivative of $u(r, \theta)$ is bounded above in absolute value by

$$\sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n (|a_n| + |b_n|),$$

which converges inside the circle. A theorem from analysis (the "Weierstrass M-series theorem") shows that $u(r, \theta)$ has as many derivatives as we want, and in fact satisfies $\Delta u = 0$ inside the circle, that is, it is harmonic inside the circle. The same applies to all subsequent formulas we got for $u(r, \theta)$; in particular, this proves that (11.32) is harmonic inside the circle, under these very weak assumptions on f.

(b) If $f(\theta)$ has many continuous derivatives, let us say three, to be on the safe side, then equality (11.37) is true, with a_n and b_n given by (11.11) and (11.12). Moreover, the coefficients a_n and b_n can be bounded by terms of the form $\frac{M}{n^3}$. This, again by the "Weierstrass M-series" result will imply that the series (11.37) converges uniformly on the *closed* circle $r \leq a$, and on the boundary is equal to $f(\theta)$. In other words, if f is sufficiently smooth, then (11.37) indeed produces a solution $u(r, \theta)$ to our problem: a function u harmonic inside the circle and equal to f on the boundary. This in turn implies that the same can be said about the function $u(r, \theta)$ defined by (11.32).

(c) If $f(\theta)$ is (only) continuous, then it can be uniformly approximated by a sequence of functions $f_k(\theta)$ that have as many derivatives as desired. This is a powerful result from analysis, the Weierstrass approximation theorem (this guy Weierstrass keeps popping up awfully often, don't you think?). In fact, one can choose f_k to be even a finite sum of trigonometric functions $\sin(m\theta)$ and $\cos(m\theta)$ (in general, it will *not* be a truncation of the Fourier series, though).

(d) For every $f_k(\theta)$, which is now as smooth as we want, the corresponding function $u_k(r, \theta)$ given by (11.32) or by (11.34) for f_k instead of f, will be a function harmonic inside the circle, and equal to f_k on the boundary of the circle. This means that $u_k - u_m$ will also be harmonic on the circle and equal to $f_k - f_m$ on the boundary of the circle, for every k and m.

(e) Remember now the Maximum Principle for harmonic functions: the maximum and the minimum of $u_k - u_m$ on the closed circle can only be attained on the boundary. This will mean that if $|f_k - f_m|$ can be made less than ϵ, then also $|u_k(r,\theta) - u_m(r,\theta)|$ will be less than ϵ for all $r \le a$ and all θ. Since f_k converges uniformly (to f) on the boundary of the circle, this will imply that u_k converges uniformly to some function u on the whole closed circle.

(f) Finally, since f_k converges uniformly to f, this means that when taking the limit as $k \to \infty$ in

$$u_k(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2} f_k(\phi)\, d\phi$$

we can switch "$\lim_{k\to\infty}$" with integration at the right-hand side. This proves that the limit function u of the sequence u_k is precisely given by (11.32), and concludes the proof.

□

The proof of the more general assertion when f is piecewise continuous is omitted. We will only observe that one important ingredient in this proof if the fact that the Poisson kernel $K(r,\theta,a,\phi)$ is *positive* for $r < a$, a fact that we invite you to prove.

Exercise 11.7. Prove that the Poisson kernel $K(r,\theta,a,\phi)$ is always positive for $r < a$.

11.5 Interpretation of the Poisson Kernel

Every time you get a solution of a linear problem in the form (11.34), where $f(\theta)$ may be either a non-homogeneous boundary conditions (as in this case) or a non-homogeneous right-hand side (as in the case of a force acting on the system), the kernel K has an important mathematical and physical interpretation.

To save a little in the notation, since the radius of the circle is fixed and the function K depends on the *difference* $\theta - \phi$ rather than on the variables θ and ϕ independently, let us call:

$$k(r,s) = \frac{1}{2\pi}\frac{a^2 - r^2}{a^2 - 2ar\cos s + r^2},$$

so that $K(r,\phi,a,\theta) = k(r,\phi - \theta)$. Then (11.34) can be rewritten as

$$u(r,\theta) = \int_{-\pi}^{\pi} k(r,\theta-\phi)f(\phi)\,d\phi. \tag{11.38}$$

Imagine now a constant boundary condition f of height $\frac{1}{\Delta}$, concentrated on a small interval of length Δ with center at a point ϕ_0.

In other words, f is zero everywhere, except near ϕ_0, at which it is constant and equal to $\frac{1}{\Delta}$ on an interval of length Δ. The height of f has been chosen so as to have $\int_{-\pi}^{\pi} f(\phi)\,d\phi = 1$. Call δ_{ϕ_0} this function. (It also depends on Δ, but let us not complicate the notation.) According to (11.38), the solution of the Dirichlet problem on the circle for this f is given by

$$u(r,\theta) = \int_{-\pi}^{\pi} k(r,\theta-\phi)\delta_{\phi_0}(\phi)\,d\phi.$$

Applying the intermediate value theorem for integrals, we can rewrite this as

$$u(r,\theta) = k(r,\theta-\phi^*)\int_{-\pi}^{\pi} \delta_{\phi_0}(\phi)\,d\phi,$$

where ϕ^* is a number on the interval with length Δ and center ϕ_0. Since the integral of δ_{ϕ_0} is equal to 1, this becomes

$$u(r,\theta) = k(r,\theta-\phi^*).$$

Taking the limit as $\Delta \to 0$, we get

$$u(r,\theta) = k(r,\theta-\phi_0).$$

As $\Delta \to 0$, the function δ_{ϕ_0} itself tends to infinity in such a way that its integral is kept equal to 1 all the time. This is the so-called *the delta function* with center ϕ_0, and is denoted as $\delta(\theta-\phi_0)$. It is not a function but rather a generalized function, what mathematicians call a *distribution*. It is one of those extremely useful beasts that was born in the deranged mind of physicists and mathematicians had to work very hard to make any sense out of them.

Lemma 11.39. *We conclude that* the Poisson kernel

$$K(r,\theta,a,\phi) = k(r,\theta-\phi) = \frac{1}{2\pi}\frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}$$

is a harmonic function inside the circle $r < a$ that is equal to zero everywhere on the boundary, except at the point ϕ, at which it is equal to ∞. *Not only that, but $k(r,\theta-\phi)$ is a harmonic function that on the boundary coincides with the delta function with center at ϕ.*

Knowing the meaning of the Poisson kernel, one can now use an heuristic argument to obtain again formula (11.38).

Assume we are given a function $f(\theta)$ on the boundary of the circle. We divide the interval $[0, 2\pi]$ into N intervals of length Δ and approximate $f(\theta)$ by a piecewise constant function, writing it down as the sum of functions $f(\phi_i)\delta_{\phi_i}\Delta$:

$$f(\phi) \approx \sum_{i=1}^{N} f(\phi_i)\delta_{\phi_i}\Delta.$$

The factor Δ is needed to get the correct height $f(\theta_i)$; recall that δ_{ϕ_i} had height $\frac{1}{\Delta}$.

Since the Dirichlet problem is linear, the solution to the boundary condition $f(\phi_i)\delta_{\phi_i}\Delta$ will be $f(\phi_i)\Delta$ times the solution to the boundary condition δ_{θ_i}. Therefore, the solution will be

$$k(r, \theta - \phi_i) f(\phi_i)\Delta.$$

Again, since the Dirichlet problem is linear, the solution of a sum of boundary conditions $f(\phi_i)\delta_{\phi_i}\Delta$ will be the sum of the solutions for each boundary condition separately, that is,

$$\sum_{i=1}^{N} k(r, \theta - \phi_i) f(\phi_i)\Delta.$$

We recognize this as a Riemann sum, so if we take the limit as $\Delta \to 0$, we get the solution to the problem with boundary condition $f(\theta)$ as

$$\lim_{\Delta \to 0} \sum_{i=1}^{N} k(r, \theta - \phi_i) f(\phi_i)\Delta = \int_{-\pi}^{\pi} k(r, \theta - \phi) f(\phi)\, d\phi,$$

Which completes the proof.

11.6 Challenge Problems for Lecture 10

Problem 11.1. *Integrating the Poisson Kernel.* Prove that

$$\int_{-\pi}^{\pi} K(r,\theta,a,\phi)\,d\phi$$

is a harmonic function $u(r,\theta)$ inside the circle $r<a$, and tends to 1 for every θ, as $r\to a^-$. *Hint:* solve the Dirichlet problem $\Delta u = 0$ inside the circle and $u=1$ on the boundary, and use uniqueness of the solution.

Problem 11.2. Solve the problem

$$\Delta u(r,\theta) = 0 \quad \text{if } r<a, \quad \textit{PDE}$$

$$u(a,\theta) = \begin{cases} 0, & \text{if} \quad -\pi<\theta<0, \\ 1, & \text{if} \quad 0<\theta<\pi. \quad \textit{BC} \end{cases}$$

Problem 11.3. Using the result you found in Problem 2, plus uniqueness of the solution of the Dirichlet problem for the Laplace equation, write down the integral

$$\int_0^{\pi} K(r,\theta,a,\phi)\,d\phi = \frac{1}{2\pi}\int_0^{\pi} \frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}\,d\phi,$$

as an infinite series, assuming $r<a$.

Problem 11.4. One can use polar coordinates, and separation of variables, for "pizza slices", that is, for sectors of a circle. As an example, find the steady-state temperature distribution of a thin plate over the sector

$$\Omega = \{(r, \theta) \,|\, 0 < r < 1,\ 0 < \theta < \frac{\pi}{3}\}$$

given that

$$u(r, 0) = 0, \qquad u\left(r, \frac{\pi}{3}\right) = 0;$$
$$u(1, \theta) = \theta\left(\frac{\pi}{3} - \theta\right).$$

We assume that the temperature at zero is bounded.

Repeat the process of separation of variables. What will be the boundary conditions for $\Theta(\theta)$? Notice that the eigenvalues will *not* be the same as for the case of the full circle.

Problem 11.5. Find the steady-state temperature distribution of a thin plate over the sector

$$\Omega = \{(r, \theta) \,|\, 0 < r < 1,\ 0 < \theta < \frac{\pi}{3}\}$$

given that

$$u(r, 0) = 0, \qquad u_\theta\left(r, \frac{\pi}{3}\right) = 0;$$
$$u(1, \theta) = \theta\left(1 - \frac{3\theta}{2\pi}\right).$$

Assume that the temperature at zero is bounded.