Twelve

Well-posed Problems: Existence, Uniqueness and Stability

When solving partial differential equations, mathematicians put a high premium on problems that are *well-posed*, that is problems which have a unique solution and which are stable in the sense that small perturbations to the initial and boundary conditions only yield small perturbations to the solution.

The heat equation, the wave equation and Laplace's Equation with appropriate boundary conditions are well-posed and this increases there usefulness when modeling physical problems. Below we will study well-posedness, concentrating primarily on the heat equation.

12.1 What is a well-posed problem?

A well-posed problem has three characteristics:

1) *Existence*: A solution exists to the problem. It satisfies the governing PDE and the associated boundary and initial conditions.

2) *Uniqueness*: The solution is the only solution satisfying the governing PDE and the associated boundary and initial conditions. That is the solution is unique.

3) *Stability*: If a small perturbation is made to the initial condition or boundary conditions, the solution changes by only a small amount.

These characteristics are physically desirable; for example, if we are modeling a physical problem, such as heat flow, we want there to be a unique solution of the governing equations and if perchance we make a small mistake in measuring the initial state we don't want the solution to change radically.

We also need to also talk about the *regularity* of the solutions. Solutions "live" in a function space. For example, for the heat equation, it is natural to talk about $u(x,t) \in C_x^2[0,L]$ - that is $u(x,t)$, $u_x(x,t)$, and the second derivative, $u_{xx}(x,t)$, are continuous, and $u(x,t) \in C_t^1[0,\infty)$ - that is $u(x,t)$ and $u_t(x,t)$ are continuous. One makes assumptions about the regularity of the solution implicitly when one solves the heat equation. After all, what would it mean for a function to solve the equation if these derivatives were undefined?

12.2 Existence

Existence can most easily be demonstrated by constructing an explicit solution. For the homogeneous Dirichlet problem,

$$\begin{array}{rccl} \mathbb{DE}: & & U_t = DU_{xx} & 0<x<L, t>0 \\ \mathbb{BC}: & U(0,t)=0, & U(L,t)=0, & t>0 \\ \mathbb{IC}: & & U(x,0)=f(x) & 0<x<L. \end{array}$$

we have already found a solution via separation of variables,

$$U(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt}. \tag{12.2}$$

Choosing

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

will satisfy the initial condition. If in addition, we specify that $f(x) \in C_x^2[0,L]$ with $f(0) = f(\pi) = 0$, then the Fourier sine series converges uniformly to the initial condition and the solution $u(x,t)$ is in $C_x^2[0,L]$ and $C_t^1[0,\infty)$. In this case, we have shown the solution exists and has the specified regularity.

Exercise 12.1. Show that for the homogeneous Neumann problem,

$$\begin{array}{rccl} \mathbb{DE}: & & U_t = DU_{xx} & 0<x<L, t>0 \\ \mathbb{BC}: & U_x(0,t)=0, & U_x(L,t)=0, & t>0 \\ \mathbb{IC}: & & U(x,0)=f(x) & 0<x<L. \end{array}$$

with $f(x) \in C_x^2[0, L]$ with $f_x(0) = f_x(L) = 0$ that a solution exists with $U(x,t)$ in $C_x^2[0, L]$ and $C_t^1[0, \infty)$. Why is the condition $f_x(0) = f_x(L) = 0$ necessary?

12.3 Energy Dissipation and Uniqueness

By looking at what is normally known as energy for the diffusion equation, we can show that the solution for the Dirichlet problem is unique. Note this energy is a mathematical construct, not to be confused with the thermal energy discussed in the derivation of the diffusion equation.

Let's define, the energy,

$$W \equiv \frac{1}{2} \int_0^L U^2 \, dx, \tag{12.4}$$

which is a function of t dependent on the particular solution $U(x,t)$ (technically it is a function of t and a *functional* of $U(x,t)$). Note that $W \geq 0$ and, assuming that U is continuous in x, $W = 0$ only for the trivial solution $U(x,t) = 0$.

If we differentiate the energy with respect to time, we find

$$\begin{aligned} \frac{dW}{dt} &= \int_0^L UU_t \, dx, \\ &= D \int_0^L UU_{xx} \, dx, \\ &= -\int_0^L (U_x)^2 \, dx + UU_x \left.\right|_{x=0}^{x=L}, \end{aligned}$$

where we have substituted the DE and used integration by parts. Now, applying the BC's, we fine that the boundary terms from the integration by parts vanish, so

$$\frac{dW}{dt} = -\int_0^L (U_x)^2 \, dx \leq 0.$$

Now, we can conclude that W is decreasing (that is energy is dissipated) *unless* $U_x = 0$, that is to say that U is constant. As the only constant solution satisfying the boundary conditions is $U = 0$, we might be tempted to conclude that the solution always decays to this trivial state. This turns out to be true, although one must invest some analysis to show it rigorously.

A second conclusion one can reach is that if $f(x) = 0$, that $U(x,t) = 0$ for all $t > 0$. This follows quickly because $W = 0$ at $t = 0$, it is non-increasing and non-negative. While this seems like a trivial result, it has a very powerful consequence.

Suppose we had two solutions to the non-homogeneous Dirichlet problem, call them U_1 and U_2. You should be able to convince yourself that there difference

$$V = U_1 - U_2$$

satisfies the homogeneous Dirichlet problem with $f(x) = 0$. Consequently, we know that $V(x,t) = 0$ for all $t > 0$, which implies $U_1 = U_2$. From this we conclude that: *The solution to the non-homogeneous Dirichlet problem is unique.* This is a powerful result indeed.

Exercise 12.2. Convince yourself the energy argument for uniqueness of solutions in the previous paragraph is correct. Show that a similar argument can be made for the Neumann problem.

12.4 The Maximum Principle

Looking at solutions to the heat equation, we note that they tend to average out maximums and minimums. We can develop some intuition for this by considering what the equation says. Basically, $U_t = DU_{xx}$ means: *The temperature is decreasing when the profile is convex down and the temperature is increasing when the profile is convex up*.

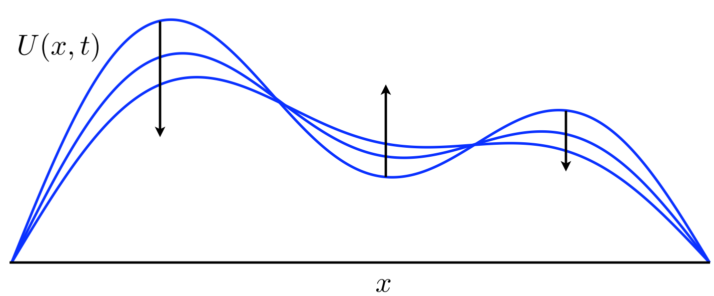

Figure 12.1: The heat equation interpreted graphically. The temperature is decreasing when the profile is convex down and the temperature is increasing when the profile is convex up.

From which we conclude that interior maximums in temperature are decreasing and interior minimums of temperature are increasing. For the Dirichlet problem,

$$\begin{array}{rlll} \mathbb{DE}: & U_t = DU_{xx} & & 0 < x < L, t > 0 \\ \mathbb{BC}: & U(0,t) = a(t), & U(L,t) = b(t), & t > 0 \\ \mathbb{IC}: & U(x,0) = f(x) & & 0 < x < L. \end{array}$$

We can give a rigorous statement of these ideas which is called the *Maximum Principle*:

Theorem 12.1 (Maximum Principle for the Diffusion Equation)**.** *Suppose $U(x,t)$ satisfies the Dirichlet problem for the diffusion equation in the rectangle $0 < x < L$, $0 < t < T$. Also, assume that $u(x,t)$ is in $C_x^2[0,L]$ and $C_t^1[0,T]$. Then it assumes its maximum value (as a function of x and t) either initially (when $t = 0$) or on the lateral boundaries (where $x = 0$ or $x = L$).*

For convenience we refer to the rectangle $0 \le x \le L$, $0 \le t \le T$ as R, and let

$$M = \max_{(x,t)\in R} U(x,t).$$

The maximum principle says that U obtains the value M either initially (when $t = 0$) or on the lateral boundaries of the rectangle R. Plausibly it could obtain the value of M at some points in the interior also.

First we prove a simple lemma about what a maximum would look like:

Lemma 12.5. *If $u(x,t)$ has a maximum $(\bar{x},\bar{t})$ in the interior of the rectangle R, then $U_t(\bar{x},\bar{t}) = U_x(\bar{x},\bar{t}) = 0$ and $U_{xx}(\bar{x},\bar{t}) \le 0$.*

Proof of Lemma. This follows from single variable calculus. First note that if we consider $U(\bar{x},t)$ as a function of t that it reaches a maximum at $\bar{t}$, and therefore U_t must vanish (where we have used the differentiability of U with respect to t). Similarly, we consider $U(x,\bar{t})$ as a twice continuously differentiable function of x. If it is a maximum, then $U_x(\bar{x},\bar{t}) = 0$. Moreover, if $U_{xx}(\bar{x},\bar{t}) > 0$, it is a strict minimum and therefore cannot be a maximum, from which we conclude $U_{xx}(\bar{x},\bar{t}) \le 0$. □

Now we proceed to the main event:

Proof of Maximum Principle. Suppose we found a maximum in the interior of the rectangle; we know that $U_t = 0$ and if $U_{xx} < 0$, we would have a contradiction because u satisfies the heat equation, $U_t = DU_{xx}$. The problem is that we could have a maximum where $U_{xx} = 0$. We deal with this by introducing the idea of a *subfunction*.

Define a new function,

$$V(x,t) = U(x,t) - \epsilon tx(L-x),$$

where ϵ is a positive constant; we call $V(x,t)$ a *subfunction* of $U(x,t)$ as it is slightly below it. Note that

(a) The function $V \leq U$ with equality only on the lateral and bottom boundary of R.

(b) The difference between U and V in R is

$$U - V = \epsilon tx(L-x) \leq \epsilon \frac{TL^2}{4},$$

which tends to zero uniformly as $\epsilon \to 0$.

(c) In the limit of decreasing ϵ,

$$\max_{x\in R} U(x,t) = \lim_{\epsilon\to 0} \left[\max_{x\in R} V(x,t)\right],$$

which follows from the uniform bound on the difference between U and V shown above.

Substituting into the $\mathbb{DE}$, we see that V satisfies

$$\begin{array}{lll} \mathbb{DE}: & V_t = DV_{xx} - \epsilon\left[2Dt + x(L-x)\right] & 0 < x < L, t > 0 \\ \mathbb{BC}: & V(0,t) = a(t), \quad V(L,t) = b(t), & t > 0 \\ \mathbb{IC}: & V(x,0) = f(x) & 0 < x < L. \end{array}$$

Note that at a maximum $(\bar{x},\bar{t})$ of V in the interior of R that $V_t = 0$, which allows us to conclude that $V_{xx} = \epsilon[2t + x(L-x)/D] > 0$ from the $\mathbb{DE}$ which is a contradiction, so V does *not* have a maximum in the interior of the rectangle R.

If the maximum occurs at a point on the top boundary, $(\bar{x},T)$, where $0 < \bar{x} < L$, we know that $V_x(\bar{x},T) = 0$ and $V_{xx}(\bar{x},T) \leq 0$, so

$$V_t = DV_{xx} - \epsilon\left[2Dt + x(L-x)\right] < 0$$

and consequently the function is decreasing in time which is a contradiction, so we know the maximum occurs on one of the other three boundaries.

So the maximum of $V(x,t)$ must occurs on the lateral boundaries or the bottom boundary where $U = V$. Consequently we have demonstrated the Maximum Principle for V, but as $\epsilon \to 0$, we know U converges uniformly to V (as does the maximum of U), so we conclude that the maximum of U must occur on the lateral or bottom boundary also, which completes the proof. □

There is a stronger version of the Maximum Principle that says if a maximum of U occurs in the interior of R then U is constant which leads us to the following exercise for the reader to contemplate.

Exercise 12.3. Show that while the Maximum Principles guarantees that the maximum of U occurs on the boundary of R, that it doesn't guarantee that it occurs *only* on the boundary of R. You may wish to do this by presenting an example.

Note the same result is also true of the minimum of $U(x,t)$, which is sometime called the *Minimum Principle*. This can be seen easily by considering $-U(x,t)$ which also satisfies the diffusion equation.

The Maximum Principle gives us another proof of uniqueness of the solution for the Dirichlet problem. Again, suppose we had two solutions to the non-homogeneous Dirichlet problem, call them U_1 and U_2 that satisfy the regularity conditions for the Maximum Principle ($U(x,t) \in C_x^2[0,L]$ and $U(x,t) \in C_t^1[0,\infty)$. The difference $V = U_1 - U_2$ satisfies the homogeneous Dirichlet problem with $f(x) = 0$. From the Maximum Principle (and the Minimum Principle) we know that $V(x,t)$ obtains its maximum (and minimum) either initially (at $t = 0$) or on the boundaries (where $x = 0$ or $x = L$) where $V = 0$. Consequently $V = 0$ for $0 < x < L$ and $t > 0$. Which means $U_1 = U_2$ and again we conclude the solution is unique.

Exercise 12.4. The Maximum Principle for the homogeneous Neumann problem,

$$\begin{array}{rccl}
\mathbb{DE}: & U_t = DU_{xx} & & 0 < x < L, t > 0 \\
\mathbb{BC}: & U_x(0,t) = 0, & U_x(L,t) = 0, & t > 0 \\
\mathbb{IC}: & U(x,0) = f(x) & & 0 < x < L,
\end{array}$$

states that:

> **Theorem 12.2** (Maximum Principle for the Diffusion Equation)**.** *Suppose $U(x,t)$ satisfies the homogeneous Neumann problem for the diffusion equation in the rectangle $0 < x < L$, $0 < t < T$. Also, assume that $u(x,t)$ is in $C_x^2[0,L]$ and $C_t^1[0,T]$. Then $U(x,t)$ assumes its maximum value (as a function of x and t) initially (when $t = 0$).*

Prove this theorem using the subfunction $V(x,t) = U(x,t) - \epsilon t$.

Exercise 12.5. Show that the solution to the Neumann problem is unique using the results of Exercise 12.4.

12.5 Stability

> *Can a butterfly flapping its wings in Beijing alter the weather in San Francisco?*
>
> \- Attributed to Ed Lorenz

Loosely speaking, we will say a system is *stable* if a small change in the initial condition induces only a small perturbation in the solution for $t > 0$

12.5.1 The backward heat equation - an example of instability

An example of instability is the backwards heat equation. Recall that in the heat equation we assume D is positive. Consider the Dirichlet problem that $D < 0$. In this case, the heat flows from cold to hot. This is the equivalent of running the heat equation backward in time.

Note that in our previous derivation of the solution, we did not make any use of the sign of D. Consequently, the solution to:

$$\begin{array}{rccl} \mathbb{DE}: & U_t = -U_{xx} & & 0 < x < L, t > 0 \\ \mathbb{BC}: & U(0,t) = 0 & U(L,t) = 0 & t > 0 \\ \mathbb{IC}: & U(x,0) = \frac{1}{n}\sin(\frac{n\pi x}{L}) & & 0 < x < L, \end{array}$$

where we have chosen $D = -1$ is

$$u(x,t) = \frac{1}{n}\sin\left(\frac{n\pi x}{l}\right) e^{\frac{n^2\pi^2}{l^2}t}.$$

Note that

$$\max_{0<x<l} u(x,0) = \frac{1}{n} \qquad 0 < x < L$$

but

$$\max_{0<x<l} u(x,t) = \frac{1}{n} e^{\frac{n^2\pi^2}{L^2}t}.$$

So given any $\epsilon > 0$, I can choose n such that $\frac{1}{n} < \epsilon$ and $|u(x,0)| < \epsilon$. But for any $T > 0$,

$$\max_{0<x<l} u(x,T) = \frac{1}{n} e^{\frac{n^2\pi^2}{l^2}T}$$

and as $n \to \infty$, this maximum of the solution tends towards infinity. Consequently, I can find solutions that, although arbitrarily small initially, that can be arbitrarily large at any fixed positive time T. In fact, for many initial conditions, the solution goes to infinity in a finite amount of time.

12.5.2 Stability for the forward heat equation

In fact, the forward heat equation is stable to perturbations of the initial condition. Suppose we consider two solutions to the homogeneous Dirichlet problem, $U_1(x,t)$ and $U_2(x,t)$ with initial conditions $U_1(x,0) = f_1(x)$ and $U_2(x,0) = f_2(x)$. If

$$\max_{0<x<l} |f_1(x,0) - f_2(x,0))| \leq \delta.$$

we can appeal to the dissipation of energy to get a bound on the difference between the solutions for all time.

Let

$$V(x,t) = U_1(x,t) - U_2(x,t)\ .$$

Note that $V(x,t)$ also satisfies a homogeneous Dirichlet problem, and we have shown previously that the energy associated with V is dissipated. Consequently,

$$W[V(x,t)] \leq W[V(x,0)]$$

or

$$\int_0^L \frac{V^2}{2}\, dx \leq \int_0^L \frac{\delta^2}{2}\, dx = \frac{\delta^2 L}{2}$$

Consequently,

$$||U_1(x,t) - U_2(x,t)||_{L^2} = ||V(x,t)||_{L^2} \leq \delta\sqrt{L}$$

so that the two solutions remain close in the L^2 or root-mean-square sense for all time.

One problem with this proof of stability is that just because solutions are close in L^2 does not mean that they remain close uniformly. We would refer to the result above as L^2 *stability to perturbations of the initial condition*.

A stronger result can be proven using the Maximum Principle. In fact, if

$$\max_{0<x<l} |f_1(x,0) - f_2(x,0))| \leq \delta$$

it follows immediately from the maximum principle that

$$\max_{0<x<l} |U_1(x,t) - U_2(x,t))| \leq \delta$$

which would be *uniform stability to perturbations of the initial condition*. The maximum principle can be used to prove stability with respect to perturbations to boundary conditions also.

Exercise 12.6. Show that the proof of L^2 stability for perturbation of the initial conditions of the Dirichlet problem using energy dissipation can be extended to the Neumann problem.

Exercise 12.7. Show that the proof of uniform stability can be extended to the Neumann problem using the results of Exercise 12.4.

Exercise 12.8. Show that the proof of uniform stability for the Dirichlet problem using the Maximum Principle can be extended to perturbations of the boundary condition also.

Exercise 12.9. Explain why uniform stability implies L^2 stability, but L^2 stability does *not* imply uniform stability.