Thirteen

The Cauchy Problem for the Wave Equation: d'Alembert's Solution

While the wave equation can be used to model the oscillations of a string, it also governs the propagation of electromagnetic waves over long distances. For these problems it makes more sense to consider the problem on the infinite line. This problem is called the *Cauchy Problem*, named in honor of Baron Augustin-Louis Cauchy (1789 – 1857) who considered a variety of problems in the absence of boundaries.

13.1 THE CAUCHY PROBLEM FOR THE WAVE EQUATION

For the wave equation on an infinite domain we have the following:

THE CAUCHY PROBLEM FOR THE WAVE EQUATION

$$\begin{array}{rlr} \mathbb{DE}: & u_{tt} = c^2 u_{xx}, & -\infty < x < \infty, t > 0 \\ \mathbb{IC}: & u(x,0) = f(x), \quad u_t(x,0) = g(x) & -\infty < x < \infty \end{array}$$

Amazingly, we have an exact solution to this problem:

Theorem 13.1 (d'Alembert's Solution)**.** *The solution to the Cauchy problem for the wave equation is:*

$$u(x,t) = \tfrac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz,$$

when $f \in \mathcal{C}^2_x$ *and* $g \in \mathcal{C}^1_x$ *on the real line,* $-\infty < x < \infty$*.*

Exercise 13.1. Verify that the d'Alembert solution satisfies the Cauchy problem for the wave equation via direct substitution into the $\mathbb{DE}$ and the $\mathbb{IC}$s.

13.2 A Derivation of the d'Alembert Solution

This solution can easily be verified by direct substitution, but let us motivate the derivation. The wave equation can be written in operator notation,

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u(x,t) = 0,$$

and we can factor the differential operator as a difference of two squares

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) &= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) \\ &= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) \end{aligned}$$

Consequently we could write the wave equation as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)[u_t + cu_x] = 0$$

or

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)[u_t - cu_x] = 0$$

From which we decide that if $u(x,t)$ satisfies either first-order wave equation,

$$u_t - cu_x = 0 \qquad \text{or} \qquad u_t - cu_x = 0$$

then $u(x,t)$ also satisfies the wave equation. That is both

$$u(x,t) = A(x-ct) \qquad \text{and} \qquad u(x,t) = B(x+ct)$$

are solutions for any functions A and B. Since the equation is linear and homogeneous, we can deduce that

$$u(x,t) = A(x-ct) + B(x+ct)$$

is the most general solution.

We can derive this more rigorously via a change of variables; let

$$\xi = x - ct, \qquad \eta = x + ct \qquad u(x,t) \equiv U(\xi, \eta).$$

From the chain rule, we see that

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \end{aligned}$$

from which we can see that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) &= -2c\frac{\partial}{\partial \xi} \\ \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) &= 2c\frac{\partial}{\partial \eta} \end{aligned}$$

Thus, we can rewrite the $\mathbb{DE}$ as

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) u(x,t) = 0 \qquad \Rightarrow \qquad -4c^2\frac{\partial^2}{\partial \eta \partial \xi} U(\xi, \eta) = 0$$

or (assuming $c \neq 0$) as just

$$\frac{\partial^2 U}{\partial \eta \partial \xi} = 0. \tag{13.1}$$

Integrating this equation with respect to η yields

$$\frac{\partial U}{\partial \xi} = C(\xi)$$

for some arbitrary function $C(\eta)$. Integrating with respect to ξ now yields

$$U(\xi, \eta) = A(\xi) + B(\eta)$$

where $A(z)$ is the indefinite integral of $C(z)$,

$$A(z) = \int C(z)\, dz,$$

which is again an arbitrary function. If we rewrite this in terms of the original variables, we obtain

$$\boxed{u(x,t) = A(x - ct) + B(x + ct)} \tag{13.2}$$

as previously advertised. Note that this solution can be interpreted as the superposition of a right-going wave and a left-going wave each propagating with a speed c. We will see this play out physically in some of the examples below.

Let us now apply the initial conditions; the initial position satisfies

$$u(x,0) = A(x) + B(x) = f(x). \tag{13.3}$$

The velocity can be computed by differentiating with respect to time

$$u_t(x,t) = -cA'(x-ct) + cB'(x+ct)$$

where A' denotes the derivate of A with respect to its argument and the factors of c appear due to the chain rule. Consequently, the initial velocity satisfies

$$u_t(x,0) = -cA'(x) + cB'(x) = g(x). \tag{13.4}$$

The initial velocity (13.4) can be integrated from 0 to x to yield

$$\int_0^x g(z)\,dz = \int_0^x [-cA'(x) + cB'(x)]\,dz = c[B(x) - A(x) - B(0) + A(0)]\,.$$

Remembering that we are trying to solve for A and B, we rewrite this as

$$-A(x) + B(x) = \frac{1}{c}\int_0^x g(z)\,dz + B(0) - A(0). \tag{13.5}$$

We can now solve (13.3) and (13.5) for $A(x)$ and $B(x)$ to yield

$$\begin{aligned} A(x) &= \tfrac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)\,dz - \tfrac{1}{2}[B(0) - A(0)] \\ B(x) &= \tfrac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(z)\,dz + \tfrac{1}{2}[B(0) - A(0)] \end{aligned}$$

substituting back into (13.2) yields

$$u(x,t) = A(x-ct) + B(x+ct) = \tfrac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\left[\int_0^{x+ct} g(z)\,dz - \int_0^{x-ct} g(z)\,dz\right]$$

and combining the two integrals finally yields the d'Alembert solution,

$$u(x,t) = \tfrac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz,$$

as described in the theorem above.

Exercise 13.2. Explain the appearance and subsequent disappearance of the constant $B(0) - A(0)$ in the above derivation; how is this related to A and B not being uniquely defined?

13.3 SOME EXAMPLES OF SOLUTION TO THE CAUCHY PROBLEM

We can use the d'Alembert solution to investigate some solutions to the Cauchy problem for the wave equation.

Example 13.1. Find a solution to the wave equation

$$\begin{array}{rlll} \mathbb{DE}: & u_{tt} = c^2 u_{xx}, & & -\infty < x < \infty, t > 0 \\ \mathbb{IC}: & u(x,0) = e^{-x^2}, & u_t(x,0) = 0 & -\infty < x < \infty \end{array}$$

Solution: From the d'Alembert's solution we know that

$$U(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi,$$

where $U(x,0) = f(x)$ and $U_t(x,0) = g(x)$. We see that

$$U(x,t) = \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right].$$

The solution consists of two gaussians, one propagating to the left and one propagating to the right. ■

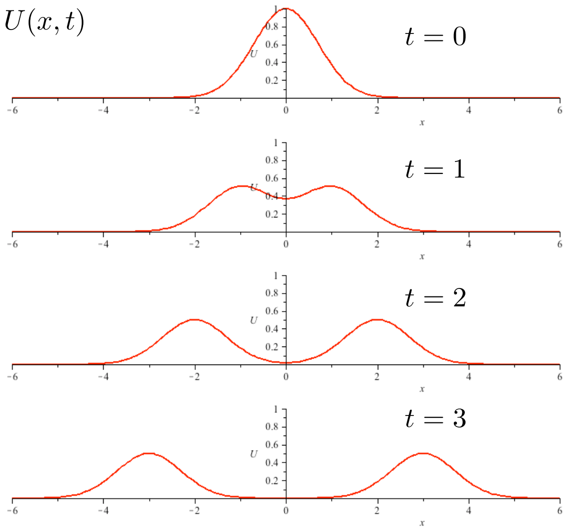

Figure 13.1: A graph of $U(x,t)$ for $t = 0, 1, 2, 3$ and $c = 1$. The initial gaussian splits into two, one propagating to the left at speed c and one propagating to the right at speed c, each of half the amplitude of the initial condition.

Example 13.2. Find a solution to the wave equation

$$\begin{array}{llll} \mathbb{DE}: & u_{tt} = c^2 u_{xx}, & & -\infty < x < \infty, t > 0 \\ \mathbb{IC}: & u(x,0) = 0, & u_t(x,0) = xe^{-x^2} & -\infty < x < \infty \end{array}$$

Show that $U(0,t) = 0$ and graph the solution at a sample time.
Solution: From the d'Alembert's solution we know that

$$U(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \xi e^{-\xi^2}\, d\xi, = -\frac{e^{-\xi^2}}{4c}\Big|_{x-ct}^{x+ct} = \frac{1}{4c}\left[e^{-(x-ct)^2} - e^{-(x+ct)^2}\right].$$

We can evaluate the solution at the origin to see that

$$U(0,t) = \frac{1}{4c}\left[e^{-(-ct)^2} - e^{-(ct)^2}\right] = 0,$$

which also follows from the odd symmetry of the solution at the origin, which is also evident in the graph below.

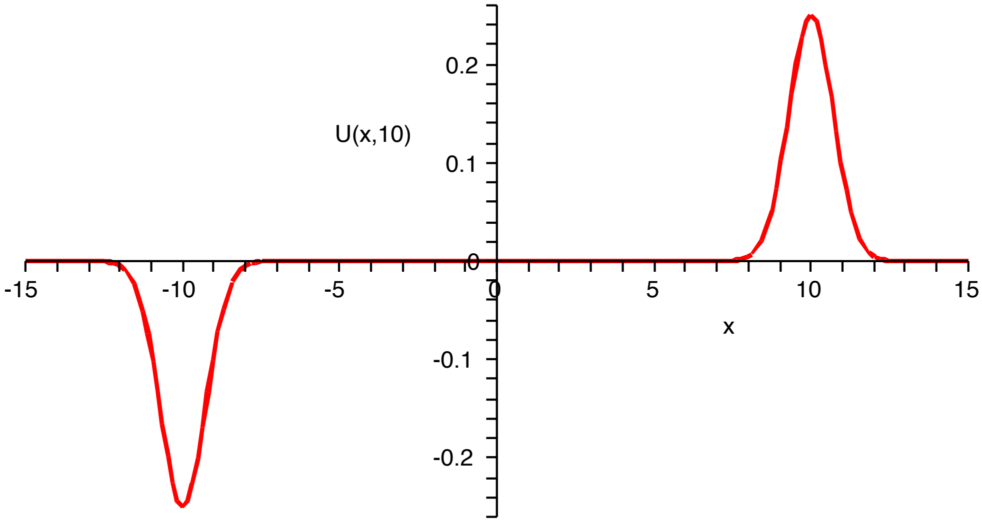

Figure 13.2: A graph of $U(x,10)$ for $c = 1$; we have a positive Gaussian propagating to the left centered at $x = 10$ and its negative image propagating to the right.

■

13.3.1 Periodic Initial Conditions

One can use the d'Alembert solution to find a solution to the wave equation on a finite interval by looking at equivalent periodic initial conditions on the infinite interval.

Example 13.3. In this problem we will use the d'Alembert solution to recover some solutions we found via separation of variables:

(a) Use the d'Alembert's solution to solve the problem

$$\begin{array}{rlr} \mathbb{DE}: & U_{tt} = c^2 U_{xx}, & -\infty < x < \infty, t > 0 \\ \mathbb{IC}: & U(x,0) = \sin(nx), \quad U_t(x,0) = g(0) & -\infty < x < \infty \end{array}$$

where n is a positive integer. Show the solution can be written in the form

$$U(x,t) = \sin(nx)\cos(\omega_n t)$$

where ω_n is to be determined and relate the solution to the one we found via separation of variables on a finite interval.

(b) Use the same idea to solve the problem

$$U_{tt} = c^2 U_{xx} \qquad 0 < x < \pi, \quad 0 < t$$

$$U(x,0) = 0, \quad U_t(x,0) = \sin(nx). \qquad 0 < x < \pi$$

$$U(0,t) = U(\pi,t) = 0, \qquad 0 < t,$$

where, once again, n is a positive integer.

Solution:

(a) From d'Alembert's formula

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi,$$

where $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, we see that

$$u(x,t) = \frac{1}{2}\left[\sin(n(x-ct)) + \sin(n(x+ct))\right].$$

Using the sum formula $\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$ yields

$$U(x,t) = \frac{1}{2}\left[\sin(n(x-ct)) + \sin(n(x+ct))\right] = \sin(nx)\cos(nct).$$

That is

$$U(x,t) = \sin(nx)\cos(\omega_n t)$$

where $\omega_n = nc$.

Note that in addition to solving the $\mathbb{DE}$ the solution satisfies

$$U(x,0) = \sin(nx)\cos(0) = \sin(nx), \quad U_t(x,0) = -\omega_n \sin(nx)\sin(0) = 0,$$

and

$$U(0,t) = \sin(0)\cos(\omega_n t) = 0, \quad U(\pi,t) = \sin(n\pi)\cos(\omega_n t) = 0,$$

when n is a positive integer. We recognize the solution as that obtained via separation of variables for the Dirichlet Problem for the wave equation,

$$\begin{array}{rlr} \mathbb{DE}: & U_{tt} = c^2 U_{xx}, & 0 < x < \pi, t > 0 \\ \mathbb{IC}: & U(x,0) = \sin(nx), \quad U_t(x,0) = g(0) & 0 < x < \pi \\ \mathbb{BC}: & U(0,t) = U(\pi,t) = 0 & t > 0. \end{array}$$

(b) We can look at the odd-periodic extension of the problem onto the real line,

$$\begin{array}{rlr} \mathbb{DE}: & U_{tt} = c^2 U_{xx}, & -\infty < x < \infty, t > 0 \\ \mathbb{IC}: & U(x,0) = 0, \quad U_t(x,0) = \sin(nx) & -\infty < x < \infty, \end{array}$$

for which we obtain the solution from the d'Alembert formula

$$U(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(n\xi)\,d\xi = \frac{1}{2nc}\left[\cos(n(x-ct)) - \cos(n(x+ct))\right].$$

Using the sum formula $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$ yields

$$U(x,t) = \frac{1}{nc}\sin(nx)\sin(nct)) = \frac{1}{\omega_n}\sin(nx)\sin(\omega_n t))$$

where $\omega_n = nc$. We leave it as an exercise for the reader to verify that the solution satisfies the wave equation, and the associated initial and boundary conditions.

■

In general, for a Dirichlet problem we can consider the odd periodic extension of the initial condition onto the real line. For the Neumann problem, we look at the even periodic extension onto the real line.

INSERT TRIANGLE WAVE EXAMPLE HERE