

# Fourteen

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## An Introduction to The Fourier Transform

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Parts of this section evolved from earlier notes due to Jon Jacobsen.

The Fourier transform is the natural generalization of the idea of Fourier series to an infinite interval. It is extremely useful when consider problems on the real line, such as the Cauchy problem for the wave or heat equation.

### 14.1 THE FOURIER TRANSFORM

The Fourier Transform can be thought of as a the limit of a Complex Fourier Series on an infinite interval. Previously we defined the Complex Fourier Series for a function  $f(x)$  defined on the interval  $x \in [-\ell, \ell]$  as

$$\text{CFS}[f(x)] = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi x}{\ell}}.$$

where

$$c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-i \frac{n\pi x}{\ell}} dx.$$

Suppose now that  $f(x)$  is a complex function defined on the real line,  $x \in \mathbb{R}$ , and it is *absolutely integrable*, that is

$$\int_{-\infty}^{\infty} |f(x)| dx \equiv \lim_{R \rightarrow \infty} \int_{-R}^R |f(x)| dx = M < \infty.$$

we can define a wavenumber

$$k_n = \frac{n\pi x}{\ell}$$

and a scaled Fourier coefficient

$$\widehat{f}_\ell(k) = 2\ell c_n = \int_{-\ell}^{\ell} f(x)e^{-ik_n x} dx.$$

If  $f(x)$  is absolutely integrable, we can let  $\ell$  go to infinity while keeping the wavenumber  $k_n$  fixed to define

$$\begin{aligned}\widehat{f}(k_n) &\equiv \lim_{\ell \rightarrow \infty} \widehat{f}_\ell(k_n), \\ &= \lim_{\ell \rightarrow \infty} \int_{-\ell}^{\ell} f(x)e^{-ik_n x} dx, \\ &= \int_{-\infty}^{\infty} f(x)e^{-ik_n x} dx.\end{aligned}$$

This is important enough that we will write a separate definition:

**Definition 14.1** (Fourier Transform). Let  $f : \mathbb{C} \rightarrow \mathbb{C}$ . The Fourier transform of  $f$  at  $k$  is defined by

$$\mathcal{F}\{f\} = \widehat{f}(k) \equiv \int_{-\infty}^{\infty} f(x)e^{-ikx} dx, \quad (14.2)$$

provided this integral exists.

In general,  $\widehat{f}(k)$  is a complex function of  $k$ , even if  $f(x)$  is a real function. Note that

$$\begin{aligned}|\widehat{f}(k)| &= \left| \int_{-\infty}^{\infty} f(x)e^{-ikx} dx \right| \\ &\leq \int_{-\infty}^{\infty} |f(x)e^{-ikx}| dx, \\ &\leq \int_{-\infty}^{\infty} |f(x)| dx.\end{aligned}$$

Thus if  $f$  is absolutely integrable, the Fourier transform  $\widehat{f}(k)$  exists. In general we will also assume that  $f \rightarrow 0$  as  $x \rightarrow \pm\infty$ .

## 14.2 INVERSE FOURIER TRANSFORM

A pleasing aspect of Fourier transforms is that there is a particularly nice form for the inverse transform:

**Definition 14.3** (Inverse Fourier Transform). Let  $u : \mathbb{C} \rightarrow \mathbb{C}$ . The inverse transform of  $u$  is defined by

$$\mathcal{F}^{-1}\{\hat{u}(k)\} = u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(k)e^{ikx} dk. \quad (14.4)$$

It is not obvious that  $\mathcal{F}^{-1}\{\mathcal{F}\{u\}\} = u$ , or for that matter that the Fourier transform is unique. However, we will eventually show that both of these things are true. The Fourier inversion formula (14.4) represents  $u$  as a continuous superposition of  $e^{-ikx}$  with amplitudes  $\hat{u}(k)$ . For instance, the transform of  $\hat{\delta}_0(k) = 1$ , implies the delta distribution contains equal strength of all modes.

The reader is warned that there are *six standard ways to define  $\hat{f}$  and  $\check{f}$* . The essential choices to make are whether to put the  $1/2\pi$  on  $\hat{f}$  or  $\check{f}$  (or  $1/\sqrt{2\pi}$  on each) and whether to put the minus sign in  $\hat{f}$  or  $\check{f}$ . Other sources will put the  $2\pi$  in the exponent. They are all equivalent, and the differences are not of any fundamental importance. On the other hand, it is important to remember this when consulting different sources.

### 14.3 SOME PROPERTIES AND EXAMPLES OF FOURIER TRANSFORMS

We can compute some examples of Fourier transforms.

**Example 14.1.** Find the Fourier transform of the top hat function

$$T(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

and relate it to the coefficients of the Complex Fourier Series.

**Solution:** We find the Fourier Transform by direct calculation

$$\hat{T}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx} dx = \int_{-1}^1 e^{-ikx} dx = \frac{2 \sin k}{k}.$$

Note that for  $\ell > 1$  if we compute the coefficients of the Complex Fourier Series,

$$c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x)e^{-i\frac{n\pi x}{\ell}} dx = \frac{1}{2\ell} \int_{-1}^1 e^{-i\frac{n\pi x}{\ell}} dx = \frac{1}{2\ell} \left[ \frac{2 \sin(\frac{n\pi}{\ell})}{n\pi/\ell} \right].$$

We see that

$$2\ell c_m = \frac{2 \sin(k_m)}{k_m} \quad k_m = \frac{m\pi x}{\ell}$$

which is exactly  $\widehat{T}(k)$ . ■

Another example that will prove useful is:

**Example 14.2.** Compute the Fourier Transform of

$$f(x) = H(x)e^{-ax}$$

for  $a > 0$ , where

$$H(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases},$$

is the Heaviside function, so

$$H(x)e^{-ax} = \begin{cases} e^{-ax} & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases}.$$

**Solution:** By definition,

$$\begin{aligned} \widehat{f}(k) &= \int_{-\infty}^{\infty} H(x)e^{-ax}e^{-ikx} dx \\ &= \int_0^{\infty} e^{-ax}e^{-ikx} dx \\ &= \int_0^{\infty} e^{-(a+ik)x} dx \\ &= \frac{e^{-(a+ik)x}}{-(a+ik)} \Big|_{x=0}^{\infty} \\ &= \frac{1}{a+ik}. \end{aligned}$$

where we have used the fact that  $a > 0$  to assure the integral converges. ■

We can use some properties to increase the size of our list of Fourier transforms.

**Theorem 14.1 (Linearity).** *The Fourier transform is a linear operator; that is*

$$\mathcal{F}\{af(x) + b(g(x))\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{(g(x))\}$$

for complex scalars  $a$  and  $b$ , assuming the Fourier transforms of  $f$  and  $g$  exist.

The proof is left for the reader.

**Theorem 14.2** (Reflection). *If the Fourier transform of  $f$  exists and  $\mathcal{F}\{f(x)\} = \widehat{f}(k)$  then*

$$\mathcal{F}\{f(-x)\} = \widehat{f}(-k).$$

*Proof.* By direct calculation

$$\begin{aligned} \mathcal{F}\{f(-x)\} &= \int_{-\infty}^{\infty} f(-x)e^{-ikx} dx && \boxed{\text{Let } y = -x, m = -k.} \\ &= \int_{-\infty}^{\infty} f(y)e^{-imy} dy \\ &= \widehat{f}(m), \end{aligned}$$

but  $m = -k$  so  $\widehat{f}(m) = \widehat{f}(-k)$ . □

**Example 14.3.** Compute the Fourier transform of  $g(x) = H(-x)e^{ax}$ ,  $a > 0$ .

**Solution:** This is the reflection of  $f(x) = H(x)e^{-ax}$ , so

$$\widehat{g}(k) = \widehat{f}(-k) = \frac{1}{a - ik}$$

■

**Example 14.4.** Compute the Fourier transform of  $h(x) = e^{-a|x|}$ ,  $a > 0$ .

**Solution:** Since  $h(x) = e^{-a|x|} = H(x)e^{-ax} + H(-x)e^{ax} = f(x) + g(x)$ , we can use linearity to see that

$$\widehat{h}(k) = \widehat{f}(k) + \widehat{g}(k) = \frac{1}{a - ik} + \frac{1}{a + ik} = \frac{2a}{a^2 + k^2}$$

■

A dilation of a function,  $f(ax)$  corresponds to a stretching of the  $x$ -axis,

**Theorem 14.3** (Dilation). *If the Fourier transforms of  $f$  exists and  $\mathcal{F}\{f(x)\} = \hat{f}(k)$  then*

$$\mathcal{F}\{f(ax)\} = \frac{1}{a} \hat{f}\left(\frac{k}{a}\right).$$

*Proof.* By direct calculation

$$\begin{aligned} \mathcal{F}\{f(ax)\} &= \int_{-\infty}^{\infty} f(ax)e^{-ikx} dx && \boxed{\text{Let } y = ax, m = k/a.} \\ &= \frac{1}{a} \int_{-\infty}^{\infty} f(y)e^{-imy} dy. \\ &= \frac{1}{a} \hat{f}(m) \end{aligned}$$

But  $m = k/a$ , so

$$\mathcal{F}\{f(ax)\} = \frac{1}{a} \hat{f}\left(\frac{k}{a}\right).$$

□

Read this and think about how varying  $a$  would change the profile of the function  $f$  and its transform. If you squish  $f$  by a factor of  $a$ , its transform expands by the same amount. This principle is the essence behind the Heisenberg Uncertainty Principle, which basically states that *a function and its transform can not be highly localized*. In quantum mechanics, if  $f$  is the wave function,  $|f|^2$  and  $|\hat{f}|^2$  are probability distributions for position and momentum and the Heisenberg Uncertainty Principle says that localization in one of these variables implies dispersion in the other.

We can also use dilation to sneak up on the idea of a  $\delta$ -function:

**Example 14.5.** For  $\delta > 0$  consider

$$\phi_\delta(x) = \begin{cases} \frac{1}{2\delta} & -\delta < x < \delta \\ 0 & \text{otherwise.} \end{cases}$$

Find the Fourier transform of  $\phi_\delta$ .

**Solution:** Note that for  $\phi_\delta(x) = \frac{1}{2\delta}T(x/\delta)$  where  $T(x)$  is the top hat function with Fourier transform

$$\hat{T}(k) = \frac{2 \sin k}{k}.$$

Using the dilation theorem (with  $a = 1/\delta$ ), we see that

$$\hat{\phi}_\delta(k) = \frac{1}{2\delta} [\delta \hat{T}(k\delta)] = \frac{1}{2} \hat{T}(k\delta) = \frac{\sin(\delta k/2)}{\delta k/2}.$$

■

In particular, as  $\delta \rightarrow 0$ , the transform  $\hat{\phi}_\delta(k)$  approaches 1 for every fixed value of  $k$ .<sup>1</sup> Note that  $\phi_\delta(x)$  has an area that is fixed at unity, but as  $\delta$  tends to zero its width gets narrower and its height gets taller. This is an example of a  $\delta$ -sequence that is used to define a Dirac  $\delta$ -function, sometimes call a unit impulse.

Formally, the Fourier transform of the delta function,

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx = 1,$$

is equal to 1 in agreement with the limit process above.

#### 14.4 TRANSFORMS OF DERIVATIVES

Suppose  $\hat{y}(k) = \mathcal{F}\{y(x)\}$ , what can we say about  $\mathcal{F}\{y'(x)\}$ ?

$$\hat{y}(k) = \int_{-\infty}^{\infty} y(x) e^{-ikx} dx$$

and

$$\mathcal{F}\{y'(x)\} = \int_{-\infty}^{\infty} y'(x) e^{-ikx} dx.$$

We integrate by parts;

$$\begin{array}{ll} y(x) = u & y'(x) dx = du \\ e^{-ikx} = v & -ik e^{-ikx} = dv. \end{array}$$

So

$$\mathcal{F}\{y'(x)\} = y(x) e^{-ikx} \Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} (-ikx) e^{-ikx} y(x) dx.$$

If  $|y(x)| \rightarrow 0$  as  $|x| \rightarrow \infty$  the first term vanishes,

$$\mathcal{F}\{y'(x)\} = ik \int_{-\infty}^{\infty} e^{-ikx} y(x) dx = ik \hat{y}(k),$$

and the Fourier Transform turns differentiation into multiplication! Let's write this as a theorem

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<sup>1</sup>Remember  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ .

**Theorem 14.4** (Differentiation). *Suppose the Fourier transforms of  $y(x)$  and  $y'(x)$  exists and  $\mathcal{F}\{y(x)\} = \hat{y}(k)$ . Then*

$$\mathcal{F}\{y'(x)\} = ik\hat{y}(k),$$

assuming that  $y(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

**Example 14.6.** Solve the differential equation for  $y(x)$ ,

$$\text{DE: } y' + y = H(x)e^{-2x} \quad -\infty < x < \infty$$

$$\text{BC: } |y(x)| \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty.$$

**Solution:** Fourier Transform both sides

$$\mathcal{F}\{y' + y\} = \mathcal{F}\{H(x)e^{-2x}\}$$

$$\mathcal{F}\{y'\} + \mathcal{F}\{y\} = \frac{1}{2 + ik}.$$

But  $\mathcal{F}\{y\} = \hat{y}$  and  $\mathcal{F}\{y'\} = ik\hat{y}$ , so

$$(1 + ik)\hat{y} = \frac{1}{2 + ik}$$

and we can solve for  $\hat{y}$ ,

$$\hat{y} = \frac{1}{(1 + ik)} \frac{1}{(2 + ik)}$$

But

$$\begin{aligned} y(x) &= \mathcal{F}^{-1}\{\hat{y}(k)\} \\ &= \mathcal{F}^{-1}\left\{\frac{1}{(1 + ik)} \frac{1}{(2 + ik)}\right\} \\ &= \mathcal{F}^{-1}\left\{\frac{1}{1 + ik} - \frac{1}{2 + ik}\right\} \\ &= H(x)e^{-x} - H(x)e^{-2x}. \end{aligned}$$

So

$$y(x) = H(x)[e^{-x} - e^{-2x}].$$

■

**Exercise 14.1.** Suppose  $y(x)$  and its first  $n$  derivatives are absolutely integrable. Show that

$$\mathcal{F}\{y^{(n)}(x)\} = (ik)^n \hat{y}(k),$$

where  $y^{(n)}(x)$  denotes the  $n^{\text{th}}$  derivative of  $y(x)$ .

## 14.5 SHIFTING THEOREMS

The shifting theorem allows us to compute the transform of a function that has been spatially translated or *shifted*. Consider:

$$\begin{aligned}\mathcal{F}\{f(x+a)\} &= \int_{-\infty}^{\infty} f(x+a)e^{-ikx} \quad z = a+x \\ &= \int_{-\infty}^{\infty} f(z)e^{-ik(z-a)} dz \quad dz = dx \\ &= e^{ika} \int_{-\infty}^{\infty} f(z)e^{-ikz} dz \\ &= e^{ika} \widehat{f}(k)\end{aligned}$$

**Example 14.7.** Use the Fourier Transform to find the solution to PDE:

$$\begin{aligned}\mathbb{DE} : u_t + cu_x &= 0 \quad -\infty < x < \infty, t > 0 \\ \mathbb{IC} : u(x, 0) &= F(x) \quad -\infty < x < \infty\end{aligned}$$

**Solution:** If we Fourier Transform in  $x$

$$\begin{aligned}\mathcal{F}_x\{u(x, t)\} &= \int_{-\infty}^{\infty} u(x, t)e^{-ikx} dx \equiv \widehat{u}(k, t) \\ \mathcal{F}_x\{u_t(x, t)\} &= \int_{-\infty}^{\infty} u_t(x, t)e^{-ikx} dx = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u(x, t)e^{-ikx} dx \\ &= \widehat{u}_t(k, t)\end{aligned}$$

Also

$$\mathcal{F}\{u_x(x, t)\} = ik\widehat{u}(k, t).$$

the Fourier Transform of the  $\mathbb{DE}$  yields

$$\mathcal{F}\{u_t + cu_x\} = \widehat{u}_t + ikc\widehat{u} = 0.$$

This is an ODE in  $t$ . We can solve this:

$$\widehat{u}(k, t) = A(k)e^{-ikct}.$$

To solve for  $A(k)$ , Fourier transform the  $\mathbb{IC}$ ,

$$\mathcal{F}\{u(x, 0)\} = \widehat{u}(k, 0) = \mathcal{F}\{F(x)\} = \widehat{F}(k).$$

Now

$$\widehat{u}(k, 0) = A(k) = \widehat{F}(k)$$

and

$$\hat{u}(k, t) = \hat{F}(k)e^{-ikct}.$$

Now by the shifting formula,

$$u(x, t) = F(x - ct).$$

■

#### 14.6 THE CONVOLUTION THEOREM

Suppose we know the Fourier transform of two functions,  $\mathcal{F}\{u\} = \hat{u}(k)$  and  $\mathcal{F}\{v\} = \hat{v}(k)$ . Can we compute the inverse transform of their product,  $\mathcal{F}^{-1}\{\hat{u}(k)\hat{v}(k)\}$ ? Amazingly, the answer is yes!! The answer is called the *convolution* of  $u$  and  $v$  and corresponds to a sort of superposition of the two functions. Let us define the convolution.

**Definition 14.5** (Convolution in  $\mathbb{R}$ ). Let  $u, v : \mathbb{R} \rightarrow \mathbb{R}$ . The convolution of  $u$  and  $v$  at  $x$  is defined by

$$u * v = \int_{-\infty}^{\infty} u(x - y)v(y) dy.$$

The fundamental theorem for convolutions is:

**Theorem 14.5** (Convolution Theorem for Fourier Transform). *The Fourier transform of the convolution of two functions is the product of the Fourier transforms of the functions, that is if*

$$\mathcal{F}\{u\} = \hat{u}(k), \quad \mathcal{F}\{v\} = \hat{v}(k)$$

then

$$\mathcal{F}\{u * v\} = \hat{u}(k)\hat{v}(k).$$

First we will do an example.

**Example 14.8.** Consider the two functions

$$f(x) = H(x)e^{-ax}, \quad g(x) = H(-x)e^{ax},$$

where  $a > 0$ .

(a) Compute the convolution  $f * g$

(b) Verify the convolution theorem, that is if

$$\mathcal{F}\{f\} = \hat{f}(k), \quad \mathcal{F}\{g\} = \hat{g}(k)$$

then

$$\mathcal{F}\{f * g\} = \hat{f}(k)\hat{g}(k).$$

**Solution:** We proceed by direct calculation.

(a) By definition,

$$\begin{aligned} f * g &= \int_{-\infty}^{\infty} f(x-y)g(y) dy, \\ &= \int_{-\infty}^{\infty} H(x-y)e^{-a(x-y)}H(-y)e^{ay} dy, \\ &= e^{-ax} \int_{-\infty}^{\infty} H(x-y)H(-y)e^{2ay} dy, \end{aligned}$$

Now note that

$$H(x-y)H(-y) = \begin{cases} H(-y) & x \geq 0 \\ H(x-y) & x \leq 0 \end{cases}$$

So

$$e^{-ax} \int_{-\infty}^{\infty} H(x-y)H(-y)e^{2ay} dy = e^{-ax} \int_{-\infty}^0 e^{2ay} dy = \frac{e^{-ax}}{2a} \quad x \geq 0$$

and

$$e^{-ax} \int_{-\infty}^{\infty} H(x-y)H(-y)e^{2ay} dy = e^{-ax} \int_{-\infty}^x e^{2ay} dy = e^{-ax} \frac{e^{2ax}}{2a} = \frac{e^{ax}}{2a} \quad x \leq 0$$

from which we decide that

$$f * g = \frac{e^{-a|x|}}{2a}.$$

(b) We've shown previously that if  $f(x) = H(x)e^{-ax}$  and  $g(x) = H(-x)e^{ax}$  then

$$\hat{f}(k) = \frac{1}{a + ik}, \quad \hat{g}(k) = \frac{1}{a - ik}.$$

Note that

$$\begin{aligned}
 \mathcal{F}\{f * g\} &= \frac{1}{2a} \mathcal{F}\{e^{-a|x|}\} \\
 &= \frac{1}{2a} \frac{2a}{a^2 + k^2} \\
 &= \frac{1}{a^2 + k^2} \\
 &= \frac{1}{a + ik} \frac{1}{a - ik} \\
 &= \hat{f}(k) \hat{g}(k)
 \end{aligned}$$

which verifies the convolution theorem. ■

*Proof of Convolution Theorem.* Proof goes here. □

#### 14.7 $\delta$ -SEQUENCES AND THE TRANSFORM OF A $\delta$ -FUNCTION

A  $\delta$ -sequence satisfies

$$\int_{-\infty}^{\infty} \delta_{\epsilon}(x) dx = 1 \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} \delta_{\epsilon}(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} .$$

**Theorem 14.6.** *Given any function  $f(x)$  that is  $f(x) > 0$ ,  $f(x)$  is continuous,*

$$\int_{-\infty}^{\infty} f(x) dx = 1,$$

and

$$\lim_{x \rightarrow \pm\infty} f(x) = 0$$

then

$$f_{\epsilon}(x) = \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right)$$

is a  $\delta$ -sequence.

*Proof.* Note that

$$\begin{aligned}\int_{-\infty}^{\infty} f_{\epsilon}(x) dx &= \int_{-\infty}^{\infty} f\left(\frac{x}{\epsilon}\right) \frac{x}{\epsilon} = z \quad \frac{dx}{\epsilon} = dz \\ &= \int_{-\infty}^{\infty} f(z) dz \\ &= 1\end{aligned}$$

Moreover, if  $f(x)$  is absolutely integrable, then □

We can now compute the Fourier Transform of the  $\delta$  function is the limit of the Fourier Transform of a  $\delta$ -sequence. Let

$$\mathcal{F}\{f_{\epsilon}(x)\} = \widehat{f}_{\epsilon}(k).$$

First note

$$\widehat{f}_{\epsilon}(0) = \int_{-\infty}^{\infty} f_{\epsilon}(x) e^{-i0x} dx = \int_{-\infty}^{\infty} f_{\epsilon}(x) dx = 1.$$

Also,

$$\begin{aligned}\mathcal{F}\{f_{\epsilon}\} &= \mathcal{F}\left\{\frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right)\right\} \\ &= \frac{1}{\epsilon} \mathcal{F}\left\{f\left(\frac{x}{\epsilon}\right)\right\} \\ &= \frac{1}{\epsilon} \mathcal{F}\left\{f\left(\frac{x}{\epsilon}\right)\right\} \\ &= \frac{1}{\epsilon} [\epsilon \widehat{f}(k\epsilon)] \\ &= \widehat{f}(k\epsilon).\end{aligned}$$

Now

$$\lim_{\epsilon \rightarrow 0} \widehat{f}_{\epsilon}(k) = \lim_{\epsilon \rightarrow 0} \widehat{f}(k\epsilon)$$

and for any fixed  $k$ ,<sup>2</sup>

$$\lim_{\epsilon \rightarrow 0} \widehat{f}_{\epsilon}(k) = \widehat{f}(0) = 1.$$

This is the Fourier Transform

$$\mathcal{F}\{\delta(x)\} = 1.$$

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<sup>2</sup>Note that we assume the Fourier Transform is continuous, which can be proven.

## 14.8 GREEN'S FUNCTIONS

The Green's function is the response to a  $\delta$ -function forcing:

$$\begin{aligned} g'' + 2g' + 5g &= \delta(x) \\ \mathcal{F}\{g'' + 2g' + 5g\} &= \mathcal{F}\{\delta(x)\} \\ (ik)^2 + 2ik + 5\hat{g} &= 1 \\ \hat{g} &= \frac{1}{(ik)^2 + 2ik + 5} \end{aligned}$$

Note that

$$\begin{aligned} (ik)^2 + 2ik + 5 &= (ik + 1)^2 + 4 \\ &= [(ik + 1 + 2i)(ik + 1 - 2i)] \end{aligned}$$

so

$$\begin{aligned} \hat{g}(k) &= \frac{1}{(ik + 1 + 2i)(ik + 1 - 2i)} \\ &= \frac{A}{ik + 1 + 2i} + \frac{B}{ik + 1 - 2i} \\ &= \frac{-\frac{1}{4i}}{ik + 1 + 2i} + \frac{\frac{1}{4i}}{ik + 1 - 2i} \\ &= \frac{1}{4i} \left[ \frac{1}{ik + 1 - 2i} - \frac{1}{ik + 1 + 2i} \right] \end{aligned}$$

We then need to figure out the inverse transform of this:

$$\begin{aligned} \mathcal{F}\{H(x)e^{-ax}\} &= \int_0^{\infty} e^{-ax-ikx} \\ &= -\frac{e^{-(a+ik)x}}{a+ik} \Big|_{x=0}^{\infty} \\ &= \frac{1}{a+ik} \quad \Re\{a\} > 0 \end{aligned}$$

Suppose  $a = \alpha + i\beta$ . Then

$$\mathcal{F}\{H(x)e^{-(\alpha+i\beta)x}\} = \frac{1}{\alpha + i\beta + ik} = e^{-\alpha x}.$$

so

$$\begin{aligned}
 g(x) &= \mathcal{F}^{-1} \left\{ \frac{1}{4i} \left[ \frac{1}{ik+1-2i} - \frac{1}{ik+1+2i} \right] \right\} \\
 &= \frac{1}{4i} [H(x)e^{-x}e^{2ix} - H(x)e^{-x}e^{-2ix}] \\
 &= \frac{1}{2}H(x)e^{-x} \sin(2x)
 \end{aligned}$$

#### 14.9 THE CAUCHY PROBLEM FOR THE HEAT EQUATION

We will solve the heat equation using the Fourier transform, but first we need to know the Fourier transform of a gaussian. We start out with a seemingly unrelated lemma which you may have seen in multivariable calculus.

**Lemma 14.6.** *Show*

$$I(a) = \int_{-\infty}^{\infty} e^{-ax^2/2} dx = \sqrt{\frac{2\pi}{a}}.$$

for  $a > 0$

**Solution:** We use polar coordinates

$$\begin{aligned}
 [I(a)]^2 &= \int_{-\infty}^{\infty} e^{-ax^2/2} dx \int_{-\infty}^{\infty} e^{-ay^2/2} dy \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{a}{2}(x^2+y^2)} dx dy \\
 &= \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-a\frac{r^2}{2}} r dr d\theta \\
 &= \frac{2\pi}{a} \int_0^{\infty} e^{-\frac{ar^2}{2}} a r dr \\
 &= \frac{2\pi}{a} (-e^{-ar^2/2}) \Big|_{r=0}^{\infty} \\
 &= \frac{2\pi}{a}
 \end{aligned}$$

Now, as  $[I(a)]^2 = \frac{2\pi}{a}$ , we conclude that

$$I(a) = \sqrt{\frac{2\pi}{a}}$$

### 14.9.1 Fourier Transform of a Gaussian

**Example 14.9.** Show that

$$\mathcal{F}\{e^{-ax^2/2}\} = e^{-\frac{k^2}{2a}} \sqrt{\frac{2\pi}{a}}$$

$$\mathcal{F}\left\{\frac{1}{2\sqrt{\pi b}} e^{-x^2/4b}\right\} = e^{-bk^2}.$$

**Solution:** From the definition of the Fourier Transform, we know that

$$\mathcal{F}\{e^{-ax^2/2}\} = \int_{-\infty}^{\infty} e^{-ax^2/2+ikx} dx.$$

Let's complete the square

$$\begin{aligned} \frac{ax^2}{2} + ikx &= \frac{a}{2} \left[ x^2 + \frac{2ik}{a}x \right] \\ &= \frac{a}{2} \left[ \left( x + \frac{ik}{a} \right)^2 - \left( \frac{ik}{a} \right)^2 \right] \\ &= \frac{a}{2} \left( x + \frac{ik}{a} \right)^2 + \frac{k^2}{2a} \end{aligned}$$

So

$$\begin{aligned} \mathcal{F}\{e^{-ax^2/2}\} &= \int_{-\infty}^{\infty} e^{-\frac{a}{2} \left( x + \frac{ik}{a} \right)^2 - \frac{k^2}{2a}} dx \\ &= e^{-\frac{k^2}{2a}} \int_{x=-\infty}^{\infty} e^{-\frac{a}{2} \left( x + \frac{ik}{a} \right)^2} dx \end{aligned}$$

Let

$$z = x + \frac{ik}{a} \Rightarrow dz = dx.$$

We need to be a little careful here, because we have shifted the integral into the complex plane an amount  $\frac{ik}{a}$ ; complex variables tell us we can just drag the contour back down to the real axis to show that

$$\begin{aligned} \mathcal{F}\{e^{-ax^2/2}\} &= e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\frac{a}{2}z^2} dz \\ &= e^{-\frac{k^2}{2a}} \sqrt{\frac{2\pi}{a}} \end{aligned}$$

So

$$\mathcal{F}\{e^{ax^2/2}\} = \sqrt{\frac{2\pi}{a}} e^{-k^2/2a}.$$

Also if we set

$$b = \frac{1}{2a} \implies a = \frac{1}{2b}$$

it follows that

$$\mathcal{F}\left\{\frac{1}{2\sqrt{\pi b}} e^{-x^2/4b}\right\} = e^{-bk^2}.$$

■

**Example 14.10.** Use the Fourier transform to solve the Cauchy problem for the heat equation

$$\begin{aligned} \text{DE} : u_t &= Du_{xx} & -\infty < x < \infty, t > 0 \\ \text{IC} : u(x, 0) &= f(x) & -\infty < x < \infty \\ \text{BC} : \lim_{x \rightarrow \pm\infty} |u(x, t)| &\text{ is bounded for all } t > 0 \end{aligned}$$

where

- (a)  $f(x) = \delta(x)$ , the  $\delta$ -function.
- (b)  $f(x) = f(x)$ , an arbitrary function
- (c)  $f(x) = H(x)$  the Heaviside function.

**Solution:** We once again use the the Fourier Transform to turn a PDE into an ODE; define

$$\mathcal{F}\{u(x, t)\} = \hat{u}(k, t)$$

then

$$\mathcal{F}\{u_t(x, t)\} = \hat{u}_t(k, t) \quad \mathcal{F}\{u_{xx}(x, t)\} = (ik)^2 \hat{u}(k, t) = -k^2 \hat{u}(k, t),$$

So the Fourier transform of the DE yields

$$u_t = Du_{xx} \implies \hat{u}_t = -Dk^2 \hat{u},$$

which is easily solved to yield

$$\hat{u}(k, t) = A(k)e^{-Dk^2 t}$$

where the constant  $A(k)$  depends on the transform variable. From the  $\mathbb{IC}$ , we see that

$$\mathcal{F}\{u(x, 0)\} = \hat{u}(k, 0) = \mathcal{F}\{f(x)\} = \hat{f}(k),$$

but from (14.10),

$$\hat{u}(k, 0) = A(k) = \hat{f}(k) \implies \hat{u}(k, t) = \hat{f}(k)e^{-Dk^2t}.$$

We need to know  $\mathcal{F}\{e^{-(Dt)k^2}\}$ ; setting  $(b = Dt)$  in the transform above, we see that

$$\mathcal{F}^{-1}\{e^{-Dk^2t}\} = G(x, t) \equiv \frac{1}{2\sqrt{\pi Dt}}e^{-x^2/4Dt}.$$

Here  $G(x, t)$  is the *Green's function* (sometimes call the *kernel*) of the heat equation.

Let's now consider the three initial conditions given above,

- (a) If  $f(x) = \delta(x)$ ,  $\mathcal{F}\{\delta(x)\} = 1$ . Then

$$u(x, t) = G(x, t) = \frac{e^{-x^2/4Dt}}{2\sqrt{\pi Dt}}.$$

This is a spreading gaussian whose width scales like  $\sqrt{Dt}$  and whose height decreases like  $\frac{1}{2\sqrt{\pi Dt}}$ . It's area = 1. The diffusion is *self-similar* - it spreads out, but maintains its characteristic shape.

- (b) If  $f(x)$  is a general function, we can still solve the problem using the convolution theorem. The solution

$$\hat{u}(k, t) = \hat{f}(k, t)e^{-Dtk^2}$$

implies  $u(x, t)$  is the convolution of the initial condition and the green's function,

$$u(x, t) = f(x) * G(x, t) = \frac{1}{2\sqrt{\pi Dt}} \int_{-\infty}^{\infty} f(x - y)e^{-y^2/4Dt} dy$$

which is known as the *Poisson Integral Formula* for the solution to the Cauchy problem of the heat equation.

- (c) Finally, let us consider the specific case when the initial condition is the Heaviside function,  $f(x) = H(x)$ . Then from the convolution formula,

$$u(x, t) = \frac{1}{2\sqrt{\pi Dt}} \int_{-\infty}^{\infty} H(x - y)e^{-y^2/4Dt} dy.$$

But

$$H(x - y) = \begin{cases} 1 & x > y \\ 0 & x < y \end{cases}.$$

So

$$u(x, t) = \frac{1}{2\sqrt{\pi Dt}} \int_{-\infty}^x e^{-y^2/4Dt} dy.$$

We can rewrite the integral in terms of the error function. Let  $z = \frac{y}{2\sqrt{Dt}}$  then  $dz = \frac{dy}{2\sqrt{Dt}}$ . So

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{Dt}}} e^{-z^2} dz.$$

Remember the error function

$$\operatorname{erf}(w) \equiv \frac{2}{\sqrt{\pi}} \int_0^w e^{-s^2} ds.$$

so

$$\begin{aligned} u(x, t) &= \frac{1}{2} \left[ \frac{2}{\sqrt{\pi}} \int_{-\infty}^0 e^{-w^2} dw + \int_0^{\frac{x}{2\sqrt{Dt}}} e^{-w^2} dw \right] \\ &= \frac{1}{2} \left[ -\operatorname{erf}(-\infty) + \operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right) \right] \\ &= \frac{1}{2} \left[ 1 + \operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right) \right] \end{aligned}$$

The reader is encouraged to graph this solution. ■

## 14.10 D'ALEMBERT SOLUTION TO THE WAVE EQUATION

In this section we will use the Fourier Transform to verify a portion of the d'Alembert solution. First, however we consider a warm-up problem

### Exercise 14.2.

Show that if

$$I_a(x) = \begin{cases} 0 & |x| > a \\ 1 & |x| \leq a \end{cases}$$

then the Fourier transform of  $I_a(x)$  is given by

$$\mathcal{F}\{I(x)\} = \frac{2 \sin(ka)}{k} .$$

**Solution:** By direct calculation we see that

$$\begin{aligned} \mathcal{F}\{I_a(x)\} &= \int_{-\infty}^{\infty} I(x)e^{-ikx} dx = \int_{-a}^a e^{-ikx} dx \\ &= \left. \frac{e^{-ikx}}{-ik} \right|_{x=-a}^{x=a} \\ &= \frac{e^{ika} - e^{-ika}}{-ik} \\ &= \frac{2 \sin(ka)}{k} . \end{aligned}$$

as advertised above. □

We now use the Fourier transform to solve the second-order wave equation for  $W(x, t)$ ,

$$W_{tt} = c^2 W_{xx} \quad -\infty < x < \infty, \quad 0 < t$$

$$W(x, 0) = f(x) \quad W_t(x, 0) = g(x)$$

**Solution:** Define  $\widehat{W}(k, t)$  to be the Fourier transform of  $W(x, t)$  with respect to  $x$ . The Fourier transform turns the partial differential equation problem into an ordinary differential equation problem:

$$\widehat{W}_{tt} + c^2 k^2 \widehat{W} = 0,$$

with

$$\widehat{W}(k, 0) = \widehat{f}(k), \quad \widehat{W}_t(k, 0) = \widehat{g}(k),$$

where  $\widehat{f}(x)$  and  $\widehat{g}(k)$  are the Fourier Transform of  $f(x)$  and  $g(x)$  respectively. The general solution to this ODE is

$$\widehat{W}(k, t) = A(k) \cos(ckt) + B(k) \sin(ckt) .$$

Applying the initial conditions yield

$$\widehat{W}(k, 0) = A(k) = \widehat{f}(k) ,$$

and

$$\widehat{W}_t(k, t) = -kcA(k) \sin(kct) + kcB(k) \cos(kct) \Rightarrow \widehat{W}_t(k, 0) = kcB(k) = \widehat{g}(k)$$

which implies  $A(k) = \widehat{f}(k)$  and  $B(k) = \widehat{g}(k)/(ck)$ . Therefore

$$\widehat{W}(k, t) = \widehat{f}(k) \cos(kct) + \widehat{f}(k) \frac{\sin(kct)}{ck}$$

We can write

$$\widehat{W}(k, t) = \widehat{W}_1(k, t) + \widehat{W}_2(k, t)$$

where

$$\widehat{W}_1(k, t) = \widehat{f}(k) \cos(kct), \quad \widehat{W}_2(k, t) = \widehat{g}(k) \frac{\sin(kct)}{ck}$$

$$W_2(x, t) = \mathcal{F}^{-1}\{\widehat{W}_2(k, t)\}$$

$$= \mathcal{F}^{-1}\{\widehat{f}(k) \cos(kct)\} = \frac{1}{2} \mathcal{F}^{-1}\{\widehat{f}(k) [e^{ikct} + e^{-ikct}]\} = \frac{1}{2} [f(x - ct) + f(x + ct)]$$

where we have used the first shifting theorem to compute the inverse transforms. Also, by the convolution theorem and the result in part (a) we see that

$$\begin{aligned} W_2(x, t) &= \mathcal{F}^{-1}\{\widehat{W}_2(k, t)\} \\ &= \frac{1}{2c} \mathcal{F}^{-1}\left\{\widehat{f}(k) \left[\frac{2 \sin(kct)}{k}\right]\right\} \\ &= \frac{1}{2c} g(x) * I_{ct}(x) = \frac{1}{2c} \int_{-\infty}^{\infty} g(y) I_{ct}(x - y) dy. \end{aligned}$$

We see that  $I_{ct}(x - y) = 1$  for  $x - ct < y < x + ct$  and 0 otherwise, from which we deduce

$$W_2(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(y) dy.$$

Now by linearity of the Fourier transform,  $W(x, t) = W_1(x, t) + W_2(x, t)$  which yields

$$W(x, t) = \frac{1}{2} [f(x - ct) + f(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

which is identical to the d'Alembert solution.