# Sixteen

# Sturm-Liouville Theory

In this lecture we will show that many eigenvalue problems can be put into Sturm-Liouville form,

$$(s(x)X'(x))' + (\lambda\rho(x) - q(x))\,X(x) = 0 \tag{16.1}$$

and that these problems generically have real eigenvalues and associated eigenvectors that are orthogonal with respect to a weighted inner-product. We begin with an example.

## 16.1 An example from population biology

Let's consider a simple model of fish populations:

**Example 16.1.** A population of trout, $u(x,t)$, is introduced into a river. The river flows at a speed $c$; a portion of the river of length $L$ is isolated between a waterfall at $x = 0$ and a dam at $x = L$. The trout reproduce at a rate $\alpha$ and also diffuse up and downstream with a diffusion constant $D$. The population is modeled by the advection-growth-diffusion equation

$$\begin{aligned}
\mathbb{DE}: &\quad u_t = Du_{xx} + cu_x + \alpha u \qquad 0 < x < L, \quad t > 0\\
\mathbb{BC}: &\quad u(0,t) = u(L,t) = 0, \quad t > 0\\
\mathbb{IC}: &\quad u(x,0) = f(x) \quad 0 < x < L.
\end{aligned}$$

where the initial population satisfies $f(x) \geq 0$. Can we find a solution for $u(x,t)$? Does the population grow or decay? ■

### 16.1.1 Separation of Variables

We look for solutions of the form

$$u(x,t) = X(x)T(t). \tag{16.2}$$

Plugging this form into the differential equation we get

$$X(x)T'(t) = [DX''(x) + cX'(x) + \alpha X(x)]\,T(t)$$

and dividing by $X(x)T(t)$ we find

$$\frac{T'(t)}{T(t)} = \frac{DX''(x) + cX'(x) + \alpha X(x)}{X(x)}.$$

Notice that the left hand side is a function of $t$ alone, while the right is a function of $x$ only. This implies that both sides must indeed be constant. We will call this *separation constant*, $-\lambda$. Thus we have

$$\frac{T'(t)}{T(t)} = \frac{DX''(x) + cX'(x) + \alpha X(x)}{X(x)} = -\lambda. \tag{16.3}$$

We can separate this equation into two equations, one involving only $t$,

$$\frac{T'(t)}{T(t)} = -\lambda,$$

and one involving only $x$,

$$\frac{DX''(x) + cX'(x) + \alpha X(x)}{X(x)} = -\lambda.$$

We will now assume that $\lambda$ is real. This assumption could plausibly cause us to lose some solutions, but eventually we will show that it is the only case that yields non-trivial solutions.

Each of these equations is now an ordinary differential equation, and thus we can draw on the theory of ordinary differential equations to solve them. The first equation,

$$T'(t) = -\lambda T(t)$$

has the solution

$$T(t) = Ae^{-\lambda t}.$$

Before we solve the second-order ordinary differential equation in $x$, we will derive some boundary conditions for this equation by apply the separation of variable ansatz to the boundary conditions on the PDE. Note that

$$u(0,t) = X(0)T(t) = 0$$

which implies either $X(0) = 0$ or $T(t) = 0$. If we choose $T(t) = 0$ then $u(x,t) = X(x)T(t) = 0$, which, while true, is just the trivial solution. Therefore we conclude that

$$u(0,t) = X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0$$

for us to find non-trivial solutions. Similarly

$$u(L,t) = X(L)T(t) = 0 \quad \Rightarrow \quad X(L) = 0$$

by analogous reasoning. Which yields the boundary value problem,

$$\mathbb{DE}: \quad DX''(x) + cX'(x) + (\alpha + \lambda)X(x) = 0 \qquad 0 < x < L \tag{16.4}$$

$$\mathbb{BC}: \quad X(0) = 0, \quad X(L) = 0. \tag{16.5}$$

This is an example of a *Sturm-Liouville Eigenvalue Problem*; we will show that this problem only has non-trivial solutions for certain values of $\lambda$, called the eigenvalues.

To solve (16.4), we look for solutions of the form $X(x) = e^{rx}$ which yields the characteristic equation

$$Dr^2 + cr + (\alpha + \lambda) = 0$$

which implies

$$r = \frac{-c \pm \sqrt{c^2 - 4(\alpha + \lambda)D}}{2D}.$$

As before, we will assume that $\lambda$ is real, a fact that we will prove later. There are now three cases:

- **Two real roots:** When

  $$c^2 - 4(\alpha + \lambda)D > 0 \quad \Rightarrow \quad \lambda < \frac{c^2}{4D} - \alpha$$

  there are two real roots

  $$r_{\pm} = \frac{-c \pm \sqrt{c^2 - 4(\alpha + \lambda)D}}{2D}.$$

In this case
$$X(x) = Be^{r_+ x} + Ce^{r_- x},$$
and to satisfy the two boundary conditions
$$X(0) = B + C = 0 \qquad X(L) = Be^{r_+ L} + Ce^{r_- L} = 0$$
which written in matrix form yields
$$\begin{bmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
which would only have a non-trivial solution if the determinant of the coefficient matrix is zero, that is if
$$e^{r_- L} = e^{r_+ L}$$
which implies $r_+ = r_-$ which is a contradiction.

- **Two equal roots:** When
$$c^2 - 4(\alpha + \lambda)D = 0 \quad \Rightarrow \quad \lambda = \frac{c^2}{4D} - \alpha$$
then the two roots are real, that is
$$r = -\frac{c}{2D}.$$
In this case
$$X(x) = Be^{-cx/2D} + Cxe^{-cx/2D},$$
and to satisfy the two boundary conditions
$$X(0) = B = 0 \qquad X(L) = Be^{-cL/2D} + CLe^{-cL/2D} = 0$$
which implies $B = C = 0$ and again there is no non-trivial solution.

- **A complex conjugate pair of roots:** When
$$c^2 - 4(\alpha + \lambda)D < 0 \quad \Rightarrow \quad \lambda > \frac{c^2}{4D} - \alpha$$
there is a complex conjugate pair of roots,
$$r_\pm = -\frac{c}{2D} \pm i\Omega \qquad \Omega = \frac{\sqrt{4(\alpha + \lambda)D - c^2}}{2D}.$$

In this case

$$X(x) = Be^{-cx/2D}\cos(\Omega x) + Ce^{-cx/2D}\sin(\Omega x),$$

the first boundary condition implies

$$X(0) = B = 0$$

while the second implies that

$$X(L) = Ce^{-cx/2D}\sin(\Omega L) = 0$$

or that

$$\Omega = \Omega_n \equiv \frac{n\pi}{L} \qquad n = 1, 2, 3 \cdots .$$

Solving

$$\Omega_n = \frac{n\pi}{L} = \frac{\sqrt{4(\alpha+\lambda)D - c^2}}{2D}$$

yields

$$\lambda = \lambda_n \equiv -\alpha + \frac{c^2}{4D} + \frac{Dn^2\pi^2}{L^2} \qquad n = 1, 2, 3 \cdots .$$

Summarizing, we have found a countable set of eigenfunctions and eigenvalues,

$$\boxed{X_n(x) = e^{-cx/2D}\sin\left(\frac{n\pi x}{L}\right) \qquad \lambda_n = -\alpha + \frac{c^2}{4D} + \frac{Dn^2\pi^2}{L^2}} \tag{16.6}$$

for $n = 1, 2, 3, \cdots$. We can associate with each eigenvalue a solution to the ODE for $T(t)$,

$$T_n(t) \equiv e^{-\lambda_n t},$$

where we note that again we have chosen the multiplicative constant $A = 1$ and the index $n$ recognizes that we restricting ourselves to the case when $\lambda = \lambda_n$.

We now have a countable set of solutions which satisfy both the differential equation, and the boundary values, namely

$$u_n(x,t) \equiv X_n(x)T_n(t) = e^{-\lambda_n t} \cdot e^{-cx/2D}\sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, ...$$

However, we know that as the differential equation and boundary condition are homogeneous the solutions form a vector space!! So the most general solution is a linear combination of the $u_n$'s.

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n u_n(x,t) \\ &= \sum_{n=1}^{\infty} a_n X_n(x) T_n(t) \\ &= \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \cdot e^{-cx/2D} \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

where the $a_n$ are arbitrary constants. So the general solution is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \cdot e^{-cx/2D} \sin\left(\frac{n\pi x}{L}\right)} \tag{16.7}$$

### 16.1.2 Solving the initial value problem

To summarize, so far we have found a general solution (16.7) that satisfies both the differential equation and the associated boundary conditions for the homogeneous Dirichlet problem for the diffusion equation. We still need to satisfy the initial condition, $u(x,0) = f(x)$, which we will argue determines the arbitrary constants $b_n$.

Apply the initial condition to the general solution yields

$$u(x,0) = \sum_{n=1}^{\infty} a_n X_n(x) = \sum_{n=1}^{\infty} a_n e^{-cx/2D} \sin\left(\frac{n\pi x}{L}\right) = f(x). \tag{16.8}$$

At first this seems daunting; we have the solution as the sum of a set of functions we have never seen before. However, note what happens if we multiply through by $e^{cx/2D}$,

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = e^{cx/2D} f(x), \qquad 0 < x < L.$$

This is just a Fourier Sine Series for the function $e^{cx/2D} f(x)$. Consequently, we know that

$$\boxed{a_n = \frac{2}{L}\int_0^{\pi} e^{cx/2D} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx \qquad n = 1, 2, 3, \ldots.}$$

### 16.1.3 Analyzing the solution: Does the population grow or decay?

Looking at the solution,

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \cdot e^{-cx/2D} \sin\left(\frac{n\pi x}{L}\right),$$

we see the $n^{th}$ term in the sum will grow only if

$$\lambda_n < 0 \quad \Rightarrow \frac{c^2}{4D} + \frac{Dn^2\pi^2}{L^2} < \alpha$$

The smallest eigenvalue is $\lambda_1$, so when the growth rate $\alpha$ satisfies

$$\alpha > \alpha_c \equiv \frac{c^2}{4D} + \frac{Dn^2\pi^2}{L^2}$$

we will have growth. Note that the threshold growth rate increases as $c$ increases (think about fish being swept downstream) and decreases as the domain gets longer. We also note that

$$a_0 = \frac{2}{L}\int_0^{\pi} e^{cx/2D} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

is positive for any non-zero initial population, $f(x) \geq 0$ (remember populations are non-negative!!).

## 16.2 Solution to a Population Problem: What Just Happened?

To summarize, we have suggested that the solution to

$$\begin{aligned}
\mathbb{DE}: &\quad u_t = Du_{xx} + cu_x + \alpha u \qquad 0 < x < L, \quad t > 0 \\
\mathbb{BC}: &\quad u(0,t) = u(L,t) = 0, \quad t > 0 \\
\mathbb{IC}: &\quad u(x,0) = f(x) \quad 0 < x < L.
\end{aligned}$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \cdot e^{-cx/D} \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{2}{L}\int_0^{\pi} e^{cx/D} f(x) \sin(nx)\, dx$$

However, the derivation has left us with a number of questions:

- Are the eigenvalues always real? Are they always positive?
- Have we found all the eigenvalues and eigenfunctions?
- Is there a systematic way to find the coefficients $a_n$? In particular, can we found an orthogonality condition for the set of eigenfunctions, $\{X_n(x)\}$?

We will attempt to answer these questions (or at least indicate the answers) below using Sturm-Liouville Theory.

## 16.3 Sturm-Liouville Eigenvalue Problems

We begin by defining the Sturm-Liouville Eigenvalue Problem:

**Definition 16.9** (Sturm-Liouville Eigenvalue Problem)**.** Let $y(x)$ be a twice continuously differentiable function on the interval $a \leq x \leq b$ (i.e. $y(x) \in C^2[a,b]$) . Let $\mathcal{L}$ be the Sturm-Liouville differential operator defined by

$$\mathcal{L}y \equiv -\left(s(x)y'\right)' + q(x)y$$

where the functions $s(x), q(x), \rho(x)$ are continuous functions on $[a,b]$ with $s(x)$ and $\rho(x)$ both positive-valued on $[a,b]$. The regular *Sturm-Liouville Eigenvalue Problem* is defined by the differential equation

$$\mathbb{DE}: \qquad \mathcal{L}y = \lambda\rho(x)y, \quad a < x < b, \tag{16.10}$$

together with the boundary conditions

$$\mathbb{BC}: \qquad \alpha y(a) + \beta y'(a) = 0, \quad \gamma y(b) + \delta y'(b) = 0 \tag{16.11}$$

where $\alpha$ and $\beta$ are not both zero and $\gamma$ and $\delta$ are not both zero. We call a constant $\lambda$ and a non-zero function $y$ that satisfy this problem an *eigenvalue/eigenfunction pair*.

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**Exercise 16.1.** Show that the Sturm-Liouville form stated at the beginning of the chapter,

$$\left(s(x)X'(x)\right)' + \left(\lambda\rho(x) - q(x)\right)X(x) = 0 \tag{16.12}$$

and the form (16.10) stated in the definition above

$$\mathcal{L}y = \lambda\rho(x)y, \tag{16.13}$$

are equivalent when $X(x)$ is replaced $y(x)$.

We will prove a set of theorems about the eigenfunctions and eigenvalues of this problem, but first we need to introduce an appropriate vector space of real functions and an inner product. Define

$$\mathcal{U} \equiv \{y(x) \in C^2[a,b],\ \alpha y(a) + \beta y'(a) = 0, \quad \gamma y(b) + \delta y'(b) = 0\},$$

The idea here is we only consider the set of functions that satisfy the boundary conditions.

**Exercise 16.2.** Show that $\mathcal{U}$ is a vector space.

In the same way that we generalized the eigenvalue problem, we also need to generalize the inner-product associated with the eigenfunctions.

**Definition 16.14** (Weighted Inner-product)**.** If $\rho(x)$ is defined on $x \in [a,b]$ with $\rho(x) > 0$ for $x \in (a,b)$ we can define the weighted inner-product,

$$\langle u, v \rangle_\rho = \int_a^b uv\ \rho(x)\ dx.$$

Note that if $\rho(x) = 1$ this is the normal $L^2$ inner-product. The vector space $\mathcal{U}$ is a real inner product space with respect to this weighted inner-product. We will show the eigenfunctions for the Sturm-Liouville eigenvalue problem are orthogonal with respect to a weighted inner-product.

**Definition 16.15** (Weighted norm)**.** We defined the weighted norm of a function, as

$$\|u\|_\rho \equiv \sqrt{\langle u, u \rangle_\rho} = \sqrt{\int_a^b u^2\ \rho(x)\ dx}.$$

### 16.3.1 Examples of Sturm-Liouville equations

(a) The Fourier Eigenvalue Problem for $X(x)$ (Dirichlet boundary conditions):

$$\begin{aligned} \mathbb{DE}: &\quad X'' + \lambda X = 0 \\ \mathbb{BC}: &\quad X(a) = 0, \quad X(b) = 0 \end{aligned}$$

for $x \in [a,b]$ satisfies the formal definition (16.10) for $s(x) = 1,\ q(x) = 0$, and weight function $\rho(x) = 1$. Note that this is a regular Sturm-Liouville problem.

(b) Bessel's equation for $\phi(x)$:

$$\begin{aligned} \mathbb{DE}: &\quad (x\phi'(x))' + \lambda\,(x\phi(x)) = 0 \\ \mathbb{BC}: &\quad \phi(1) = 0, \quad \phi(2) = 0 \end{aligned}$$

is a Sturm-Liouville equation with $s(x) = x,\ q(x) = 0$, and weight function $\rho(x) = x$. This is a regular Sturm-Liouville problem.

Sturm-Liouville equations naturally result from separation of variables applied to many problems of physical and mathematical interest. Fortunately, these equations with appropriate boundary conditions provide a wealth of orthogonal functions. In fact, we will see that regular Sturm-Liouville problems have an infinite number of eigenvalues, and the corresponding eigenfunctions form a *complete*, orthogonal set.

### 16.3.2 The Self-adjoint Sturm Liouville operator

Remember the differential operator $\mathcal{P}$ acting on elements in an inner product space is *self-adjoint* if

$$\langle u, \mathcal{P}v\rangle = \langle \mathcal{P}u, v\rangle\,.$$

We now show the Sturm-Liouville operator

$$\mathcal{L}y \equiv -\left(s(x)y'\right)' + q(x)y$$

is self-adjoint in $\mathcal{U}$.

**Theorem 16.1.** *The differential operator $\mathcal{L}$ is self-adjoint in the inner-product space $\mathcal{U}$ with the standard $L^2$ inner-product.*

*Proof.* The proof follows by integration by parts; suppose $u$ and $v$ are elements of one of the inner product spaces. Then

$$\begin{aligned} \langle u, \mathcal{L}v\rangle &= -\int_a^b u[(sv')']\,dx + \int_a^b u[qv]\,dx \\ &= \int_a^b su'v'\,dx - suv'|_{x=a}^{x=b} + \int_a^b quv\,dx \\ &= -\int_a^b su''v\,dx + s(u'v - uv')|_{x=a}^{x=b} + \int_a^b quv\,dx \\ &= \langle \mathcal{L}u, v\rangle + s(u'v - uv')|_{x=a}^{x=b}\,. \end{aligned}$$

Let us consider the boundary at $x = a$. Remember that $u$ and $v$ are in $\mathcal{U}$ and therefore satisfy the boundary conditions

$$\alpha u(a) + \beta u'(a) = 0, \quad \alpha u(a) + \beta u'(a) = 0$$

with $\alpha$ and $\beta$ not both zero. To show the boundary term vanishes at $x = a$ it suffice to show that $u'(a)v(a) - u(a)v'(a) = 0$, If $\alpha = 0$ (which means $\beta \neq 0$) then $u'(a) = v'(a) = 0$ and the boundary term vanishes. Otherwise

$$\begin{aligned} \alpha(u'(a)v(a) - u(a)v'(a)) &= u'(a)[\alpha v(a)] - v'(a)[\alpha u(a)] \\ &= u'(a)[-\beta v'(a)] - v'(a)[-\beta u'(a)] \\ &= \beta[-u'(a)v'(a) + v'(a)u'(a)] \\ &= 0 \end{aligned}$$

and again the boundary term vanishes. A similar argument holds at $x = b$. Therefore

$$\langle u, \mathcal{L}v \rangle = \langle \mathcal{L}u, v \rangle ,$$

and we have shown the operator is self-adjoint. □

### 16.3.3 The Eigenvalues of the Sturm-Liouville Problem are Real

The proof that the eigenvalues of the Sturm-Liouville problem are real follows analogously to the proof for the Fourier Eigenvalue problem.

**Theorem 16.2.** *Suppose $\mathcal{L}$ is self-adjoint linear operator associated with the inner product space $\mathcal{U}$. Then the eigenvalue problem*

$$\mathcal{L}y = \lambda \rho y$$

*where*

$$y = p + iq \qquad p, q \in \mathcal{U}$$

*has only real eigenvalues, $\lambda$.*

**Remark.** Note that we have temporarily expanded the eigenvalue problem to allow $y$ to be a complex function with real part $p$ and imaginary part $q$ (we know $p$ and $q$ are real functions because they are in the real inner product space $\mathcal{U}$). Also, for the Sturm-Liouville Eigenvalue Problem remember that the boundary conditions are hidden in the definition of the vector space.

*Proof.* First we need to be clear about how $\mathcal{L}$ acts on the complex function $y$; it is linear so

$$\mathcal{L}y = \mathcal{L}(p + iq) = \mathcal{L}p + i\mathcal{L}q.$$

next define the complex conjugate of $y$,

$$\bar{y} = p - iq.$$

We now use linearity to extend the definition of the real inner product to complex functions. Consider

$$\begin{aligned}
\langle \bar{y}, \mathcal{L}y \rangle &= \langle p - iq, \mathcal{L}p + i\mathcal{L}q \rangle \\
&= \langle p, \mathcal{L}p \rangle + \langle q, \mathcal{L}q \rangle + i(\langle p, \mathcal{L}q \rangle - \langle q, \mathcal{L}p \rangle) \\
&= \langle p, \mathcal{L}p \rangle + \langle q, \mathcal{L}q \rangle
\end{aligned}$$

where we have used the fact that the operator $\mathcal{L}$ is self-adjoint and the symmetry of the inner product to see that $\langle p, \mathcal{L}q \rangle = \langle q, \mathcal{L}p \rangle$. From the eigenvalue problem and linearity we also know

$$\begin{aligned}
\langle \bar{y}, \mathcal{L}y \rangle &= \langle \bar{y}, \lambda \rho y \rangle \\
&= \lambda \langle \bar{y}, y \rangle_\rho \\
&= \lambda \langle p - iq, p + iq) \rangle_\rho \\
&= \lambda \left[ \langle p, p \rangle_\rho + \langle q, q \rangle_\rho + i(\langle p, q \rangle_\rho - \langle q, p \rangle_\rho) \right] \\
&= \lambda(\|p|_\rho^2 + \|q\|_\rho^2)
\end{aligned}$$

Now, solving for $\lambda$ from these expressions yields

$$\lambda = \frac{\langle p, \mathcal{L}p \rangle + \langle q, \mathcal{L}q \rangle}{\|p\|_\rho^2 + \|q\|_\rho^2}.$$

The quotient on the righthand side is real, therefore $\lambda$ is real. As a side note, if we now rewrite the eigenvalue problem,

$$\mathcal{L}y = \lambda \rho y \Rightarrow \mathcal{L}(p + iq) = \lambda(p + iq),$$

and equate real and imaginary parts,

$$\mathcal{L}p = \lambda \rho p, \qquad \mathcal{L}q = \lambda \rho q,$$

we see that both the real and imaginary parts of $y$ are eigenfunctions; that is, the real eigenvalue $\lambda$ can be associated with a real eigenfunction (either $p$ and/or $q$ which can't both be zero). □

### 16.3.4 Orthogonality of Eigenfunctions

Eigenfunctions associated with self-adjoint operators inherit a natural orthogonality from the inner product space.

**Theorem 16.3.** *Suppose $\mathcal{L}$ is self-adjoint linear operator associated with the inner product space $\mathcal{U}$. If $y_n$ and $y_m$ are eigenfunctions with distinct associated eigenvalues $\lambda_n \neq \lambda_m$ for the eigenvalue problem*

$$\mathcal{L}y = \lambda \rho y$$

*then the eigenfunctions are orthogonal, that is*

$$\langle y_m, y_n \rangle_\rho = 0.$$

*Proof.* From the self-adjointness of $\mathcal{L}$ we see that

$$\langle y_m, \mathcal{L}y_n \rangle = \langle \mathcal{L}y_m, y_n \rangle$$

and from the eigenvalue problem and linearity this implies

$$\lambda_n \langle y_m, y_n \rangle_\rho = \lambda_m \langle y_m, y_n \rangle_\rho .$$

Rearranging yields

$$(\lambda_n - \lambda_m) \langle y_m, y_n \rangle_\rho = 0.$$

As $\lambda_n \neq \lambda_m$ we conclude $\langle y_m, y_n \rangle_\rho = 0$, that is the eigenfunctions are orthogonal. □

The fact that the eigenvalues are real and the eigenfunctions are orthogonal depended solely on the operator $\mathcal{L}$ being self-adjoint. Like for the Fourier Eigenvalue Problem we can sometimes deduce some results about the sign of the eigenvalues.

**Theorem 16.4.** *Suppose $y$ and $\lambda$ are an eigenvalue/eigenfunction pair for the Sturm-Liouville Eigenvalue Problem with $q(x) \geq 0$, $\alpha = 0$ or $\beta = 0$, and $\gamma = 0$ or $\delta = 0$. Then $\lambda \geq 0$. Moreover, $\lambda = 0$ is an eigenvalue if and only if $q = 0$ and the associated eigenfunction is constant.*

**Remark.** The condition that $\alpha = 0$ or $\beta = 0$, and $\gamma = 0$ or $\delta = 0$ implies that the eigenfunction satisfies Dirichlet or Neumann boundary conditions at each end point. It is possible to show positivity of the eigenvalue for some other boundary conditions also.

*Proof.* We know that

$$\langle y, \mathcal{L}y\rangle = \lambda \langle y, \rho y\rangle = \lambda \|y\|_\rho^2$$

however, using integration by parts, we also know that

$$\begin{aligned}
\langle y, \mathcal{L}y\rangle &= -\int_a^b y[(sy')']\,dx + \int_a^b y[qy]\,dx \\
&= \int_a^b sy'y'\,dx - syy'|_{x=a}^{x=b} + \|y\|_q^2 \\
&= \|y'\|_s^2 + \|y\|_q^2 - syy'|_{x=a}^{x=b} \\
&= \|y'\|_s^2 + \|y\|_q^2
\end{aligned}$$

where we have used the fact that the boundary term $yy'$ vanishes at each end point for Dirichlet or Neumann boundary conditions. Now, solving for $\lambda$ from these expressions yields

$$\lambda = \frac{\|y'\|_s^2 + \|y\|_q^2}{\|y\|_\rho^2}\ .$$

Clearly, the right-hand side is non-negative. Moreover, if $\lambda = 0$ then the numerator must be zero which implies $\|y\|_q^2 = 0$ which means $q = 0$ and also that $\|y'\|_s^2 = 0$ for $a < x < b$, that is $y$ is constant. So we conclude that $\lambda \geq 0$ and if $\lambda = 0$, then $q = 0$ and $y$ is constant. □

### 16.3.5 Putting equations in Sturm-Liouville form

The general second-order differential equation for $y(x)$,

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + [b(x) + \lambda c(x)]\,y = 0,$$

where $a(x), b(x)$ and $c(x)$ are arbitrary functions, can be put into Sturm-Liouville form,

$$\frac{d}{dx}\left(s(x)\frac{dy}{dx}\right) + [\lambda\rho(x) - q(x)]\,y(x) = 0.$$

By choosing

$$s(x) = \exp[\int a(x')dx'], \qquad q(x) = -b(x)s(x), \qquad \rho(x) = c(x)s(x)$$

which transforms the Sturm-Liouville problem into the general second-order DE. We leave this as an exercise for the reader

**Exercise 16.3.** Consider the general second-order differential equation above.

(a) Show that choosing $s(x) = \exp[\int a(x')dx']$, $q(x) = -b(x)s(x)$, and $\rho(x) = c(x)s(x)$ transforms the Sturm-Liouville problem into the general second-order DE.

(b) Write the following equations in Sturm-Liouville form:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left[\lambda - \frac{n^2}{x^2}\right] y = 0 \quad x > 0 \qquad \text{(Bessel's Equation)}$$

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{\lambda}{1-x^2} y = 0 \quad 1 > x > -1 \qquad \text{(Legendre's Equation)}$$

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0 \quad \infty > x > -\infty \qquad \text{(Hermite's Equation)}$$

Returning to our population example, consider the boundary value problem,

$$\begin{aligned} \mathbb{DE}: &\quad DX''(x) + cX'(x) + (\alpha + \lambda)X(x) = 0 \qquad 0 < x < L \qquad (16.16) \\ \mathbb{BC}: &\quad X(0) = 0, \quad X(L) = 0. \qquad (16.17) \end{aligned}$$

If we multiply (16.16) by $\exp(cx/D)$ we can rewrite it as

$$\frac{d}{dx}\left(e^{cx/D}\frac{dX}{dx}\right) + \left[\lambda e^{cx/D} + \alpha e^{cx/D}\right] X(x) = 0.$$

which is now in standard Sturm-Liouville form. Consequently we know the eigenvalues are real (as previously claimed). Note also that $\rho(x) = e^{cx/D}$. Consequently, we have the orthogonality condition (with a weighted inner-product)

$$\langle X_m, X_n \rangle_\rho = 0$$

when $m \neq n$.

Now, to solve for $a_n$ when

$$f(x) = \sum_{n=1}^{\infty} a_n X_n(x) = \sum_{n=1}^{\infty} a_n e^{-cx/2D} \sin\left(\frac{n\pi x}{L}\right)$$

we use orthogonality to yield

$$
\begin{aligned}
a_n &= \frac{\langle X_n(x), f(x)\rangle_\rho}{\|X_n(x)\|_\rho^2} \\
&= \frac{\int_0^L e^{-cx/2D}\sin\left(\frac{n\pi x}{L}\right)\cdot f(x)\cdot e^{cx/D}\,dx}{\int_0^L \left[e^{-cx/2D}\sin\left(\frac{n\pi x}{L}\right)\right]^2\cdot e^{cx/D}\,dx} \\
&= \frac{\int_0^L e^{cx/2D}\sin\left(\frac{n\pi x}{L}\right)\cdot f(x)\,dx}{\int_0^L \left[\sin\left(\frac{n\pi x}{L}\right)\right]^2\,dx} \\
&= \frac{2}{L}\int_0^L e^{cx/2D}\sin\left(\frac{n\pi x}{L}\right)\cdot f(x)\,dx
\end{aligned}
$$

which is exactly the result we found above. Consequently, the population example is a nice example of a Sturm-Liouville Eigenvalue Problem and we have a justification for several of the ad hoc assumptions made above.