Seventeen

Playing the Bongos: Bessel Functions and Oscillations of a Circular Membrane

Parts of this section evolved from earlier notes due to Darryl Yong.

17.1 An Introduction to Bessel's Equation and Bessel Functions

Bessel's Equation,

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0, \tag{17.1}$$

arises in many problems in mathematical physics such as the vibration of a circular membrane or the oscillations of a swinging chain. The constant n here is called the *order* of the equation, and the case where n is a positive integer arises most frequently. The equation has a regular singular point at $x = 0$, which suggest that we can find solutions using the method of Frobenius. For this discussion we will assume that $x > 0$.

17.1.1 Frobenius Series Solutions

We will now look for a Frobenius series solutions to Bessel's equation at $x = 0$. Let's multiply by x^2 to obtain

$$x^2y'' + xy' + \left(x^2 - n^2\right)y = 0, \tag{17.2}$$

We substitute the usual ansatz,

$$y = x^p \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+p}$$

which upon differentiation yields

$$y' = \sum_{k=0}^{\infty} (k+p) a_k x^{k+p-1}, \qquad xy' = \sum_{k=0}^{\infty} (k+p) a_k x^{k+p},$$

and a second differentiation yields

$$y'' = \sum_{k=0}^{\infty} (k+p)(k+p-1) a_k x^{k+p-2}, \qquad x^2 y' = \sum_{k=0}^{\infty} (k+p)(k+p-1) a_k x^{k+p},$$

Finally, we need to shift the sum for $x^2 y$,

$$x^2 y = \sum_{k=0}^{\infty} a_k x^{k+p+2} \quad \overset{k'=k+2}{=} \quad \sum_{k'=2}^{\infty} a_{k'-2} x^{k'+p}$$

into (17.2) to get

$$\begin{aligned} 0 &= x^2 y'' + xy' + \left(x^2 - n^2\right) y \\ &= \sum_{k=0}^{\infty} (k+p)(k+p-1) a_k x^{k+p} + \sum_{k=0}^{\infty} (k+p) a_k x^{k+p} + \sum_{k=2}^{\infty} a_{k-2} x^{k+p} - n^2 \sum_{k=0}^{\infty} a_k x^{k+p}, \\ &= \sum_{k=0}^{\infty} a_k \left[(k+p)^2 - n^2\right] x^{k+p} + \sum_{k=2}^{\infty} a_{k-2} x^{k+p}. \end{aligned}$$

Now we combine the sums and collect the leftover initial ($k = 0, 1$) terms,

$$a_0(p^2-n^2)x^p + a_1\left[(p+1)^2 - n^2\right] x^{p+1} + \sum_{k=2}^{\infty} \left\{ a_k \left[(k+p)^2 - n^2\right] + a_{k-2} \right\} z^{k+p+2} = 0 \tag{17.3}$$

Finally, we set all coefficients to zero:

$$\begin{aligned} a_0(p^2 - n^2) &= 0, \\ a_1\left[(p+1)^2 - n^2\right] &= 0, \\ a_m\left[(m+p)^2 - n^2\right] + a_{m-2} &= 0 \qquad \text{for } m = 2, 3, 4, \cdots . \end{aligned}$$

The value p is supposed to represent the lowest exponent present in the series, so a_0 should not be zero (or else p would not be the lowest exponent present in the series). Since $a_0 \neq 0$, equation (17.4a) implies that $p^2 - n^2 = 0$, which is the *indicial equation*. The indicial equation has roots $p = \pm n$. For $n > 0$, the indicial equation indicates that one possible solution to the ODE has a power series that begins with z^n, and one that begins with z^{-n}. The former would be well-defined (and bounded) at $z = 0$, the other would blow up at $z = 0$. Let's try and find a solution for $p = n$.

Note that equation (17.4b) now implies that

$$a_1 \left[(n+1)^2 - n^2\right] = a_1 \left[n(2n+1)\right] = 0$$

Consequently, we choose[1] $a_1 = 0$, which will imply that the odd terms of the series vanish.

Equation (17.4c), can be rewritten as

$$a_k = -\frac{a_{k-2}}{(k+n)^2 - n^2} = -\frac{a_{k-2}}{(k)(k+2n)},$$

is a *recurrence relation* for $k = 2, 3, 4 \cdots$. Because it relates a_k to a_{k-2}, it effectively links all odd coefficients together, and all even coefficients together. As a result, $a_1 = 0$ causes $a_3 = a_5 = \cdots = 0$, so all odd coefficients are zero. To determine the even coefficients, we use the recurrence relation repeatedly,

$$a_2 = -\frac{a_0}{2(2n+2)} = -\frac{a_0}{2^2(n+1)},$$

and

$$a_4 = -\frac{a_2}{4(4+2n)} = -\frac{a_2}{2^2 \cdot 2(n+2)} = \frac{a_0}{2^4 \cdot 2 \cdot 1 \cdot (n+2) \cdot (n+1)},$$

until the general pattern can be inferred:

$$a_{2j} = \frac{(-1)^j a_0}{2^{2j} j!(n+j) \cdots (n+1)}.$$

So,

$$y = a_0 x^n \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{2^{2j} j!(n+j) \cdots (n+1)}.$$

We note that this solution is well defined as long as n is **not** a negative integer; if n is a negative integer we discover that the coefficient a_{-2n} is undefined as we end up dividing by zero!! We now examine a set of cases.

[1]When $n = -1/2$, we are not forced to choose $a_1 = 0$ but in this case a_1 would just multiply the second solution generated by the index $n = 1/2$.

17.1.2 The Bessel Function $J_n(x)$ where $n = 0, 1, 2, 3 \cdots$.

Arguably, this case is the most important and commonly occurs in physical problems. If n is a positive integer, by tradition we choose $a_0 = 1/(2^n n!)$ to obtain the specific solution $y = J_n(x)$ where

$$J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+n)!} \left(\frac{x}{2}\right)^{2j+n},$$

which is known as the n^{th} *order Bessel function of the first kind.* Note that J_0 has a particularly simple form,

$$J_0(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j!)^2} \left(\frac{x}{2}\right)^{2j},$$

and that $J_0(0) = 1$ (which is clear below).

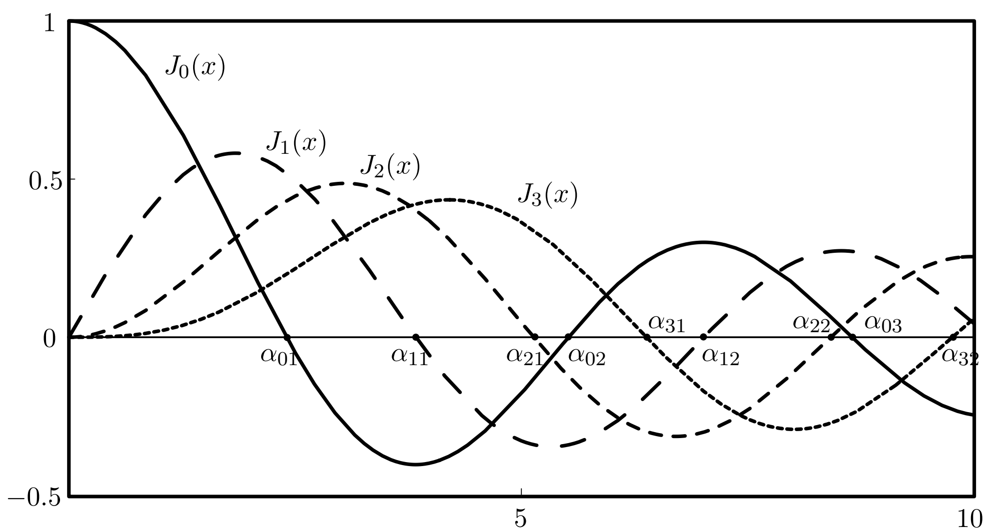

Figure 17.1: The first four Bessel functions: $J_n(x)$ for $n = 1, 2, 3$ and 4.

In general, this expansion above shows that

$$J_n(x) \sim \frac{1}{n!} \left(\frac{x}{2}\right)^n,$$

for small values of x. which is evident in Figure 17.1. Note that J_n has an infinite number of positive zeroes; the m^{th} positive zero is denoted α_{nm},

$$J_n(\alpha_{nm}) = 0 \qquad 0 < \alpha_{n1} < \alpha_{n2} < \alpha_{n3} \cdots .$$

As m tends to infinity, the distance between successive zeroes of J_n tends to π. The zeroes of the Bessel functions play a critical role in the eigenvalue problems in which they arise.

17.1.3 The Bessel Function $Y_n(x)$, $n = 0, 1, 2 \cdots$.

When $\alpha = -n$ a negative integer, the coefficients in the Frobenius series for J_α become infinite. In addition, when $n = 0$, the inidicial equation for the Frobenius series, $p^2 - n^2 = 0$, has a double root at zero. In these cases we must rely on other methods to generate the second independent solution to Bessel's equation.

One can use reduction of order[2] to obtain

$$\begin{aligned} Y_n(z) = \frac{2}{\pi}\left[\ln\left(\frac{z}{2}\right) + \gamma\right] J_n(z) - \frac{(z/2)^{-n}}{\pi}\sum_{j=0}^{n-1}\frac{(n-j-1)!(z/2)^{2j}}{j!} \\ -\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\left[\left(1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{k}\right)+\left(1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{k+n}\right)\right]\frac{(z/2)^{2j+n}}{k!(n+k)!} \end{aligned}$$

The feature of $Y_n(z)$ that will generally be most relevant to us is that as z decreases to zero, $Y_n(z)$ tends to minus infinity because of the n^{th} order pole and logarithmic singularity at $z = 0$.

To summarize, the general solution to (17.2) is

$$y(x) = c_1 J_n(x) + c_2 Y_n(x),$$

where the functions J_n and Y_n are the nth order Bessel functions of the first and second kind, respectively. For more information, a good reference is *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by Abramowitz and Stegun.

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[2] If $y_1(x)$ is a known solution to a linear n-order differential equation, the substitution $y(x) = u(x)y_1(x)$ will result in a linear differential equation of order $n-1$ for $u'(x)$.

17.2 Separation of Variables for the Axisymmetric Heat Equation

Suppose we wish to solve the axisymmetric heat equation for the temperature, $u(r,t)$, in a disc of radius d and initial temperature $f(r)$ where the outer edge of the disc is held at $u = 0$. Then

$$u_t = u_{rr} + \frac{1}{r}u_r \qquad 0 < r < d$$

$$u(d,t) = 0 \qquad u(r,0) = f(r)$$

In addition, we know that the temperature at the origin, $u(0,t)$, is bounded.

As usual, we solve for $u(r,t)$ using separation of variables. We plug $u(r,t) = R(r)T(t)$ into the PDE, and obtain

$$\frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R}$$

For this equality to hold for all r and t, each side of the equation must be a constant. Therefore, we let

$$\frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda$$

where λ is a constant. We choose the separation constant in this particular way, because we will find that the radial eigenvalue problem has only non-negative eigenvalues. The solution to the $T(t)$ differential equation is $T(t) = \exp(-\lambda t)$.

The differential equation for $R(r)$,

$$r^2R'' + rR' + \lambda r^2 R = 0,$$

which we recognize as a form of Bessel's equation of order 0, which can be written in Sturm-Liouville form

$$(rR')' + \lambda rR = 0.$$

In addition,

$$u(d,t) = R(d)T(t) = 0 \quad \Rightarrow R(d) = 0$$

which means we have the singular Sturm-Liouville Eigenvalue Problem

$$DE: \quad (rR')' + \lambda rR = 0 \quad 0 < r < d \tag{17.5}$$

$$BC: \quad R(d) = 0 \quad \text{and} \quad R(0) \text{ is bounded.} \tag{17.6}$$

we can show that the eigenvalues to this problem are positive.

We can find the solutions to (17.5,17.6) by first making a change of variables. Let $z = \sqrt{\lambda} r$, then the equation becomes

$$(zR_z)_z + zR = 0.$$

This equation has two independent solutions, best defined via their Frobenius series

$$J_0(z) = 1 - \frac{z^2}{(1!)^2 2^2} + \frac{z^4}{(2!)^2 2^4} - \frac{z^6}{(3!)^2 2^6} \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{(n!)^2}$$

$$Y_0(z) = \frac{2}{\pi} \log(z) + \cdots \qquad z > 0$$

where J_0 is a Bessel function of the first kind and Y_0 is a Bessel function of the second kind. Undoing the change of variables, we find that the general solution for $R(r)$ is

$$R(r) = CJ_0(\sqrt{\lambda} r) + DY_0(\sqrt{\lambda} r)$$

but as Y_0 has a logarithmic singularity at $r = 0$, we must choose $D = 0$. So we let

$$R(r) = CJ_0(\sqrt{\lambda} r)$$

and to satisfy $R(d) = 0$ we see that we must choose λ such that

$$J_0(\sqrt{\lambda} d) = 0.$$

Fortunately, we know that J_0 has an infinite number of zeros; that is

$$J_0(\alpha_n) = 0 \qquad 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$$

from which we deduce that

$$\sqrt{\lambda} d = \alpha_n \quad \Rightarrow \quad \lambda = \lambda_n \equiv \frac{\alpha_n^2}{d^2} \qquad n = 1, 2, 3 \cdots$$

So this equation has eigenvalues and eigenfunctions,

$$R_n(r) = J_0\left(\frac{\alpha_n r}{d}\right) \qquad \lambda_n = \frac{\alpha_n^2}{d^2},$$

where $n = 1, 2, 3, \ldots$.

To satisfy the initial condition, we need to use a linear combination of these solutions:

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_n}{d} r\right) \exp\left(-\frac{\alpha_n^2}{d^2} t\right). \tag{17.7}$$

The initial condition requires that

$$u(r,0) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_n}{d} r\right) = f(r).$$

Because the differential equation for $R(r)$ is a Sturm-Liouville eigenvalue problem, we know that the weighted inner-product of any two distinct eigenfunctions will be zero. Define

$$\langle P(r), Q(r)\rangle_r \equiv \int_0^d P(r)Q(r)\, r\, dr \qquad \|P(r)\|_r^2 \equiv \langle P(r), P(r)\rangle_r$$

Then the orthogonality condition for the eigenfunctions is

$$\langle R_m(r), R_n(r)\rangle_r = \int_0^d J_0\left(\frac{\alpha_m}{d} r\right) J_0\left(\frac{\alpha_n}{d} r\right) r\, dr = \begin{cases} 0 & m \neq n \\ \|R_n(r)\|_r^2 = \frac{d^2}{2} J_1(\alpha_n)^2 & m = n. \end{cases}$$

where the $\|R_n(r)\|_r^2$ is evaluated above from integral identities for Bessel functions.

We use the orthogonality condition to determine that

$$A_n = \frac{\langle f(r), R_n(r)\rangle_r}{\|R_n(r)\|_r^2} = \frac{\int_0^d f(r) J_0(\alpha_n r/d)\, r\, dr}{\frac{d^2}{2} J_1(\alpha_n)^2}.$$

See the MAPLE worksheet for plots of this solution.

17.3 Playing the Bongos: A Vibrating Circular Membrane

Drums from small bongos to orchestral kettle drums consist of a membrane tautly stretched over a circular frame. As a first approximation, the vibrations of the membrane can be modeled by the wave equation,

$$u_{tt} = c^2 \nabla^2 u,$$

where u is the vertical displacement of the membrane and c is the speed of waves traveling on the membrane. As we are describing the vibrations of a circular membrane, it is convenient to use polar coordinates. Let the displacement of the membrane be $u = u(r, \theta, t)$, in which case the wave equation can be more explicitly written as

$$u_{tt} = c^2 \left[u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right], \qquad r < a. \tag{17.8}$$

Since the membrane is fixed a the boundary of the drumhead, where $r = a$, we specify

$$u(a, \theta, t) = 0.$$

We will look for solutions of a particular form in which the variables *separate*. One can look for solutions that are periodic in θ and oscillatory in time. Let

$$u(r, \theta, t) = R(r) \cos(n\theta) \cos(\omega t)$$

where $n = 0, 1, 2, 3 \cdots$ is an non-negative integer and ω is the oscillation frequency which is to be determined. Substituting into the wave equation (17.8) yields

$$-\omega^2 [R \cos(n\theta) \cos(\omega t)] = c^2 \left[R_{rr} + \frac{1}{r} R_r - \frac{n^2}{r^2} R \right] \cos(n\theta) \cos(\omega t)$$

which after dividing by c^2 and rearranging can be rewritten as

$$\left[R_{rr} + \frac{1}{r} R_r + \left(\lambda^2 - \frac{n^2}{r^2} \right) R \right] \cos(n\theta) \cos(\omega t) = 0$$

where we have let $\lambda = \omega / c$. Which means that $R(r)$ must satisfy

$$R_{rr} + \frac{1}{r} R_r + \left(\lambda^2 - \frac{n^2}{r^2} \right) R = 0, \qquad r < a. \tag{17.9}$$

In addition, from the boundary condition we see that we must have $R(a) = 0$.

At this point we make a change of variables; let $r = x/\lambda$ and let $R(r) = R(x/\lambda) \equiv y(x)$. This yields Bessel's Equation,

$$\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{n^2}{x^2} \right) y = 0, \tag{17.10}$$

for which the general solution is

$$y(x) = c_1 J_n(x) + c_2 Y_n(x),$$

where J_n and Y_n are Bessel's functions of the first and second kind, respectively. Changing variables back to $R(r)$, we see that the general solution to (17.9) is

$$R(r) = CJ_n(\lambda r) + DY_n(\lambda r),$$

Bessel's functions of the second kind, $Y_n(\lambda r)$ have singularities at $r = 0$, so for $R(0)$ to remain finite we must choose $D = 0$. The other boundary condition $R(a) = 0$ requires that

$$CJ_n(\lambda a) = 0.$$

which means that λa must be a zero of the Bessel function. This is an *eigenvalue problem*; remembering that we denote the m^{th} zero of the n^{th} Bessel function as α_{nm}, that is $J_n(\alpha_{nm}) = 0$, we see that we must choose

$$\lambda = \lambda_{nm} = \frac{\alpha_{nm}}{a}$$

which corresponds to an oscillation frequency

$$\omega_{nm} \equiv c\lambda_{nm} = \alpha_{nm}\frac{c}{a}.$$

Consequently, we conclude that

$$u(r, \theta, t) = J_n\left(\alpha_{nm}\frac{r}{a}\right)\cos(n\theta)\cos(\omega_{nm}t), \qquad \omega_{nm} = \alpha_{nm}\frac{c}{a}$$

is a solution to the wave equation for $n = 0, 1, 2, 3\cdots$ and $m = 1, 2, 3\cdots$.

Geometrically, n is an angular wavenumber indicating the number of oscillations in θ. The second wavenumber m indicates the number of radial maxima and minima. The figure on the next page illustrates some sample eigenmodes, each which oscillates with it's own characteristic frequency, $\omega_{nm} = \alpha_{nm}\frac{c}{a}$ determined by the wavenumbers.

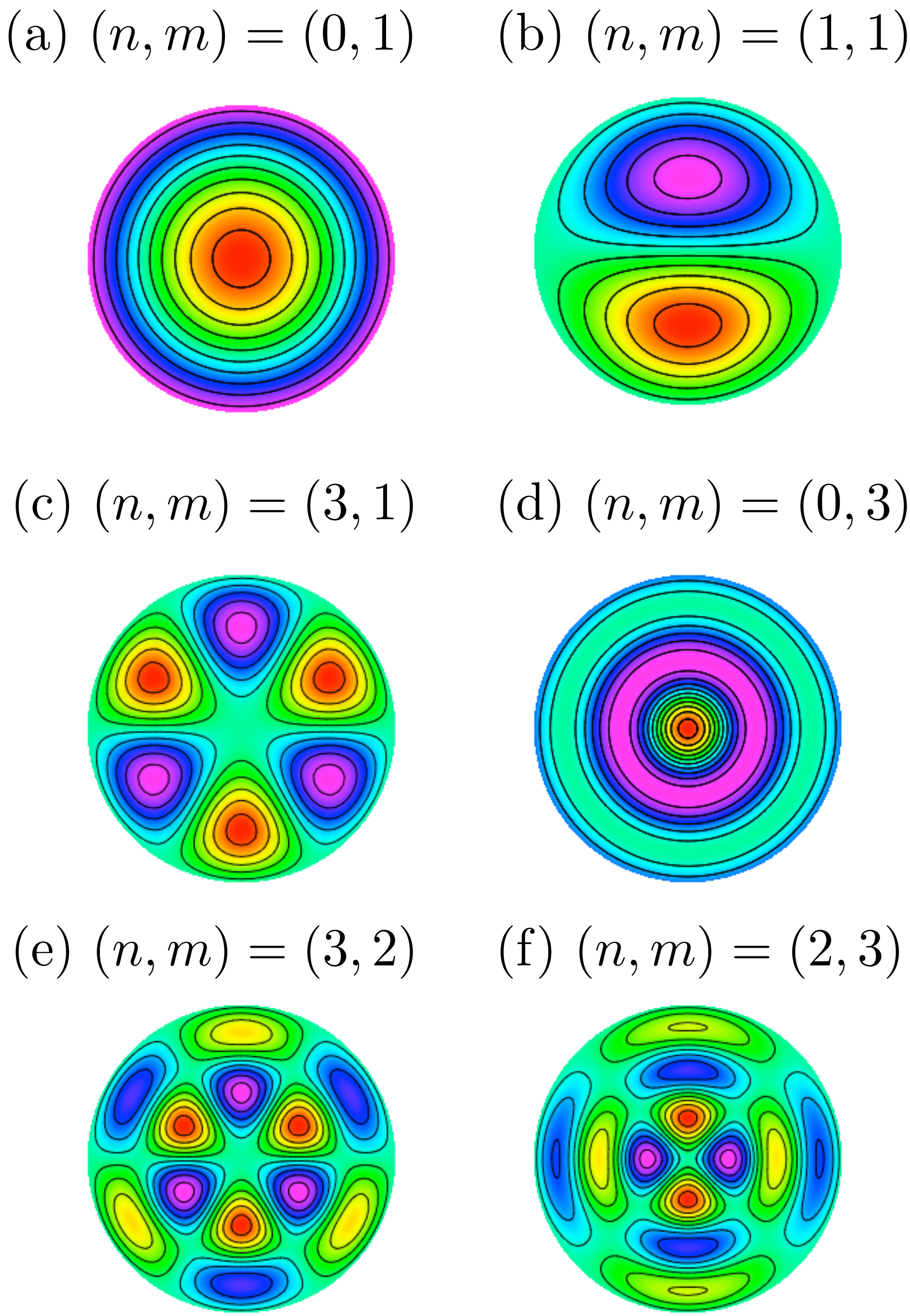

Figure 17.2: Six eigenmodes for the oscillation of a circular membrane. The displacement is proportional to $J_n\left(\alpha_{nm}\frac{r}{a}\right)\cos(n\theta)$ where the wavenumbers (n, m) are indicated on each figure.