Three

A Motivational Example: Solving the Diffusion Equation via Separation of Variables

In Chapter 2, we derived the homogeneous Dirichlet problem for the diffusion equation.

THE DIRICHLET PROBLEM FOR THE DIFFUSION EQUATION
(HOMOGENEOUS BOUNDARY CONDITIONS)

$$\begin{aligned} \mathbb{DE}: &\quad u_t = \kappa u_{xx} && 0 < x < \pi, t > 0 && (3.1)\\ \mathbb{BC}: &\quad u(0,t) = 0, \quad u(\pi,t) = 0 && t > 0 && (3.2)\\ \mathbb{IC}: &\quad u(x,0) = f(x) && 0 < x < \pi. && (3.3) \end{aligned}$$

This equation, also called the *heat equation*, governs the heat distribution in a finite metal bar of length π, where we keep the endpoints at a fixed temperature, in our case 0. The initial temperature at time $t = 0$ is given by $f(x)$. We saw a few solutions to this system, but we didn't have a systematic way of solving the problem given a particular $f(x)$.

In this section, we will present a method to solve the equation for (essentially) any initial $f(x)$. However, along the way we will make a few assumptions and take a few shortcuts that will hopefully leave the reader with the appreciation that a deeper structure (namely Fourier Series with a soupçon of Sturm-Liouville Theory) lurks beneath the surface. These ideas will be explored in the subsequent few chapters.

3.1 THE METHOD OF SEPARATION OF VARIABLES

In 1807 Jean Baptiste Joseph Fourier caused a big stir when he managed to solve a problem of heat dispersion using what are now called Fourier series. We will use the method he developed to solve our homogeneous Dirichlet problem[].

When solving a differential equation, it is frequently advantageous to first look for special solutions that might be easier to find then the general case. Fourier's first step was to look for solutions in the special form

$$u(x,t) = X(x)T(t). \tag{3.4}$$

Plugging this form into the differential equation $u_t = \kappa u_{xx}$, we get

$$X(x)T'(t) = \kappa X''(x)T(t)$$

and dividing by $\kappa X(x)T(t)$ we find

$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)}.$$

Notice that the left hand side is a function of t alone, while the right is a function of x only. This implies that both sides must indeed be constant! We will call this constant $-\lambda$. It is known as the *separation constant*. The reason for the negative sign in front of the λ will be apparent shortly. Thus we have

$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} = -\lambda. \tag{3.5}$$

We can separate this equation into two equations, one involving only x, one involving only t:

$$\frac{T'(t)}{\kappa T(t)} = -\lambda,$$

and

$$\frac{X''(x)}{X(x)} = -\lambda.$$

We will now assume that λ is real and positive. This assumption could plausibly cause us to lose some solutions, but eventually we will show that it is the only case that yields non-trivial solutions.

Each of these equations is now an ordinary differential equation, and thus we can draw on the theory of ordinary differential equations to solve them. The first equation,

$$T'(t) = -\lambda\kappa T(t)$$

has the solution

$$T(t) = Ce^{-\lambda\kappa t}.$$

Before we solve the second-order ordinary differential equation in x, we will derive some boundary conditions for this equation by apply the separation of variable ansatz to the boundary conditions on the PDE. Note that

$$u(0,t) = X(0)T(t) = 0$$

which implies either $X(0) = 0$ or $T(t) = 0$. Note that if we choose $T(t) = 0$ that $u(x,t) = X(x)T(t) = 0$, which, while true, is just the trivial solution. Therefore we conclude that

$$u(0,t) = X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0$$

for us to find non-trivial solutions. Similarly

$$u(\pi,t) = X(\pi)T(t) = 0 \quad \Rightarrow \quad X(\pi) = 0$$

by analogous reasoning. We have now derived the boundary value problem,

$$\mathbb{DE}: \quad X''(x) + \lambda X(x) = 0 \qquad 0 < x < \pi \tag{3.6}$$

$$\mathbb{BC}: \quad X(0) = 0, \quad X(\pi) = 0. \tag{3.7}$$

this is our first example of a *Sturm-Liouville Eigenvalue Problem*; we will show that this problem only has non-trivial solutions for certain special values of λ called *eigenvalues*.

Solving (5.1) yields

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

where we have used the fact that $\lambda > 0$. Applying the first boundary condition yields

$$X(0) = 0 \quad \Rightarrow \quad A = 0$$

so now

$$X(x) = B\sin(\sqrt{\lambda}x).$$

Applying the second boundary condition implies

$$X(\pi) = 0 \quad \Rightarrow \quad B\sin(\sqrt{\lambda}\pi) = 0.$$

so either $B = 0$ which again yields the trivial solution or $\sin(\sqrt{\lambda}\pi) = 0$ which is true only when

$$\sqrt{\lambda} = n \qquad \text{for} \quad n = 1, 2, 3, ...$$

This now yields a countable set of eigenvalues (λ_n) and associated eigenfunctions ($X_n(x)$),

$$\boxed{X_n(x) = \sin(nx) \qquad \lambda_n = n^2} \tag{3.8}$$

which satisfy the eigenvalue problem. We have essentially chosen $B = 1$ in the solution we found above – don't worry, we will bring back the arbitrary constant later.

We can associated with each eigenvalue a solution to the ODE for $T(t)$,

$$T_n(t) \equiv e^{-\lambda_n \kappa t} = e^{-\lambda_n \kappa t}$$

where we note that again we have chosen the multiplicative constant $C = 1$ and the index n recognizes that we restricting ourselves to the case when $\lambda = \lambda_n$.

To summarize, we now have a countable set of solutions which satisfy both the differential equation, and the boundary values, namely

$$u_n(x,t) \equiv X_n(x)T_n(t) = e^{-n^2\kappa t}\sin(nx) \qquad n = 1, 2, 3, ...$$

However, we know that the differential equation and boundary condition are homogeneous, the solutions form a vector space!! So the most general solution is a linear combination of the u_n's.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\kappa t}\sin(nx)} \tag{3.9}$$

where the b_n are arbitrary constants.

3.2 SOLVING THE INITIAL VALUE PROBLEM

To summarize, so far we have found a general solution (9.7) that satisfies both the differential equation and the associated boundary conditions for the homogeneous Dirichlet problem for the diffusion equation. We still need to satisfy the initial condition, $u(x, 0) = f(x)$ which we will argue determines the arbitrary constants b_n.

Apply the initial condition to the general solution yields

$$u(x,0)=\sum_{n=1}^{\infty} b_n u_n(x,0)=\sum_{n=1}^{\infty} b_n X_n(x)=\sum_{n=1}^{\infty} b_n \sin(nx)=f(x). \tag{3.10}$$

This infinite sum of sine functions is called a *Fourier Sine Series*. The numbers b_n are called the *Fourier Coefficients* of $f(x)$. In general the question of whether and how a Fourier Sine Series converges is rather subtle. For the moment we will ignore the technical details and blithely compute a plausible answer. First, we consider an example where the answer is straightforward; namely when the initial condition is a finite linear combination of terms of the form $\sin(nx)$.

Example 3.1. Find the solution to

$$\begin{array}{rlll} \mathbb{DE}: & \multicolumn{2}{c}{u_t=\kappa u_{xx}} & 0<x<\pi, t>0 \\ \mathbb{BC}: & u(0,t)=0 & u(\pi,t)=0 & t>0 \\ \mathbb{IC}: & \multicolumn{2}{l}{u(x,0)=4\sin x+3\sin 4x} & 0<x<\pi. \end{array}$$

Solution: The solution can be found by inspection; looking at the initial condition (9.12) associated with the general solution (9.7),

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin(nx)=4\sin x+3\sin 4x$$

we see that choosing $b_1=4$ and $b_4=3$ and setting all the remaining terms to zero yields

$$\boxed{u(x,t)=4e^{-\kappa t}\sin(x)+3e^{-16\kappa t}\sin(4x).}$$

Note that as we showed in Chapter 2, this solution is the unique solution to this problem. ■

To solve the general problem we first introduce an *inner product*,

$$\langle u,v\rangle \equiv \int_0^{\pi} uv\,dx.$$

which takes as arguments two real functions (here u and v) on the $x\in[0,\pi]$ and associates with them a real number. We will define the characteristics of an inner product in detail in the next section, but for the moment we will concentrate on one amazing fact, namely that the functions X_n are *orthogonal* with respect to this inner product,

$$\begin{aligned}\langle X_n(x), X_m(x)\rangle &= \langle \sin(nx), \sin(mx)\rangle \\ &= \int_0^\pi \sin(nx)\sin(mx)\,dx \\ &= \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}\end{aligned}$$

or to summarize

ORTHOGONALITY CONDITION

$$\langle X_n(x), X_m(x)\rangle = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases} \tag{3.11}$$

where the diligent reader may check the evaluation of the integral above.

We can use this identity to *project* the infinite sum of functions in the Fourier Sine Series onto a single function and thus determine the value of the Fourier coefficients, b_n. To calculate the Fourier coefficients, we compute the inner product of $X_m(x)$ with the initial condition,

$$f(x) = \sum_{n=1}^{\infty} b_n X_n(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

This yields

$$\begin{aligned}\langle X_m(x), f(x)\rangle &= \left\langle X_n(x), \sum_{n=1}^{\infty} b_n X_n(x)\right\rangle \\ &= \sum_{n=1}^{\infty} b_n \langle X_m(x), X_n(x)\rangle \\ &= b_m \langle X_m(x), X_m(x)\rangle \\ &= \frac{\pi}{2} b_m\end{aligned}$$

where we have first used the fact that the inner product is linear in its arguments (you can check this yourself or wait for the discussion in the next chapter), then noted from the orthogonality condition that $\langle X_n(x), X_m(x)\rangle$ vanishes unless $n = m$.

Solving for b_m now yields a formula to calculate the Fourier coefficients,

$$\boxed{b_m = \frac{2}{\pi}\langle X_m(x), f(x)\rangle = \frac{2}{\pi}\int_0^\pi f(x)\sin(mx)\,dx \qquad m = 1, 2, 3, \ldots.}$$

The astute reader will notice that the calculation above exchanges an infinite sum and integral. Moreover, there is no guarantee that with this choice of b_m that the Fourier sine series converges (or if that it does that it converges to the function $f(x)$). Nonetheless, we will persevere with this plausible result for the moment and show, at least numerically, that the results make sense for an example problem.

Example 3.2. Find the solution to

$$\begin{array}{rcl} \mathbb{DE}: & u_t = \kappa u_{xx} & 0 < x < \pi, t > 0 \\ \mathbb{BC}: & u(0,t) = 0 \quad u(\pi,t) = 0 & t > 0 \\ \mathbb{IC}: & u(x,0) = x(\pi - x) & 0 < x < \pi. \end{array}$$

Solution: From the solution above, we see that

$$\begin{aligned} b_m &= \frac{2}{\pi}\int_0^\pi f(x)\sin(mx)\,dx \\ &= \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin(mx)\,dx \\ &= \begin{cases} 0 & n \quad \text{even} \\ \dfrac{8}{\pi n^3} & n \quad \text{odd} \end{cases} \end{aligned}$$

This suggests the full solution is

$$\boxed{u(x,t) = \frac{8}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{e^{-n^2\kappa t}}{n^3}\sin(nx)\ .}$$

■

It is easy to show that this sum converges absolutely, and it certainly satisfies the diffusion equation and the associated boundary condition. To satisfy the initial condition, we need to check that

$$u(x,0) = x(\pi - x) = \frac{8}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{e^{-n^2\kappa t}}{n^3}\sin(nx) \qquad \text{for} \quad 0 < x < \pi.$$

We graph the solution below:

INSERT GRAPH HERE.

3.3 SOLUTION TO THE DIRICHLET PROBLEM: WHAT JUST HAPPENED?

To summarize, we have suggested that the solution to

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is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\kappa t} \sin(nx) \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx.$$

However, the derivation has left us with a number of questions:

- Are the eigenvalues always real? Are they always positive?
- Have we found all the eigenvalues and eigenfunctions?
- Why are the eigenfunctions orthogonal?
- When does the Fourier Sine Series for the initial condition converge?
- How should we define convergence here?

We will attempt to answer these questions (or at least indicate the answers) in the next few chapters. As we will see the answers are surprisingly deep and nuanced.

3.4 PROBLEMS FOR CHAPTER 3

Problem 3.1. Find the exact solution for Dirichlet problem with initial conditions:

(a) $f(x) = \sin(x) + 5\sin(2x) + 3\sin(3x)$.

(b) $f(x) = \sin^3 x$.

Your answer should only contain a finite set of functions.

Problem 3.2. Rework the solution to the homogeneous Dirichlet problem for a bar of length L instead of length π. That is, solve

$$\begin{array}{rccl} \mathbb{DE}: & U_t = \kappa U_{xx} & & 0 < x < L, t > 0 \\ \mathbb{BC}: & U(0,t) = 0 & U(L,t) = 0 & t > 0 \\ \mathbb{IC}: & U(x,0) = f(x) & & 0 < x < L. \end{array}$$