Five

Sturm-Liouville Eigenvalue Problems

In Chapter (3), we showed the boundary value problem

$$\mathbb{DE}: \quad X''(x) + \lambda X(x) = 0 \qquad 0 < x < \pi \tag{5.1}$$

$$\mathbb{BC}: \quad X(0) = 0, \quad X(\pi) = 0. \tag{5.2}$$

yielded a countable set of real positive eigenvalues $\{\lambda_n\}$ and associated eigenfunctions $\{X_n(x)\}$,

$$X_n(x) = \sin(nx) \qquad \lambda_n = n^2 \qquad n = 1, 2, 3, \cdots \tag{5.3}$$

that were orthogonal in the $L^2[0, \pi]$ inner-product,

$$\langle X_n(x), X_m(x)\rangle \equiv \int_0^\pi X_n(x)X_m(x)\,dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases} \tag{5.4}$$

In this section, we will show that the existence of this orthogonal set is not a coincidence, but a consequence of the form of this boundary value problem. This is an example of a *Sturm-Liouville Eigenvalue Problem* or SLEP; we will explore some of the properties associated with these problems below. We will concentrate on a class of problems associated with Fourier Series which we call the *Fourier Eigenvalue Problem*.

To understand SLEPs perhaps the best analogy is the matrix eigenvalue problem from Linear Algebra. Remember, if we have a real symmetric $N \times N$ matrix A, that the eigenvalue problem

$$A\vec{x} = \lambda\vec{x} \qquad \vec{x} \in \mathbb{R}^N$$

has the following properties:

- The matrix A has N real eigenvalues, each associated with a real eigenvector,
- Eigenvectors associated with different eigenvalues are orthogonal with respect to the dot product in $\mathbb{R}^N$,
- If the matrix is positive definite, the eigenvalues are positive.

We explore the analogous problem for SLEPs below.

5.1 The Fourier Eigenvalue Problem

In this Section we consider an example of a Sturm-Liouville Eigenvalue Problem associated with Fourier Series:

Definition 5.5 (Fourier Eigenvalue Problem)**.** Let $y(x)$ be a twice continuously differentiable function on the interval $a \leq x \leq b$ (i.e. $y(x) \in C^2[a, b]$) . Let $\mathcal{L}$ be a differential operator defined by $\mathcal{L}y \equiv -y''$. The *Fourier Eigenvalue Problem* is defined by the differential equation

$$\mathbb{DE}: \qquad \mathcal{L}y = \lambda y, \quad a < x < b. \tag{5.6}$$

Together with one of the boundary conditions

$$\begin{array}{llcl} \text{(D)} & \mathbb{BC}: & y(a) = 0, \quad y(b) = 0 & \text{Dirichlet} \\ \text{(N)} & \mathbb{BC}: & y'(a) = 0, \quad y'(b) = 0 & \text{Neumann} \\ \text{(P)} & \mathbb{BC}: & y(a) = y(b), \quad y'(a) = y'(b) & \text{Periodic} \end{array}$$

we call a constant λ and a non-zero function y that satisfy this problem an *eigenvalue/eigenfunction pair*.

We will prove a set of theorems about the eigenfunctions and eigenvalues of this problem, but first we need to introduce an appropriate vector space of real functions and an inner product. Define

$$\mathcal{U}_D \equiv \{y(x) \in C^2[a, b], y(a) = 0, y(b) = 0\}, \qquad \text{Dirichlet} \tag{5.7}$$

$$\mathcal{U}_N \equiv \{y(x) \in C^2[a, b], y'(a) = 0, y'(b) = 0\}, \qquad \text{Neumann} \tag{5.8}$$

$$\mathcal{U}_P \equiv \{y(x) \in C^2[a, b], y(a) = y(b), y'(a) = y'(b)\}. \qquad \text{Periodic} \tag{5.9}$$

The idea here is we only consider the set of functions that satisfy the boundary conditions.

Exercise 5.1. Show that $\mathcal{U}_D$, $\mathcal{U}_N$, and $\mathcal{U}_P$ are vector spaces.

If we associate the usual L^2 inner product

$$\langle u, v\rangle = \int_a^b uv\, dx.$$

with these spaces, we see that they are also real inner product spaces.

5.2 Self-adjoint Operators

We say a differential operator $\mathcal{P}$ acting on elements in an inner product space is *self-adjoint* if

$$\langle u, \mathcal{P}v\rangle = \langle \mathcal{P}u, v\rangle .$$

If we consider the above statement in the inner product space of $\mathbb{R}^N$ with the usual dot product, this is exactly the condition for $\mathcal{P}$ to be an $N \times N$ symmetric matrix. Let us apply this idea to the Fourier Eigenvalue Problem.

Theorem 5.1. *The differential operator $\mathcal{L}$ is self-adjoint in the inner product spaces $\mathcal{U}_D$, $\mathcal{U}_N$, or $\mathcal{U}_P$.*

Proof. The proof follows by integration by parts; suppose u and v are elements of one of the inner product spaces. Then

$$\begin{aligned}
\langle u, \mathcal{L}v\rangle &= -\int_a^b uv''\, dx \\
&= \int_a^b u'v'\, dx - uv'|_{x=a}^{x=b} \\
&= -\int_a^b u''v\, dx + u'v - uv'|_{x=a}^{x=b} \\
&= \langle \mathcal{L}u, v\rangle + u'v - uv'|_{x=a}^{x=b} .
\end{aligned}$$

If u and v satisfy the Neumann or Dirichlet boundary conditions then $u'v - uv' = 0$ at each endpoint. Similarly if u and v satisfy the periodic boundary conditions, then $u'v - uv'$ is the same at $x = a$ and $x = b$ so $u'v - uv'|_{x=a}^{x=b}$ vanishes. Therefore

$$\langle u, \mathcal{L}v\rangle = \langle \mathcal{L}u, v\rangle ,$$

and we have shown the operator is self-adjoint. □

5.2.1 The Eigenvalues of a Self-adjoint Operator are Real

In a moment, we will show the eigenvalues of a self-adjoint operator are real. Perhaps the first question to ask is:

Why might we consider the possibility that the eigenvalues could be complex?

If we think about the matrix eigenvalue problem in $\mathbb{R}^N$, we know that for a general matrix A the eigenvalues satisfy the *characteristic polynomial*, $P(\lambda) = \det(A - \lambda I)$. This real polynomial may have roots that are real or that occur in complex conjugate pairs. Only for special classes of matrices, such as symmetric matrices, do we know *a priori* that the eigenvalues are real. To show the eigenvalues of a self-adjoint operator are real, we will use an argument that is analogous to that used to show the eigenvalues of a real symmetric matrix are real.

Theorem 5.2. *Suppose $\mathcal{L}$ is self-adjoint linear operator associated with the inner product space $\mathcal{U}$. Then the eigenvalue problem*

$$\mathcal{L}y = \lambda y$$

where

$$y = p + iq \qquad p, q \in \mathcal{U}$$

has only real eigenvalues, λ.

Remark. Note that we have temporarily expanded the eigenvalue problem to allow y to be a complex function with real part p and imaginary part q (we know p and q are real functions because they are in the real inner product space $\mathcal{U}$). Also, for the Fourier Eigenvalue Problem remember that the boundary conditions are hidden in the definition of the vector space.

Proof. First we need to be clear about how $\mathcal{L}$ acts on the complex function y; it is linear so

$$\mathcal{L}y = \mathcal{L}(p + iq) = \mathcal{L}p + i\mathcal{L}q.$$

next define the complex conjugate of y,

$$\bar{y} = p - iq.$$

We now use linearity to extend the definition of the real inner product to complex functions. Consider

$$\begin{aligned} \langle \bar{y}, \mathcal{L}y \rangle &= \langle p - iq, \mathcal{L}p + i\mathcal{L}q \rangle \\ &= \langle p, \mathcal{L}p \rangle + \langle q, \mathcal{L}q \rangle + i(\langle p, \mathcal{L}q \rangle - \langle q, \mathcal{L}p \rangle) \\ &= \langle p, \mathcal{L}p \rangle + \langle q, \mathcal{L}q \rangle \end{aligned}$$

where we have used the fact that the operator $\mathcal{L}$ is self-adjoint and the symmetry of the inner product to see that $\langle p, \mathcal{L}q\rangle = \langle q, \mathcal{L}p\rangle$. From the eigenvalue problem and linearity we also know

$$\begin{aligned}
\langle \bar{y}, \mathcal{L}y\rangle &= \langle \bar{y}, \lambda y\rangle \\
&= \lambda \langle \bar{y}, y\rangle \\
&= \lambda \langle p - iq, p + iq)\rangle \\
&= \lambda \left[\langle p, p\rangle + \langle q, q\rangle + i(\langle p, q\rangle - \langle q, p\rangle)\right] \\
&= \lambda(\|p\|^2 + \|q\|^2)
\end{aligned}$$

Now, solving for λ from these expressions yields

$$\lambda = \frac{\langle p, \mathcal{L}p\rangle + \langle q, \mathcal{L}q\rangle}{\|p\|^2 + \|q\|^2}.$$

The quotient on the righthand side is real, therefore λ is real. As a side note, if we now rewrite the eigenvalue problem,

$$\mathcal{L}y = \lambda y \Rightarrow \mathcal{L}(p + iq) = \lambda(p + iq),$$

and equate real and imaginary parts,

$$\mathcal{L}p = \lambda p, \qquad \mathcal{L}q = \lambda q,$$

we see that both the real and imaginary parts of y are eigenfunctions; that is, the real eigenvalue λ can be associated with a real eigenfunction (either p and/or q which can't both be zero). □

Exercise 5.2. Use this proof to convince yourself that a real $N \times N$ symmetric matrix has real eigenvalues.

5.2.2 Orthogonality of Eigenfunctions

Eigenfunctions associated with self-adjoint operators inherit a natural orthogonality from the inner product space.

Theorem 5.3. *Suppose $\mathcal{L}$ is self-adjoint linear operator associated with the inner product space $\mathcal{U}$. If y_n and y_m are eigenfunctions with distinct associated eigenvalues $\lambda_n \neq \lambda_m$ for the eigenvalue problem*

$$\mathcal{L}y = \lambda y$$

then the eigenfunctions are orthogonal, that is

$$\langle y_m, y_n\rangle = 0.$$

Proof. From the self-adjointness of $\mathcal{L}$ we see that

$$\langle y_m, \mathcal{L}y_n \rangle = \langle \mathcal{L}y_m, y_n \rangle$$

and from the eigenvalue problem and linearity this implies

$$\lambda_n \langle y_m, y_n \rangle = \lambda_m \langle y_m, y_n \rangle .$$

Rearranging yields

$$(\lambda_n - \lambda_m) \langle y_m, y_n \rangle = 0.$$

As $\lambda_n \neq \lambda_m$ we conclude $\langle y_m, y_n \rangle = 0$, that is the eigenfunctions are orthogonal. □

The fact that the eigenvalues are real and the eigenfunctions are orthogonal depended solely on the operator $\mathcal{L}$ being self-adjoint. If we specialize to the Fourier Eigenvalue Problem we can also deduce some results about the sign of the eigenvalues.

5.3 Solving the Fourier Eigenvalue Problem

In this Section we will solve the Fourier Eigenvalue Problem for the three different boundary conditions (Dirichlet, Neumann and Periodic). First however we can prove a theorem about the non-negativity of the eigenvalues which will save us some work.

Theorem 5.4. *Suppose y and λ are an eigenvalue/eigenfunction pair for the Fourier Eigenvalue Problem. Then $\lambda \geq 0$. Moreover, If $\lambda = 0$ is an eigenvalue then then the associated eigenfunction is constant.*

Proof. We know that

$$\langle y_n, \mathcal{L}y_n \rangle = \lambda \langle y_n, y_n \rangle = \lambda \|y\|^2$$

however, using integration by parts, we also know that

$$\begin{aligned}
\langle y_n, \mathcal{L}y_n \rangle &= -\int_a^b yy'' \, dx, \\
&= \int_a^b y'y' \, dx - yy'|_{x=a}^{x=b}, \\
&= \|y'\|^2,
\end{aligned}$$

where we have used the fact that the boundary term yy' vanishes at each end point for the Dirichlet and Neumann problems, and the endpoint contributions cancel for the periodic problem. Now, solving for λ from these expressions yields

$$\lambda = \frac{\|y'|^2}{\|y\|^2} \, .$$

Clearly, the right-hand side is non-negative. Moreover, if $\lambda = 0$ then $y' = 0$ for $a < x < b$, that is if y is constant. So we conclude that $\lambda \geq 0$ and if $\lambda = 0$, then y is constant. □

We can now quickly solve the Fourier Eigenvalue Problem for the three boundary conditions

5.3.1 The Fourier Eigenvalue Problem: Dirichlet Boundary Conditions

We essentially solved this problem already in Chapter 3. For simplicity we will solve the problem on the interval $0 < x < \ell$. We wish to find non-zero, twice differentiable eigenfunctions $y_n(x)$ and associated eigenfunctions λ_n solving

$$\begin{aligned} \mathbb{DE}: \qquad & -y'' = \lambda y, \quad 0 < x < \ell, \\ \mathbb{BC}: \qquad & y(0) = 0, \quad y(\ell) = 0. \end{aligned}$$

we know that λ is real and non-negative from Theorem 5.4. If $\lambda = 0$, y is constant, but as $y(0) = 0$ this yields only the trivial solution $y(x) = 0$. If $\lambda > 0$, then

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Applying the first boundary condition yields

$$y(0) = 0 \quad \Rightarrow \quad A = 0$$

so now

$$y(x) = B\sin(\sqrt{\lambda}x).$$

Applying the second boundary condition implies

$$y(\ell) = 0 \quad \Rightarrow \quad B\sin(\sqrt{\lambda}\ell) = 0.$$

so either $B = 0$ which again yields the trivial solution or $\sin(\sqrt{\lambda}\pi) = 0$ which is true only when

$$\sqrt{\lambda} = \frac{n\pi}{\ell} \qquad \text{for} \quad n = 1, 2, 3, \ldots .$$

This now yields a countable set of eigenvalues and associated eigenfunctions,

$$\boxed{y_n(x) = \sin\left(\frac{n\pi x}{\ell}\right) \qquad \lambda_n = \frac{n^2\pi^2}{\ell^2} \qquad \text{for} \quad n = 1, 2, 3, \ldots.} \tag{5.10}$$

which satisfy the eigenvalue problem. We have set the arbitrary constant $B = 1$ in the solution we found above – don't worry, when we look for a solution or construct a Fourier Series, it is a linear combination of the elements of this orthogonal set so we will bring back the arbitrary constant later. In the next Chapter we will show that the orthogonal set generated here is associated with the *Fourier Sine Series*.

5.3.2 The Fourier Eigenvalue Problem: Neumann Boundary Conditions

Again, for simplicity we will solve the problem on the interval $0 < x < \ell$. We wish to find non-zero, twice differentiable eigenfunctions $y_n(x)$ and associated eigenfunctions λ_n solving

$$\begin{aligned} \mathbb{DE}: &\qquad -y'' = \lambda y, \quad 0 < x < \ell, \\ \mathbb{BC}: &\qquad y'(0) = 0, \quad y'(\ell) = 0. \end{aligned}$$

we know that λ is real and non-negative from Theorem 5.4. If $\lambda = 0$, y is a constant, which satisfies both the DE and the Neumann boundary conditions. This yields

$$y_0(x) = \frac{1}{2} \qquad \lambda_0 = 0$$

where we have chosen the value $y_0 = \frac{1}{2}$ to agree with a convention that arises in the definition of a Fourier Series. If $\lambda > 0$, then

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

and

$$y'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

Applying the first boundary condition yields

$$y'(0) = 0 \quad \Rightarrow \quad B = 0$$

so now

$$y(x) = A\cos(\sqrt{\lambda}x).$$

Applying the second boundary condition implies

$$y'(\ell) = 0 \quad \Rightarrow \quad -A\sqrt{\lambda}\sin(\sqrt{\lambda}\ell) = 0.$$

so either $A = 0$ which again yields the trivial solution or $\sin(\sqrt{\lambda}\pi) = 0$ which is true only when

$$\sqrt{\lambda} = \frac{n\pi}{\ell} \qquad \text{for} \quad n = 1, 2, 3, \ldots.$$

Putting this together with the constant solution now yields a countable set of eigenvalues and associated eigenfunctions,

$$\boxed{\begin{array}{lll} y_0(x) = \frac{1}{2} & \lambda_0 = 0 & \\ y_n(x) = \cos\left(\frac{n\pi x}{\ell}\right) & \lambda_n = \frac{n^2\pi^2}{\ell^2} & \text{for} \quad n = 1, 2, 3, \ldots. \end{array}} \tag{5.11}$$

which satisfy the eigenvalue problem. In the next section will show that the orthogonal set generated here is associated with the *Fourier Cosine Series*.

5.3.3 The Fourier Eigenvalue Problem: Periodic Boundary Conditions

We solve the periodic problem on the interval $-\ell < x < \ell$. We wish to find non-zero, twice differentiable eigenfunctions $y_n(x)$ and associated eigenfunctions λ_n solving

$$\mathbb{DE}: \qquad -y'' = \lambda y, \quad -\ell < x < \ell, \tag{5.12}$$

$$\mathbb{BC}: \qquad y(-\ell) = y(\ell), \quad y'(-\ell) = y'(\ell). \tag{5.13}$$

we know that λ is real and non-negative from Theorem 5.4. If $\lambda = 0$, y is a constant, which satisfies both the DE and the periodic boundary conditions. This yields

$$y_0(x) = \frac{1}{2} \qquad \lambda_0 = 0$$

where again we have chosen the value $y_0 = \frac{1}{2}$ to agree with a convention that arises in the definition of a Fourier Series. If $\lambda > 0$, then

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

and

$$y'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

Applying the first boundary condition yields

$$y(-\ell) = y(\ell) \quad \Rightarrow \quad A\cos(\sqrt{\lambda}\ell) - B\sin(\sqrt{\lambda}\ell) = A\cos(\sqrt{\lambda}\ell) + B\sin(\sqrt{\lambda}\ell)$$

which implies

$$2B\sin(\sqrt{\lambda}\ell) = 0$$

Applying the second boundary condition implies

$$y'(-\ell) = y'(\ell) \quad \Rightarrow \quad A\sqrt{\lambda}\sin(\sqrt{\lambda}\ell) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\ell) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\ell) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\ell).$$

which implies

$$2A\sqrt{\lambda}\sin(\sqrt{\lambda}\ell) = 0$$

If A and B are not both zero and remember that $\lambda > 0$, we see that the eigenvalue condition reduces to

$$\sin(\sqrt{\lambda}\ell) = 0$$

or

$$\sqrt{\lambda} = \frac{n\pi}{\ell} \qquad \text{for} \quad n = 1, 2, 3, \ldots.$$

Note that in this case there are two linear independent eigenfunctions $\cos\left(\frac{n\pi x}{\ell}\right)$ and $\sin\left(\frac{n\pi x}{\ell}\right)$ associated with the eigenvalue. Putting this together with the constant solution now yields a countable set of eigenvalues and associated eigenfunctions,

$$\boxed{\begin{array}{lll} y_0(x) = \dfrac{1}{2} & \lambda_0 = 0 & \\ y_n^c(x) = \cos\left(\dfrac{n\pi x}{\ell}\right) & \lambda_n = \dfrac{n^2\pi^2}{\ell^2} & \text{for} \quad n = 1, 2, 3, \ldots. \\ y_n^s(x) = \sin\left(\dfrac{n\pi x}{\ell}\right) & \lambda_n = \dfrac{n^2\pi^2}{\ell^2} & \text{for} \quad n = 1, 2, 3, \ldots. \end{array}} \tag{5.14}$$

which satisfy the eigenvalue problem. Strictly speaking, we need to check the orthogonal of the eigenfunctions $y_n^c(x)$ and $y_n^s(x)$ because they are associated with the same eigenvalue. In practice,

$$\langle y_n^c(x), y_n^s(x)\rangle \equiv \int_{-\ell}^{\ell} \cos\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right) dx = 0$$

is guaranteed as one of the functions is odd and the other is even. In the next section will show that the orthogonal set generated here is associated with the general *Fourier Series*.