Six

Fourier Series I: Definitions and Least Squares Approximation

A Fourier Series approximates a periodic function by a sum of sines and cosines of the same period. From the oscillation of a guitar string to the compression of images, an idea Joseph Fourier employed to approximate the solution to the heat equation has grown into a valuable tool in many disciplines include math, science and engineering. Below we develop the basic tools need to expand a function into a Fourier Series. In the next two chapters we study the nature of convergence for these expansions.

6.1 Fourier Series: The Basics

In the previous Chapter we used Sturm-Liouville Theory to generate three sets of orthogonal functions that will be the bases for studying Fourier Series. For periodic functions on the interval $x \in [-\ell, \ell]$ we showed the set of functions

$$\left\{ \frac{1}{2},\ \cos\left(\frac{n\pi x}{\ell}\right),\ \sin\left(\frac{n\pi x}{\ell}\right) \right\}$$

are orthogonal in the L^2 inner-product. This allows us to propose the *Fourier Series* expansion for a function $f(x)$,

$$\mathbb{FS}[f(x)] \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right). \tag{6.1}$$

Note that as the Fourier Series is the sum of periodic functions with period 2ℓ, it also is 2ℓ-periodic. We know that the choice of coefficients

$$\begin{aligned} a_0 &= \frac{\left\langle \frac{1}{2}, f(x) \right\rangle}{\left\| \frac{1}{2} \right\|^2} = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx, \\ a_n &= \frac{\left\langle \cos\left(\frac{n\pi x}{\ell}\right), f(x) \right\rangle}{\left\| \cos\left(\frac{n\pi x}{\ell}\right) \right\|^2} = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx, \\ b_n &= \frac{\left\langle \sin\left(\frac{n\pi x}{\ell}\right), f(x) \right\rangle}{\left\| \sin\left(\frac{n\pi x}{\ell}\right) \right\|^2} = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx, \end{aligned}$$

will minimize the L^2-error between the function and the Fourier Series approximation. We can also now explain the curious choice of a constant term of $\frac{1}{2}$. This choice means that the formula for a_n is exactly the same as the formula for a_0 when $n = 0$.

Example 6.1. Find the Fourier series for

$$g(x) = x^2 \qquad -\pi \le x \le \pi\ .$$

Graph some partial sums of the series and examine their convergence.
Solution: Note that we take $\ell = \pi$ in the formulae above, because the interval is $x \in [-\pi, \pi]$. The sine terms in the Fourier series will vanish,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(nx)\ dx = 0,$$

which is easily seen because $g(x)$ is an even function and $\sin(nx)$ is odd. So the Fourier series for $g(x)$ will only involve cosines,

$$\mathbb{FS}[g(x)] = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \tag{6.2}$$

The coefficients are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)\ dx.$$

We see that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\ \cos xt\,dx = \frac{1}{\pi}\left[\frac{2x\ \cos nx}{n^2} + \frac{(n^2x^2-2)\sin nx}{n^3}\right]_{x=-\pi}^{\pi} = \frac{4(-1)^n}{n^2},$$

when $n = 1, 2, 3, \cdots$. When $n = 0$,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \, dt = \frac{2\pi^2}{3}.$$

Therefore,

$$\mathbb{FS}[g(x)] = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx.$$

We can define the *partial sums* of this series,

$$S_N(x) = \frac{\pi^2}{3} + \sum_{n=1}^{N} \frac{4(-1)^n}{n^2}\cos nx.$$

which we graph below, for $N = 1, 2, 4, 8$ and 16 terms along with $g(x) = x^2$. Note that the partial sums appear to converge quickly and uniformly to $g(x) = x^2$.

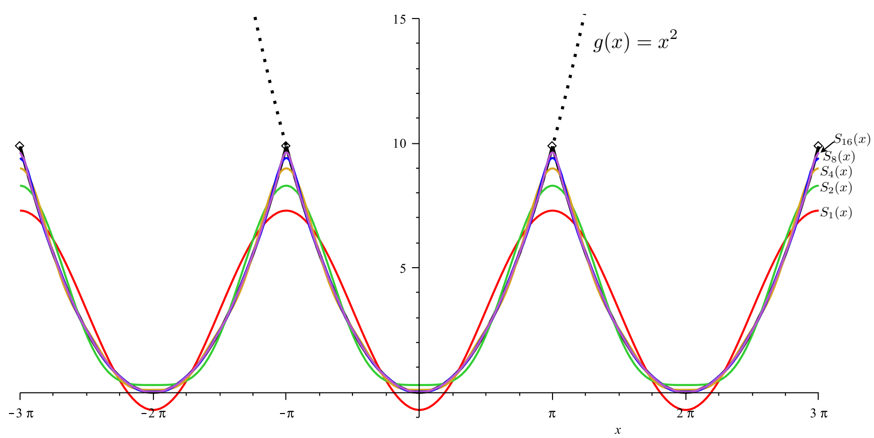

Figure 6.1: The partial sums, $S_N(x)$, for $N = 1, 2, 4, 8$ and 16. Note the partial sums converge to the function $g(x) = x^2$ (dotted) in the range $-\pi < x < \pi$. Outside this range the Fourier series converges to the 2π-periodic extension of $g(x)$ which is indicated in black.

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The Fourier Series (6.1) is 2ℓ-periodic and although it appears to converge to $g(x)$ on the interval $x \in [-\pi, \pi]$, it periodically repeats to either side of this function. We call this function the *periodic extension* of f.

Definition 6.3. Suppose a function, $f(x)$, is defined on an interval $x \in [-\ell, \ell)$. We define the *periodic extension* of $f(x)$, $\tilde{f}(x)$ to be the unique function such that $\tilde{f}(x) = f(x)$ for $x \in [-\ell, \ell)$ and $\tilde{f}(x + 2\ell) = \tilde{f}(x)$.

The first condition assures agreement on the *basic period*, $x \in [-\ell, \ell)$, and the second condition assures the function is 2ℓ-periodic.

We know that from Theorem 4.1 that the Fourier series is the best approximation of a periodic function in the least squares sense on the interval $x \in [-\ell, \ell]$. We want to say something like:

Slogan 1. The Fourier Series

$$\mathbb{FS}[f(x)] \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

with coefficients

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\, dx \qquad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\, dx,$$

converges to $\tilde{f}(x)$, the periodic extension of $f(x)$ from $x \in [-\ell, \ell)$.

The difficulty here is the meaning of the word *converges*. We will investigate some details of the convergence of Fourier series in the next two chapters.

6.2 The Fourier Sine Series

Similarly, for functions on the interval $x \in [0, \ell]$ which vanish at the endpoint we showed the set of functions

$$\left\{\sin\left(\frac{n\pi x}{\ell}\right)\right\}$$

are orthogonal in the L^2 inner-product. This allows us to propose the *Fourier Sine Series* expansion for a function $f(x)$,

$$\mathbb{FSS}[f(x)] \equiv \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right). \tag{6.4}$$

as a natural expansion for functions that vanish at the endpoints of the interval $[0, \ell]$. We know that the choice of coefficients

$$b_n \quad = \quad \frac{\left\langle \sin\left(\frac{n\pi x}{\ell}\right), f(x)\right\rangle}{\left\|\sin\left(\frac{n\pi x}{\ell}\right)\right\|^2} = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\, dx,$$

will minimize the L^2-error between the function and the Fourier Sine Series approximation.

Example 6.2. Expand

$$f(x) = x(\pi - x) \qquad 0 \le x \le \pi$$

in a Fourier Sine Series. Graph some partial sums of the series and examine their convergence.
Solution: We recognize as the initial condition problem from Example 3.2. We take $\ell = \pi$ in the formulae above, because the interval is $x \in [0, \pi]$. Consequently,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin(nx)\,dx \\ &= \begin{cases} 0 & n \quad \text{even} \\ \dfrac{8}{\pi n^3} & n \quad \text{odd} \end{cases} \end{aligned}$$

So the full series is

$$\mathbb{FSS}[f(x)] = \frac{8}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{\sin(nx)}{n^3}\ .$$

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If we extend the Fourier Sine Series (6.4) beyond the interval $x \in [0, \ell]$ we expect it to converge to an odd 2ℓ-periodic function because the basis elements $\left\{\sin\left(\frac{n\pi x}{\ell}\right)\right\}$ are odd and 2ℓ-periodic. Let's be a bit more specific about this.

Definition 6.5. Suppose a function, $f(x)$, is defined on an interval $x \in [0, \ell]$ with $f(0) = f(\ell) = 0$. We define the *odd extension* of $f(x)$, $f_o(x)$ to be the unique function such that $f_o(-x) = -f(x)$ for $x \in [-\ell, \ell]$.

Exercise 6.1. Suppose $f(x)$ is defined on the interval $x \in [0, \ell]$ with $f(0) = f(\ell) = 0$. Show that

$$\mathbb{FSS}[f(x)] = \mathbb{FS}[f_o(x)]$$

Explain in words what this means.

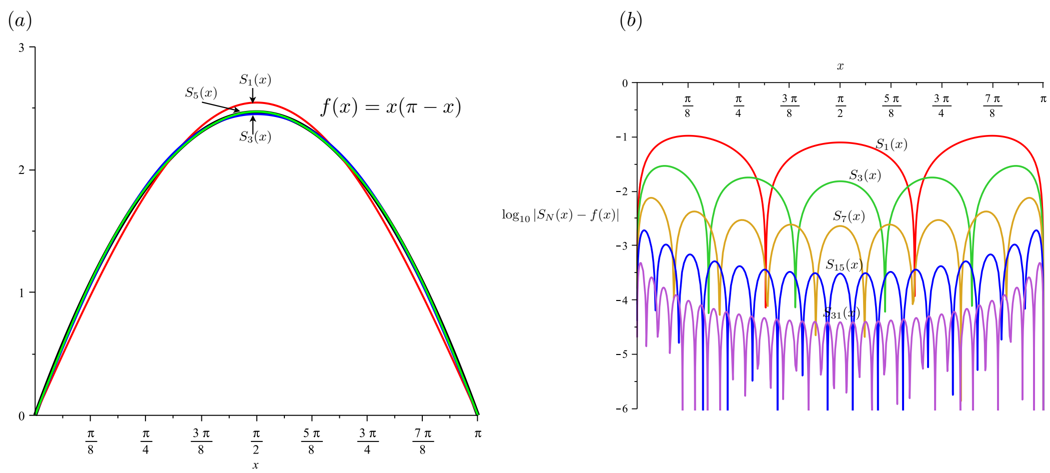

Figure 6.2: The partial sums, $S_N(x)$, for a Fourier Sine Series. (a) Note the partial sums $S_1(x)$,$S_3(x)$ and $S_5(x)$ converge rapidly to the function $f(x) = x(\pi - x)$(which is the black curve) in the range $0 < x < \pi$. (b) The log of the absolute value of the pointwise error, $\log_{10}|S_N(x) - f(x)|$ for $N = 1, 3, 7, 15$ and 31. Note that maximum error monotonically decreases as the number of terms in the approximation increases.

In addition, we can now define the *odd periodic extension* of $f(x)$, $\widetilde{f_o}(x)$ which is exactly what it sounds like – the periodic extension of the odd extension of $f(x)$. Note that $\widetilde{f_o}(x)$ is defined for all x

Exercise 6.2. Show that $\widetilde{f_o}(n\ell) = 0$ for any integer n.

Exercise 6.3. Show that $\widetilde{f_o}(2n\ell - x) = -\widetilde{f_o}(x)$ for any integer n.

We can now talk about the appropriate slogan for Fourier Sine Series:

Slogan 2. The Fourier Sine Series

$$\mathbb{FSS}[f(x)] \equiv \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) \qquad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx,$$

converges to $\widetilde{f_o}$, the odd periodic extension of $f(x)$ from $x \in [0, \ell]$.

Again, we will need to be cognizant of what *converges* means here.

6.3 The Fourier Cosine Series

Finally, for functions on the interval $x \in [0, \ell]$ whose derivatives vanish at the endpoint we showed the set of functions

$$\left\{ \frac{1}{2}, \ \cos\left(\frac{n\pi x}{\ell}\right), \right\}$$

are orthogonal in the L^2 inner-product. This allows us to propose the *Fourier Cosine Series* expansion for a function $f(x)$,

$$\mathbb{FCS}[f(x)] \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right). \tag{6.6}$$

as a natural expansion for functions that vanish at the endpoints of the interval $[0, \ell]$. We know that the choice of coefficients

$$\begin{aligned} a_0 &= \frac{\left\langle \frac{1}{2}, f(x) \right\rangle}{\left\| \frac{1}{2} \right\|^2} = \frac{2}{\ell} \int_0^\ell f(x)\, dx, \\ a_n &= \frac{\left\langle \cos\left(\frac{n\pi x}{\ell}\right), f(x) \right\rangle}{\left\| \cos\left(\frac{n\pi x}{\ell}\right) \right\|^2} = \frac{2}{\ell} \int_0^\ell f(x) \cos\left(\frac{n\pi x}{\ell}\right)\, dx, \end{aligned}$$

will minimize the L^2-error between the function and the Fourier Series approximation.

Example 6.3. Verify that the function

$$f(x) = 1 - 2x^2 + x^4$$

satisfies $f'(0) = f'(1) = 0$ Expand $f(x)$ in a Fourier Sine Series for the interval $x \in [0, 1]$. Graph some partial sums of the series and examine their convergence.
Solution: A simple calculation shows that

$$f'(x) = 4x^3 - 4x$$

so $f'(0) = f'(1) = 0$ and $f(x)$ is a natural candidate to expand as a Fourier Cosine Series. From the formulae above (with $\ell = 1$) we see that

$$\begin{aligned} a_0 &= 2\int_0^1 1 - 2x^2 + x^4\, dx = \frac{16}{15}, \\ a_n &= 2\int_0^1 \left(1 - 2x^2 + x^4\right) \cos\left(n\pi x\right)\, dx, = (-1)^{1+n} \frac{48}{\pi^4 n^4} \end{aligned}$$

so

$$\mathbb{FCS}[f(x)] \equiv \frac{8}{5} + \frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{1+n}}{n^4}\cos(n\pi x). \tag{6.7}$$

■

For the Fourier Cosine Series (6.6) beyond the interval $x \in [0, \ell]$ we expect it to converge to an even 2ℓ-periodic function because the basis elements $\left\{\frac{1}{2}, \cos\left(\frac{n\pi x}{\ell}\right)\right\}$ are even and 2ℓ-periodic. Let's be a bit more specific about this.

Definition 6.8. Suppose a function, $f(x)$, is defined on an interval $x \in [0, \ell]$. We define the *even extension* of $f(x)$, $f_e(x)$ to be the unique function such that $f_e(-x) = f(x)$ for $x \in [-\ell, \ell]$.

Exercise 6.4. Suppose $f(x)$ is defined on the interval $x \in [0, \ell]$. Show that

$$\mathbb{FCS}[f(x)] = \mathbb{FS}[f_e(x)]$$

Explain in words what this means.

In addition, we can now define the *even periodic extension* of $f(x)$, $\widetilde{f_e}(x)$ – the periodic extension of the even extension of $f(x)$. Note that $\widetilde{f_e}(x)$ is defined for all x. The appropriate slogan for Fourier Cosine Series:

Slogan 3. The Fourier Cosine Series

$$\mathbb{FCS}[f(x)] \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) \qquad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\, dx,$$

converges to $\widetilde{f_e}$, the even periodic extension of $f(x)$ from $x \in [0, \ell]$.

Again, we will need to be cognizant of what *converge* means here.

6.4 Bessel's Inequality, Parseval's Identity and Least Squares Convergence

In Chapter 4 we derive Bessel's Inequality which states that if F is an element of the inner product space $\mathcal{V}$ and $\{e_i\}$ are an orthogonal set in $\mathcal{V}$, then

$$\|F\|^2 \geq \sum_{n=1}^{N}(c_n)^2\|e_n\|^2 \tag{6.9}$$

where

$$c_n \equiv \frac{\langle e_n, F \rangle}{\|e_n\|^2}.$$

is the projection of F on e_n. Let us apply this idea to Fourier Series.

First, we define the *partial sums* of a Fourier series,

$$S_N \equiv \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

with coefficients

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\, dx \qquad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\, dx.$$

Bessel's Inequality now states that

$$\|f\|^2 \geq a_0^2 \left\|\tfrac{1}{2}\right\|^2 + \sum_{n=1}^{N} a_n^2 \left\|\cos\left(\frac{n\pi x}{\ell}\right)\right\|^2 + b_n^2 \left\|\sin\left(\frac{n\pi x}{\ell}\right)\right\|^2$$

evaluating the norms yields

$$\left\|\tfrac{1}{2}\right\|^2 = \frac{\ell}{2} \qquad \left\|\cos\left(\frac{n\pi x}{\ell}\right)\right\|^2 = \left\|\sin\left(\frac{n\pi x}{\ell}\right)\right\|^2 = \ell$$

so Bessel's inequality now states that

$$\|f\|^2 \geq \frac{\ell}{2}a_0^2 + \sum_{n=1}^{N} \ell\left(a_n^2 + b_n^2\right),$$

or if we divide by ℓ

$$\frac{1}{\ell}\|f\|^2 \geq \tfrac{1}{2}a_0^2 + \sum_{n=1}^{N} a_n^2 + b_n^2\,,$$

which should be true for the partial sums of any Fourier Series, independent of any notion of convergence of the series.

Example 6.4. Let's verify Bessel's inequality for the first example in the chapter; we found that for $g(x) = x^2$ on the interval $x \in [-1, 1]$ that

$$\mathbb{FS}[g(x)] = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx.$$

Note that

$$\frac{1}{\ell}\|g\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2)^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}x^4\,dx = \frac{2}{5}\pi^4$$

So that Bessel's inequality states that

$$\frac{2}{5}\pi^4 \geq \tfrac{1}{2}\left(\frac{2\pi^2}{3}\right)^2 + \sum_{n=1}^{N}\left(\frac{4(-1)^n}{n^2}\right)^2 = \frac{2\pi^4}{9} + \sum_{n=1}^{N}\frac{16}{n^4}$$

moving the a_0 term to the lefthand side and dividing by 16 yields

$$\frac{\pi^4}{90} \geq \sum_{n=1}^{N}\frac{1}{n^4}.$$

We invite the reader to verify this fact numerically. ■

Exercise 6.5. Write down Bessel's Inequality for the Fourier Sine Series and the Fourier Cosine Series.

6.4.1 Least Squares and Convergence in L^2.

In Chapter 4 we related the L^2 error of the N^{th} partial sum to Bessel's Inequality. For a Fourier Series this implies

$$(E_N)^2 \equiv \|f(x) - S_N\|^2 = \|f\|^2 - \ell\left[\tfrac{1}{2}a_0^2 + \sum_{n=1}^{N}a_n^2 + b_n^2\right]. \tag{6.10}$$

and Bessel's Inequality follows from the fact that the error is positive.

We would say the function *converges in* L^2 or in a mean-square sense if $\lim_{N\to\infty}E_N = 0$. An amazing result from analysis states that

Theorem 6.1. *If $f \in L^2[-\ell,\ell]$ (without smoothness assumptions), then for almost every $x \in [-\ell,\ell]$ we have that $S_N(x)$ converges to $f(x)$. That is* $\lim_{N\to\infty}E_N = 0$ *for all $f \in L^2[-\ell,\ell]$.*

Remark. This result is due to Lennart Carleson (in 1966) and has been called one of the hardest theorems in analysis. The notion of *almost every* x is precisely defined in terms of the Lebesgue integral.

This theorem means that one way the slogan above for convergence of the Fourier Series can be made rigorous by specifying that f is in $L^2[-\ell,\ell]$ and saying that convergence is in the mean-square sense. However, there are other notions of convergence that we will discuss in the next two chapters which apply for different classes of functions.

6.4.2 Parseval's Identity

A corollary to this theorem is:

Theorem 6.2. *(Parseval's Identity for Fourier Series) If $f \in L^2[-\ell, \ell]$ then*

$$\frac{1}{\ell}\|f\|^2 = \tfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

where a_n and b_n are the Fourier coefficients defined above.

The proof follows directly from the fact that $\lim_{N\to\infty} E_N = 0$ which turns Bessel's Inequality into an equality as $N \to \infty$.

So we can conclude from the example above that

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4},$$

an amazing result that is difficult to prove without the aid of Fourier Series.

Exercise 6.6. Write down Parseval's Theorem for the Fourier Sine Series and the Fourier Cosine Series.

Exercise 6.7. Apply Parseval's identity to the function $f(x) = x$ and conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$