Seven

Fourier Series II: Uniform Convergence, Pointwise Convergence and Gibb's Phenomena

In the previous Chapter we saw that Fourier Series are the best approximation to a function in a least square sense. However, this only tells us that the series converges to a function in some average sense. In fact we can say something much more concrete; Fourier series converge uniformly for sufficiently smooth functions.

For convenience in this Chapter we will use Fourier Series on the interval $x \in [-\pi, \pi]$; these results generalize quickly to an arbitrary interval. The big theorem we wish to state is:

Theorem 7.1. *(Uniform Convergence of Fourier Series) Let $f(x)$ be a continuous 2π-periodic function whose derivative is piecewise continuous. Then the Fourier Series, $\mathbb{FS}[f(x)]$, converges uniformly to f.*

Colloquially, this theorem says as the number of terms in the Fourier Series increases, the maximum difference between the partial sum and the function it approximates decays uniformly to zero.

To be more rigorous, we first need to define a piecewise continuous function,

Definition 7.1. A function $f(x)$ is *piecewise continuous* on $x \in [a, b]$ if:

(a) The function $f(x)$ is continuous on all but a finite number of points.

(b) For all $x_0 \in (a, b)$ the righthand limit,

$$f(x_0^-) \equiv \lim_{\epsilon \nearrow 0} f(x - \epsilon)$$

and the lefthand limit

$$f(x_0^+) \equiv \lim_{\epsilon \nearrow 0} f(x + \epsilon)$$

exist.

(c) The limits at the endpoints, $f(a^+)$ and $f(b^-)$ exist also.

Remember the partial sum of the Fourier Series (on the interval $x \in [-\pi, \pi]$) is defined as

$$S_N \equiv \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\ dx \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\ dx,$$

and now, we can define uniform convergence

Definition 7.2. We say a Fourier Series *converges uniformly* If for every $\epsilon > 0$ there exists N_ϵ such that

$$|f(x) - S_N| < \epsilon \qquad \text{for} \quad N > N_\epsilon$$

for all $x \in [-\pi, \pi]$.

Colloquially, this says we retain a sufficient number of terms of the Fourier Series, we can reduce the error to be less than any positive constant ϵ at every point in the interval.

7.1 The Riemann-Lebesgue Lemma and the Decay of Fourier Coefficients

In the previous sections we have seen that as the wavenumber n increases the Fourier coefficients a_n and b_n decrease to zero, which is clearly a necessary condition for a Fourier Series to converge. We will now prove that this is the case when $f(x)$ is a piecewise differentiable function.

Theorem 7.2 (Riemann-Lebesgue Lemma for Piecewise Differentiable Functions)**.** *Define*

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ dx \qquad B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\ dx,$$

where $f(x)$ is a piecewise differentiable function. Then

$$\lim_{n\to\infty} a_n = 0, \qquad \lim_{n\to\infty} b_n = 0.$$

Note that this theorem does *not* depend on n being an integer or $f(x)$ being periodic. Moreover, the restriction to piecewise differentiable functions can be considerably weakened.

Proof. We begin by using integration by parts and exploiting the differentiability of $f(x)$. Note that

$$\begin{aligned} B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ &= \frac{1}{n\pi}\left\{ f(x)\sin(nx)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)\sin nx\,dx \right\} \end{aligned}$$

Define

$$P = \max_{x\in[-\pi,\pi]} |f(x)|, \qquad Q = \sup_{x\in[-\pi,\pi]} |f'(x)|.$$

We know that since $f(x)$ is continuous and periodic that P exists. For the derivative, Q is the *supremum* or least upper bound of $|f'(x)|$; it is the maximum of the absolute value of the function and its left- and right-hand limits at the discontinuities.

Now,

$$\begin{aligned}
|B_n| &\leq \frac{1}{n\pi}\left\| f(x)\sin(nx)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)\sin nx\,dx \right\| \\
&\leq \frac{2}{n\pi}P + \frac{1}{n\pi}\int_{-\pi}^{\pi} |f'(x)\sin nx|\,dx \\
&\leq \frac{2}{n\pi}P + \frac{1}{n\pi}\int_{-\pi}^{\pi} |f'(x)|\,dx \\
&\leq \frac{2}{n\pi}P + \frac{1}{n\pi}\int_{-\pi}^{\pi} Q\,dx \\
&\leq \frac{2}{n\pi}P + \frac{2}{n}Q
\end{aligned}$$

and as n tends to infinity the righthand side vanishes, proving the result for B_n. A nearly identical calculation shows $\lim_{n\to\infty}|A_n| \to 0$. □

Exercise 7.1. Prove the following theorem:

Theorem 7.3. *Let f be a 2π-periodic function with $k-1$ continuous derivatives, and whose k-th derivative is piecewise continuous. The Fourier coefficients of f decay like C/n^k, that is*

$$|A_n| < \frac{C}{n^k} \qquad |B_n| < \frac{C}{n^k}$$

for a constant C which depends only on f.

Show that if $k = 2$ that this implies the Fourier Series converges uniformly (although this doesn't imply that it converges to $f(x)$!!).

Note: In the previous section we showed that when $f(x)$ was the periodic extension of x^2, which is piecewise differentiable , the Fourier coefficients decayed like C/n^2 which suggests that this theorem isn't sharp. In fact, one can improve the bound on the coefficients to C/n^{k+1}.

A discussion of Gibb's phenomena goes here.