Eight

Complex Fourier Series

In this Chapter we will discuss the natural extension of the idea of Fourier Series to the space of complex functions. It turns out that expanding a function as a sum of complex exponentials can yield a more elegant formulation of the idea of Fourier Series, useful in many contexts.

8.1 Complex Vector Spaces

Previously we defined a real vector space as being a set of functions closed under addition and scalar multiplication, where it was understood that the scalars were real valued. We can extend this idea to a vector space closed under multiplication my complex scalars.

Definition 8.1. A *complex vector space* $\mathcal{V}$ is a set of functions that is closed under addition and scalar multiplication, that is if u and v are elements $\mathcal{V}$, so is $\alpha u + \beta v$ where α and β are complex scalars.

Note that if we have a function of x, f that is an element of a complex vector space that f will naturally have real and imaginary parts

$$f = u + iv$$

where u and v are real. We will refer to $u = \Re\{f\}$ as the *real part* of f and $v = \Im\{f\}$ as the *imaginary part* of f. We can also define the *complex conjugate* of the function as

$$\overline{f} = u - iv.$$

and the magnitude, $|f|$, as the positive value satisfying

$$|f|^2 \equiv \overline{f} f = (u - iv)(u + iv) = u^2 + v^2.$$

For most of this chapter we will consider the case when f is a function of x defined on some interval $[a, b]$.

8.2 The Complex Inner-Product

In Chapter 4 we introduced the inner product of two real functions p and q on an interval $x \in [a, b]$,

$$\langle p, q\rangle \equiv \int_a^b \bar{u}v \, dx. \tag{8.2}$$

Now, let us define a new inner-product appropriate for two complex functions u and v defined on the inteval $x \in [a, b]$,

$$[u, v] \equiv \int_a^b \bar{u}v \, dx. \tag{8.3}$$

This *complex inner product* is a function from two elements of a complex vector space to the complex numbers that satisfies three properties

- **Hermitian Symmetry:** The inner product is *hermitian symmetric* in its arguments; that is

  $$[u, v] = \overline{[v, u]}$$

  for all elements u and v in the vector space.

- **Linearity:** The inner product is linear in the second argument; that is

  $$[u, \alpha v + \beta w] = \alpha\,[u, v] + \beta\,[u, w]$$

  for all complex scalars α and β and any u, v and w in the vector space. Note the hermitian symmetry and linearity in the second argument guarantees that

  $$[\alpha v + \beta w, u] = \overline{\alpha}\,[v, u] + \overline{\beta}\,[w, u]$$

- **Positive Definiteness:** We say an inner-product is positive definite if

  $$[u, u] \geq 0$$

  with equality when $u = 0$ only[1]. This allows to define the *norm* of a complex vector,

  $$\|u\| \equiv \sqrt{[u, u]}.$$

[1] This assumes the function is continuous; if one chooses to venture into the intricate machinery of measure theory and allow for discontinuous functions one would write "$u = 0$ *almost everywhere*".

Remember an *inner product space* is simply a vector space together with an inner product defined on that vector space. We can nowdefine an inner product space that is the set of *complex square-integrable functions*. For our inner product (8.3) above, the set of complex functions u whose norm is finite,

$$\|u\| \equiv \sqrt{\int_a^b u^2 \, dx},$$

is the vector space of square integrable functions.

Exercise 8.1. Show that if u and v are real functions that

$$[u, v] = \langle u, v \rangle .$$

8.3 Complex Fourier Series: The Fourier Series Revisited

For periodic functions on the interval $x \in [-\ell, \ell]$ we showed the set of functions

$$\left\{ \frac{1}{2}, \; \cos\left(\frac{n\pi x}{\ell}\right), \; \sin\left(\frac{n\pi x}{\ell}\right) \right\}$$

are orthogonal in the real L^2 inner-product. Let us now expand the space they describe by considering them as a basis of a complex vector space. This allowed us to propose that the Fourier Series expansion can be expanded to a complex function $f(x)$,

$$\mathbb{FS}[f(x)] \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right). \tag{8.4}$$

Where now the coefficients are complex numbers, but still given by the same formulae

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) \, dx, \quad b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) \, dx. \tag{8.5}$$

This choice of coefficients still minimize the L^2-error between the function and the Fourier Series approximation when $f(x)$ is a complex function.

Exercise 8.2. If $f(x)$ is a complex function, show that the real part of the Fourier Series, $\mathbb{FS}[f(x)]$, is the Fourier Series of the real part of $f(x)$ and that the imaginary part of the Fourier Series is the Fourier Series of the imaginary part of $f(x)$. Use this to explain why the coefficient formulae for a_n and b_n remain the same. What does this mean for the convergence of a Fourier Series for a complex function?

Remembering Euler's Formula,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

we can introduce a new set of basis functions

$$y_n(x) = e^{i\frac{n\pi x}{\ell}} \qquad \text{for} \quad n = 0, \pm 1, \pm 2, \cdots \tag{8.6}$$

that span the same space. To see this note that

$$\begin{aligned} \tfrac{1}{2} &= \tfrac{1}{2}\cdot 1 = \tfrac{1}{2}y_0(x) \\ \cos\left(\frac{n\pi x}{\ell}\right) &= \frac{e^{i\frac{n\pi x}{\ell}} + e^{-i\frac{n\pi x}{\ell}}}{2} = \frac{y_n(x) + y_{-n}(x)}{2} \\ \sin\left(\frac{n\pi x}{\ell}\right) &= \frac{e^{i\frac{n\pi x}{\ell}} - e^{-i\frac{n\pi x}{\ell}}}{2i} = \frac{y_n(x) - y_{-n}(x)}{2i} \end{aligned}$$

The orthogonality of these functions under the complex inner product is easy to check

$$[y_n(x), y_m(x)] \equiv \int_{-\ell}^{\ell} \overline{e^{i\frac{n\pi x}{\ell}}} e^{i\frac{m\pi x}{\ell}}\, dxdx = \int_{-\ell}^{\ell} e^{i\frac{(m-n)\pi x}{\ell}}\, dx = \begin{cases} 0 & n \neq m, \\ 2\ell & n = m. \end{cases}$$

This allows us to propose the *Complex Fourier Series* expansion for a function $f(x)$,

$$\mathbb{CFS}[f(x)] = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{\ell}}. \tag{8.7}$$

Note that as the Fourier Series is the sum of periodic functions with period 2ℓ, it also is 2ℓ-periodic. We know that the choice of coefficients

$$c_n = \frac{\left[e^{i\frac{n\pi x}{\ell}}, f(x)\right]}{\left\|e^{i\frac{n\pi x}{\ell}}\right\|^2} = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-i\frac{n\pi x}{\ell}}\, dx, \tag{8.8}$$

will minimize the L^2-error between the function and the Fourier Series approximation.

8.4 Relationship Between Real and Complex Fourier Series

In general, using the fact that

$$e^{\pm i\frac{n\pi x}{\ell}} = \cos\left(\frac{n\pi x}{\ell}\right) + i\sin\left(\frac{n\pi x}{\ell}\right)$$

we can rewrite the complex Fourier series in terms of a real Fourier Series, with complex coefficients

$$\begin{aligned} CFS[f(x)] &= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{\ell}}, \\ &= \sum_{n=-\infty}^{\infty} c_n \left[\cos\left(\frac{n\pi x}{\ell}\right) \pm i \sin\left(\frac{n\pi x}{\ell}\right)\right], \\ &= c_0 + \sum_{n=1}^{\infty} (c_n + c_{-n}) \cos\left(\frac{n\pi x}{\ell}\right) + i\,(c_n - c_{-n}) \sin\left(\frac{n\pi x}{\ell}\right), \end{aligned}$$

Note that for $n > 0$

$$\begin{aligned} c_n &= \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x) e^{-i\frac{n\pi x}{\ell}}\,dx \\ &= \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right) - i\sin\left(\frac{n\pi x}{\ell}\right)\,dx \\ &= \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx - i\frac{1}{2\ell}\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right)\,dx \\ &= a_n - ib_n \end{aligned}$$

where a_n and b_n are the coefficients of the real Fourier Series. A similar calculation shows that $c_{-n} = a_n + ib_n$ and $c_0 = a_0/2$, so in summary

$$c_0 = \frac{a_0}{2} \qquad c_n = \frac{a_n - ib_n}{2} \qquad c_{-n} = \frac{a_n + ib_n}{2}$$

Solving for a_n and b_n yields

$$a_0 = 2c_0 \qquad a_n = c_n + c_{-n} \qquad b_n = i\,(c_n - c_{-n})$$

and one can quickly convince yourself from the expression above that the real and the complex Fourier series are equivalent.

Exercise 8.3. Show that if $f(x)$ is real that $c_n = \overline{c_{-n}}$.

Example 8.1. Consider a complex periodic function, $h(x)$, with period 2π and

$$h(x) = e^{ix/2} \qquad -\pi < x < \pi.$$

(a) Find a complex Fourier series for $h(x)$.

(b) Compute Fourier series for the real periodic functions, $C(x)$ and $S(x)$, with period 2π,

$$C(x) = \cos(x/2) \qquad -\pi < x < \pi$$

$$S(x) = \sin(x/2) \qquad -\pi < x < \pi$$

by taking the real and imaginary parts of the series you found in part (a).

(c) Show that $C'(x) = -S(x)/2$ but $S'(x) \neq C(x)/2$. Why is this?

Solution:

(a) The function $h(x)$ has the complex Fourier series representation

$$h(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x) e^{-inx} dx$$

Evaluating, we see that

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(1/2-n)x}\, dx \\ &= \frac{1}{i\pi(1-2n)} e^{i(1/2-n)x} \Big|_{x=-\pi}^{x=\pi} \\ &= \frac{2\sin\left((1/2-n)\pi\right)}{\pi(1-2n)} \\ &= \frac{2(-1)^n}{\pi(1-2n)}. \end{aligned}$$

Or we can reconstitute the series to obtain

$$h(x) = e^{ix/2} = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1-2n} e^{inx}, \qquad -\pi < x < \pi.$$

(b) In general, using the fact that $e^{\pm inx} = \cos(nx) \pm i\sin(nx)$ we can rewrite the complex Fourier series in terms of a real Fourier Series, with complex coefficients

$$h(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, = c_0 + \sum_{n=1}^{\infty} (c_n + c_{-n})\cos(nx) + i\,(c_n - c_{-n})\sin(nx),$$

Noting from above that

$$c_0 = \frac{2}{\pi},$$

$$c_n + c_{-n} = \frac{2(-1)^n}{\pi(1-2n)} + \frac{2(-1)^{-n}}{\pi(1+2n)} = \frac{4(-1)^n}{\pi(1-4n^2)},$$

$$c_n - c_{-n} = \frac{2(-1)^n}{\pi(1-2n)} - \frac{2(-1)^{-n}}{\pi(1+2n)} = \frac{8n(-1)^n}{\pi(1-4n^2)},$$

we see that the series for $h(x)$ is

$$h(x) = e^{ix/2} = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi(1-4n^2)} \cos(nx) + i\frac{8n(-1)^n}{\pi(1-4n^2)} \sin(nx), \qquad -\pi < x < \pi.$$

Taking the real part of the series yields a Fourier cosine series for the even periodic function $C(x)$,

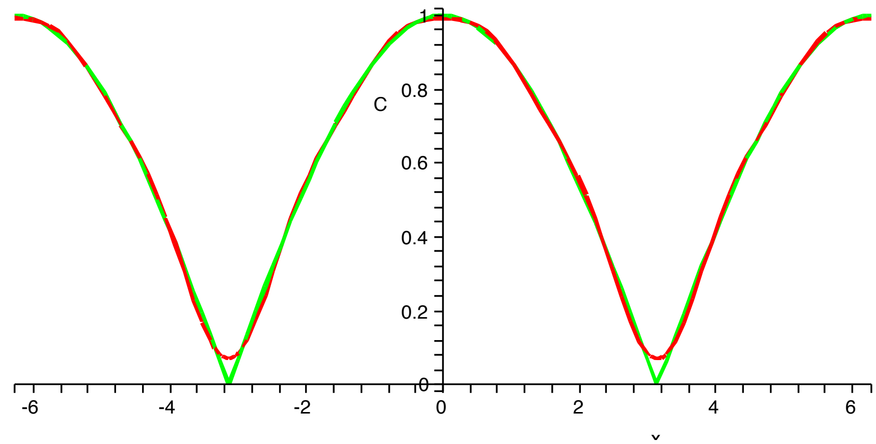
$$C(x) = \cos(x/2) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi(1-4n^2)} \cos(nx), \qquad -\pi < x < \pi,$$

whereas taking the imaginary part yields a Fourier sine series for the odd periodic function $S(x)$,

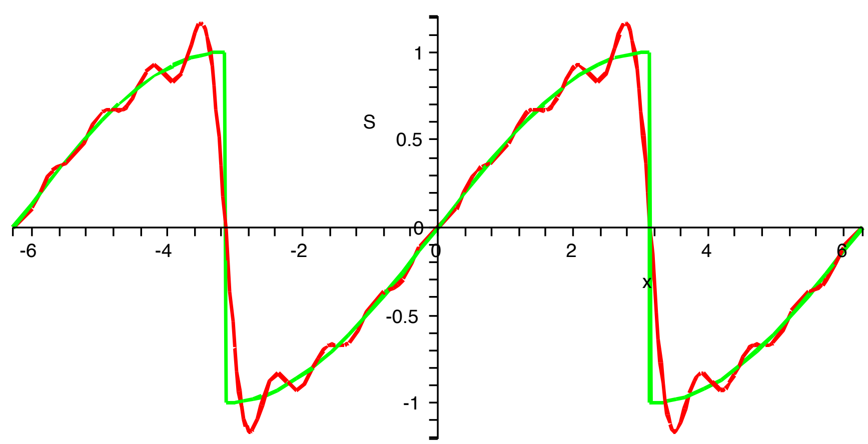
$$S(x) = \sin(x/2) = \sum_{n=1}^{\infty} \frac{8n(-1)^n}{\pi(1-4n^2)} \sin(nx), \qquad -\pi < x < \pi.$$

We graph a partial sum of five terms for $C(x)$ below.

Note that the sum is converging uniformly as the function is continuous and piecewise differentiable.



By contrast, consider the partial sum with ten terms for for $S(x)$.



The discontinuity at $x = \pm\pi$ leads to the persistent overshoot of Gibb's Phenomena.

(c) We can differentiate the fourier series for $C(x)$ term-by-term, using the fact that $(\cos(nx))' = -n\sin(nx)$ to obtain

$$C'(x) = -\sum_{n=1}^{\infty} \frac{4n(-1)^n}{\pi(1-4n^2)} \sin(nx), \qquad -\pi < x < \pi,$$

which is exactly $-S(x)/2$. However, differentiating $S(x)$ term-by-term,
using $(\sin(nx))' = n\cos(nx)$ yields

$$S'(x) = \sum_{n=1}^{\infty} \frac{8n^2(-1)^n}{\pi(1-4n^2)} \cos(nx), \qquad -\pi < x < \pi.$$

which not only is not $C'(x)$, but which is clearly divergent at $x = 0$ where

$$S'(0) = \sum_{n=1}^{\infty} \frac{8n^2(-1)^n}{\pi(1-4n^2)}$$

and the terms in the series approach $\pm 2/\pi$ as $n \to \infty$. The difference between these series is the fact that $C(x)$ is continuous and piecewise differentiable, so we expect it's derivative to have a convergent Fourier series. However, the same discontinuity in $S(x)$ that leads to Gibb's Phenomena is also associated with the derivative of $S(x)$ approaching infinity at $x = \pm\pi$, and the resulting Fourier Series diverging.

■

8.5 Galerkin Method for Solving the Heat Equation

A *Galerkin Method* is a method by which the solution a partial differential equation is expressed as an expansion in orthogonal functions in space with time-dependent coefficients. Let us consider an example.

Let $u(x,t)$ describe the temperature of a metal ring, where x parameterizes the angle, which we choose for convenience to span the interval $[-\pi, \pi]$. Suppose that the ring has some internal heating that is angle-dependent, so that $u(x,t)$ satisfies the inhomogeneous heat equation,

$$\begin{aligned} &\mathbb{DE}: \quad u_t = Du_{xx} + f(x) && -\pi \le x \le \pi, \quad t > 0 \\ &\mathbb{IC}: \quad u(x,0) = 0 && -\pi \le x \le \pi, \end{aligned}$$

where D is the thermal diffusivity and $f(x)$ describes the internal heating. Furthermore, we have assume that the temperature of the ring is initially zero.

Because the temperature $u(x,t)$ is parameterized by the angle x, the temperature must be a 2π-periodic function, $u(x,t) = u(x+2\pi,t)$, This suggests that we should write the temperature as a complex Fourier expansion with time-dependent coefficients,

$$u(x,t) = \sum_{n=-\infty}^{\infty} A_n(t)e^{inx}. \tag{8.9}$$

Let's substitute this expression into the heat equation, $u_t = Du_{xx} + f(x)$,

$$\sum_{n=-\infty}^{\infty} \frac{dA_n(t)}{dt} e^{inx} = -D \sum_{n=-\infty}^{\infty} A_n(t)n^2 e^{inx} + f(x)$$

If we further assume that $f(x)$ has the complex Fourier series representation

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{inx},$$

where

$$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

then we obtain

$$\sum_{n=-\infty}^{\infty} \left[\frac{dA_n(t)}{dt} + Dn^2 A_n(t)\right] e^{inx} = \sum_{n=-\infty}^{\infty} f_n e^{inx}.$$

The only way for the two Fourier series on either side of this equation to match is if the coefficients of both series are identical. Therefore,

$$\frac{dA_n(t)}{dt} + Dn^2 A_n = f_n,$$

for all positive integers n. Effectively, we have used the assumption in (8.9) to turn the original PDE into an infinite system of ODEs. The initial condition for each of these ODEs is $A_n(0) = 0$, since the initial heat distribution is zero. The solution to each of these ODEs is

$$A_n(t) = \begin{cases} \dfrac{f_n}{Dn^2}\left[1 - e^{-Dn^2 t}\right] & \text{if } n \neq 0 \\ f_0\, t & \text{if } n = 0, \end{cases}$$

and all that is left to do is to plug these into (8.9).

Example 8.2. Suppose $f(x) = \cos^2(x)$ for the problem above. Find an explicit solution for $u(x,t)$.
Solution: Suppose $f(x) = \cos^2(x)$. Since we already know the general solution, all that needs to be done is to figure out the complex Fourier series for $f(x)$. We'll do this using simple trigonometric identities:

$$\begin{aligned} f(x) = \cos^2(x) &= \frac{1}{2}\cos 2x + \frac{1}{2} \\ &= \frac{1}{4}\left[e^{2ix} + e^{-2ix}\right] + \frac{1}{2} \\ &= \frac{1}{4}e^{-2ix} + \frac{1}{2} + \frac{1}{4}e^{2ix}. \end{aligned}$$

So we see that the complex Fourier series for $f(x)$ only has three nonzero coefficients, which are $f_{-2} = f_2 = 1/4$, and $f_0 = 1/2$. Therefore, the solution in this particular case is

$$\begin{aligned} u(x,t) &= \frac{f_{-2}}{4D}\left[1 - e^{-4Dt}\right]e^{-2ix} + f_0\, t\, e^{0ix} + \frac{f_2}{4D}\left[1 - e^{-4Dt}\right]e^{2ix} \\ &= \frac{1}{16D}\left[1 - e^{-4Dt}\right]e^{-2ix} + \frac{t}{2} + \frac{1}{16D}\left[1 - e^{-4Dt}\right]e^{2ix} \\ &= \frac{t}{2} + \frac{1}{8D}\left[1 - e^{-4Dt}\right]\cos(2x). \end{aligned}$$

The temperature $u(x,t)$ grows linearly in time and without bound. One interpretation of this is that the internal heating is always positive and therefore the total heat must continually increase. ■