An Applied Mathematician's Guide to

Complex Variables:

Analytic Functions, the Residue Theorem, and the Laplace Transform

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Foreword

These notes were developed over multiple years of teaching the course Math 115. I am grateful to Darryl Yong who provided many of the examples and Michael Davis who transcribed some the of notes and created most the figures. I, of course, take responsibility for any mistakes and omissions.

– Andrew J. Bernoff, Spring 2019

Part I

Complex Arithmetic and Complex Functions

One

Complex Arithmetic

CHAPTER OUTLINE

- Complex numbers, complex conjugates.
- Polar representation and Euler's formula.
- Regions of the complex plane.
- Roots of complex numbers.

1.1 Complex Numbers

You remember that the imaginary number, $i = \sqrt{-1}$, is a useful tool when evaluating expressions that contain negative numbers under a radical symbol. From previous classes, we know that

$$\begin{aligned} i &= \sqrt{-1} \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \\ &\vdots \end{aligned}$$

Also remember that $\mathbb{C}$ represents the set of all complex numbers and can be notated as

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\} \tag{1.1}$$

There is a *bijection* between a pair of real numbers (a, b) (which can be thought of as points in the plane $\mathbb{R}^2$) and complex numbers. We can recover the Cartesian representation of a complex numbers by identifying the real and imaginary part,

$$\text{Re}[a + bi] = a \qquad \text{Im}[a + bi] = b$$

which yields the point (a, b) in the plane $\mathbb{R}^2$.

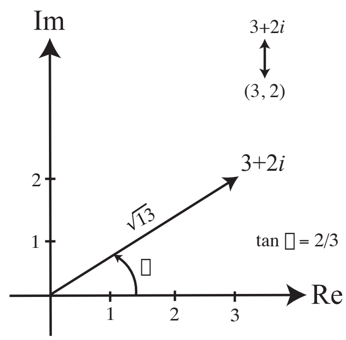


Figure 1.1: The Cartesian, complex and polar representation of $z = 3 + 2i$.

1.2 Complex Conjugate

Definition 1.2. The *complex conjugate* of a complex number $z = a + bi$ is defined as $\bar{z} = a - bi$.

Exercise 1.1. Show that graphically, the complex conjugate of a number z is its reflection in the real axis.

We can also use the complex conjugate to compute the distance of $z = x + yi$ from the origin. Note that

$$\begin{aligned} z\bar{z} &= (x + yi)(x - yi) \\ &= x^2 + xyi - xyi - y^2 i^2 \\ &= x^2 + y^2 \\ &\equiv |z|^2 \end{aligned}$$

This allows us to define the *magnitude* of a complex number.

Definition 1.3. The *magnitude* or *length* of $z = x + iy$ is defined as

$$|z| \equiv \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

Note that it is the distance of the point (x, y) from the origin in the plane.

1.3 Polar Representation

Complex numbers can also be represented using polar coordinates. If the point (x, y) is a distance r and an angle θ from the polar axis, then we know that

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

We write the complex number $z = x + iy$ as

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) \equiv re^{i\theta}.$$

That is we define

$$e^{i\theta} \equiv \cos\theta + i\sin\theta$$

which is commonly known as Euler's formula

Just as with polar coordinates, we can extract the magnitude r and the angle θ, sometimes called the *argument*, associated with a complex number $z = x + iy$.

$$\begin{aligned} r = |z| = |x + iy| = r &= \sqrt{x^2 + y^2} \\ \tan\theta &= \frac{y}{x}\ . \end{aligned}$$

Here we write $\theta = arg(z)$ and one must be careful to choose θ in the proper quadrant of the complex plane; θ is also arbitrary up to integer multiples of 2π, an issue we will address in detail in this and later lectures.

Example 1.1. Write $-3 + 3i$ in polar coordinates.
Solution: We see that $r = \sqrt{(-3)^2 + 3^2} = 3\sqrt{2}$ and $\tan\theta = 3/(-3) = -1$.

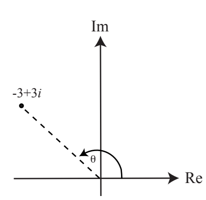


Figure 1.2: The point $z = -3 + 3i$.

Since z lies in the second quadrant, we choose that $\theta = 3\pi/4$ and

$$\boxed{-3 + 3i = 3\sqrt{2}e^{i\frac{3\pi}{4}}}$$

■

So what happens geometrically when you multiply two complex numbers together? We compute

$$\begin{aligned} z_1 &= r_1 e^{i\theta_1} \\ z_2 &= r_2 e^{i\theta_2} \\ z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1+\theta_2)} \end{aligned}$$

where we leave it as an exercise to show from Euler's formula that

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

We see that the magnitude of the product is the product of the magnitudes and the argument of the product is the sum of the arguments.

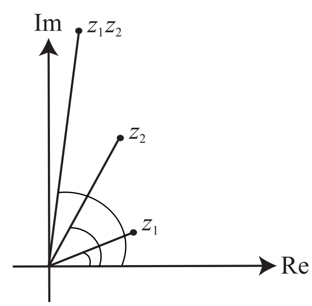


Figure 1.3: Multiplication of complex numbers

Example 1.2. Compute $(-3+3i)^{40}$.
Solution: We could compute this by multiplying out the product

$$(-3+3i)^{40} = (-3+3i)(-3+3i)\cdots(-3+3i)$$

which is a very tedious way to find the product. However, we can use the polar form; remember $-3+3i = 3\sqrt{2}e^{\frac{i3\pi}{4}}$, so

$$\begin{aligned} (-3+3i)^{40} &= \left(3\sqrt{2}e^{\frac{i3\pi}{4}}\right)^{40} \\ &= \left(3\sqrt{2}\right)^{40}\left(e^{\frac{i3\pi}{4}}\right)^{40} \\ &= 3^{40}2^{20}e^{i30\pi} \\ &= 3^{40}2^{20}. \end{aligned}$$

■

1.4 Regions of the Complex Plane

There are many different regions of the complex plane. Three important ones are

- The *upper-half plane* (UHP), $Im[z] \geq 0$
- The *right-half plane* (RHP), $Re[z] \geq 0$
- The *unit disc*, $|z| \leq 1$ centered at the origin

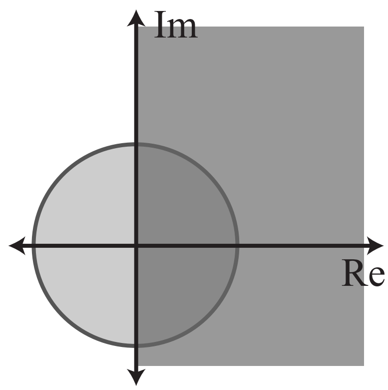


Figure 1.4: The unit disc and right-half plane regions.

We can also describe curves in the complex plane. Two examples:

Circle: Let's construct a circle centered at z_0 or radius a:

(a) A circle is the set of points z a fixed distance, a, from the point z_0 in the complex plane

$$|z - z_0| = a \quad (non - parametric)$$

(b) A circle is the set of points z that are displaced by a distance a at an angle θ in the complex plane

$$z = z_0 + ae^{i\theta} \quad (parametric)$$

where θ is a parameter that varies from 0 to 2π for one rotation around the circle.

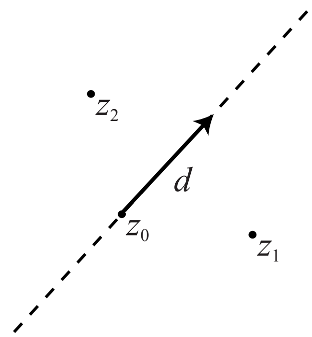


Figure 1.5: A line in the complex plane.

Line: We can construct a line in several ways.

(a) The set of points z equidistant from z_1 and z_2 - that is the perpendicular bisector.

$$|z - z_1| = |z - z_2| \quad (non - parametric)$$

(b) The set of points z displaced from z_0 be a constant multiple, r of the complex number d

$$z = z_0 + rd \quad (parametric)$$

where $r \in \mathbb{R}$ is the parameter.

1.5 Roots of Complex Numbers

A classic mathematical brain teaser asks if $i = \sqrt{-1}$, what is $\sqrt{i}$? The well-known science fiction writer, Isaac Asimov, claimed that this was a hypercomplex number. Fortunately, we know better! Computing roots of complex numbers is easy if we use the polar representation. Using the polar representation $i = e^{i\pi/2}$, we see

$$\begin{aligned} \sqrt{i} &= (e^{i\pi/2})^{1/2} \\ &= e^{i\pi/4} \\ &= \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \end{aligned}$$

This is one answer, but in fact there are two distinct solutions. To see this, one must first remember that we can write $1 = e^{2\pi in}$ for any integer n. So

$$\begin{aligned} i &= e^{i\pi/2+2\pi in} \\ \sqrt{i} &= e^{i\pi/4+\pi in} \end{aligned}$$

Remember that $e^{2\pi i} = 1$, we see this yields two distinct solutions for $n = 0, 1$, that is $\{e^{\pi/4},\ e^{5i\pi/4}\}$ where

$$\begin{aligned} e^{5i\pi/4} &= \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) \\ &= -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \end{aligned}$$

and the remaining values of n just alternate between these two values. Therefore, any nonzero complex number has two distinct square roots.

Example 1.3. Find all solutions to $z^3 + 8 = 0$.

Solution: We rewrite the problem in the form

$$z^3 = -8$$

and the first step is to write -8 as a complex number in polar coordinates. Remember $-1 = e^{i\pi}$ or, more generally, $-1 = e^{i(\pi+2\pi n)}$ for any integer n. So,

$$-8 = 8 \cdot e^{i(\pi+2\pi n)}$$

for any integer n. The equation can now be written as

$$z^3 = -8 = 8 \cdot e^{i(\pi+2\pi n)}$$

so

$$\begin{aligned} z &= (-8)^{1/3} = (8)^{1/3} e^{i(\pi/3+2\pi n/3)} \\ &= \quad (8)^{1/3} e^{i\pi/3}, \quad (8)^{1/3} e^{i3\pi/3}, \quad (8)^{1/3} e^{i5\pi/3} \cdots \\ &= 2 \cdot \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), \quad -2, \quad 2 \cdot \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \cdots \\ &= 1 + i\sqrt{3}, \quad -2, \quad 1 - i\sqrt{3} \cdots \end{aligned}$$

You might ask yourself the question, what happens for other values of n?

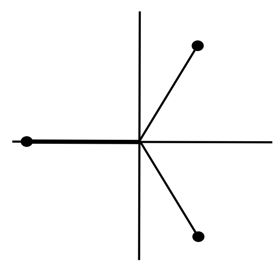


Figure 1.6: The cube roots of -8. Note that the roots lie on a circle of radius 2 at angles $\theta = \pi/3 + 2\pi n/3$.

Do you think we found all the solutions? ■

Example 1.4. What is the value of i^i ?
Solution: Remember that

$$\begin{aligned} i &= e^{i\pi/2} \quad \text{so} \\ i^i &= \left(e^{i\pi/2}\right)^i = e^{i^2\pi/2} = e^{-\pi/2} \\ &\approx 0.20788\ldots \end{aligned}$$

but also

$$\begin{aligned} i &= e^{i(\pi/2+2n\pi)} \qquad n \in \mathbb{Z} \\ i^i &= e^{-(\pi/2+2n\pi)} \end{aligned}$$

So there are infinitely many values (all real) of i^i. Weird, huh? ■

Two

Complex Functions

CHAPTER OUTLINE

- Complex functions.
- Real and imaginary components of a complex function.
- Multi-valued functions.
- Exponential & logarithm functions.
- The derivative of a complex function.

2.1 COMPLEX FUNCTIONS

A complex function is a map from a complex number to another complex number. For example if we define

$$f(z) = z^3$$

then

$$f(1+i) = (1+i)^3 = -2+2i.$$

It is useful for visualizing a complex function to break it into real and imaginary components.

2.1.1 Real & Imaginary Components

One way to visualize $f(z)$ is to look at its real and imaginary parts.

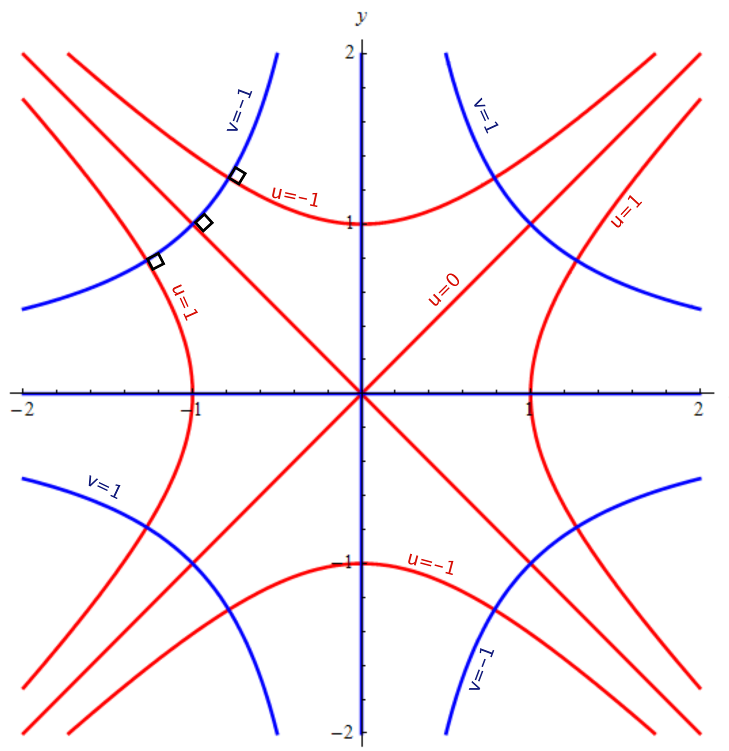


Figure 2.1: Real and imaginary parts of the function $f(z) = z^2$. We found that $f(z) = u(x, y) + iv(x, y)$ where $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. The level curves of $u(x, y)$ are drawn in red and the level curves of $v(x, y)$ in blue. The level curves are mutually orthogonal in this case.

Example 2.1. Consider the function$f(z) = z^2$. Show that if $z = x + iy$ that the real and imaginary parts of $f(z)$ are quadratic functions of x and y.
Solution: Let $z = x + iy$ and define $f(z) = u + iv$ where u and v are real functions. Then

$$\begin{aligned} f(z) &= u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy \\ \text{Re}[f] &= u(x, y) = x^2 - y^2 \\ \text{Im}[f] &= v(x, y) = 2xy \end{aligned}$$

In general, every complex function $f(z)$ can be written in terms of two real functions of two variables, $f(z) = u(x, y) + iv(x, y)$. ■

2.1.2 Multi-valued Functions

We've seen previously that when we take an nth root of a complex number we generally get n answers. This leads to the idea of a "multi-valued" function which is a bit of an oxymoron as, by definition, functions should return a single value. Let's look at an example and then show how to define a function as single-valued.

Example 2.2. Consider $f(z) = \sqrt{z}$. The function $f(z)$ is "multi-valued" and associates two values to (almost) every number. For example

$$f(i) = \sqrt{i} = \pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right).$$

How can we make this choice unique? ■

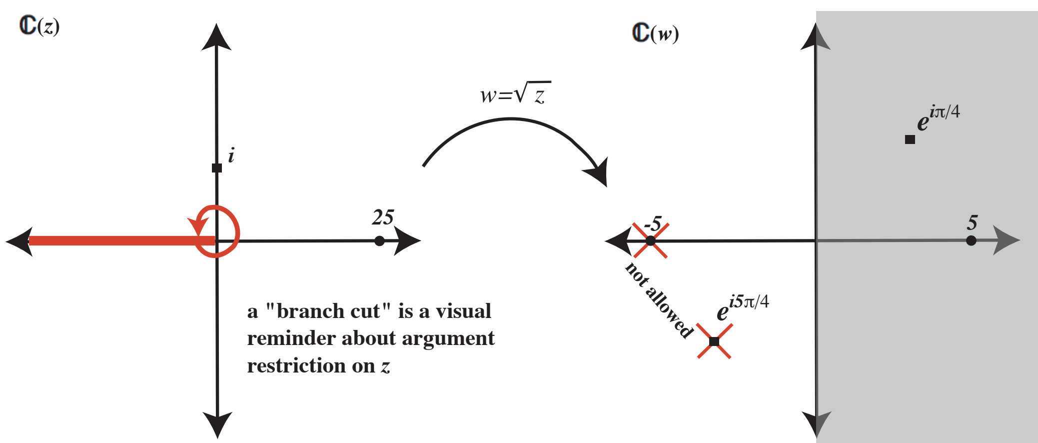


Figure 2.2: Note $f(z) = \sqrt{z}$ maps the complex plane to the right half plane.

To do analysis on these types of functions, we usually have to make a decision about which values we want. We do this by placing restrictions on the *argument* of $f(z)$ to specify a unique value. For example, we can define a unique value for the argument by specifying that it is in a given range. This *principle value* of the argument, $Arg(z)$ is defined as

$$Arg(z) = arg(z) \quad \text{where } arg(z) \in (-\pi, \pi]$$

If $\theta = Arg(z)$ then

$$w = \sqrt{z} = \left(\sqrt{re^{i\theta}}\right) = \left(\sqrt{r}e^{i\theta/2}\right)$$

where we see that $\frac{\theta}{2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Note that along the negative real axis the value of z jumps discontinuously. This is known as a *branch cut*.

2.1.3 Exponential & Logarithm Functions

Using Euler's Formula, we can define the exponential function. Consider

$$w = f(z) = e^z = \exp(z)$$

If $\quad z = x + iy, \quad$ then

$$\begin{aligned} e^z = e^{x+iy} &= e^x e^{iy} \\ &= e^x(\cos y + i \sin y) \end{aligned}$$

So

$$\begin{aligned} u(x,y) &= \text{Re}[f(z)] = e^x \cos y \\ v(x,y) &= \text{Im}[f(z)] = e^x \sin y \end{aligned}$$

Note the answer is defined for all z.

2.1.4 Logarithm Function

To define

$$w = \log z$$

Think of a logarithm as the inverse of the exponential function. If $w = \log z$ then $z = e^w$. Let $w = u + iv$ and $z = x + iy$, then

$$\log z = \log(x + iy) = w = u + iv \Leftrightarrow z = x + iy = e^w = e^{u+iv} = e^u e^{iv}$$

Now we can identify e^u as the magnitude of z and v as the argument of z, so

$$e^u = |x + iy| = \sqrt{x^2 + y^2} \Rightarrow u = \ln \sqrt{x^2 + y^2}$$

and

$$v = \arg(x + iy) = \arg(z)$$

from which we conclude

$$\boxed{\log(z) = \ln|z| + i\arg(z)}$$

Example 2.3. Compute $\log(10),\ \log(-10)$ and $\log(1+2i)$
Solution: Using the formula above, we see that

$$\log(10) = \ln|10| + i\arg(10) = \ln 10 + i(2n\pi), \quad n \in \mathbb{Z}$$

$$\log(-10) = \ln|-10| + i\arg(-10) = \ln 10 + i(\pi + 2n\pi), \quad n \in \mathbb{Z}$$

$$\log(1+2i) = \ln|1+2i| + i\arg(1+2i) = \ln\sqrt{5} + i\left[\tan^{-1}\left(\frac{2}{1}\right) + 2n\pi\right], \quad n \in \mathbb{Z}$$

■

We can define a unique value of $\log(z)$ by choosing the principal value for the $\arg$ function. Remember that Arg(z), the principle value of $\arg(z)$, is defined as

$$\text{Arg}(z) = \arg(z) \qquad \text{where } z \in (-\pi, \pi]$$

then we define Log(z), the principle value of the logarithm as

$$\text{Log}(z) = \ln|z| + i\,\text{Arg}(z)$$

Notice the difference between $\arg(z)$ and Arg(z), and $\log(z)$ and Log(z).

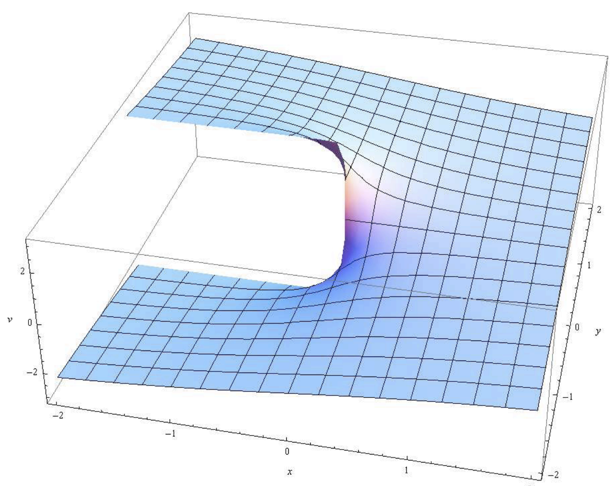


Figure 2.3: The imaginary part of the logarithm, Im[Log(z)] $=$ Arg(z). Note the branch cut along the negative real axis

2.2 The Derivative of a Complex Function

The derivative of a complex function is similar to that of a regular real-valued function. There are some important differences to understand however.

Definition 2.1. We define the derivative of a complex function, $f(z)$, at a point z_0 as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

when the limit exists and is independent of the path along which Δz tends to zero in the complex plane. If $f'(z_0)$ exists, then we say $f(z)$ is *differentiable* at $z = z_0$.

Example 2.4. Compute the derivative of $f(z) = z^2$ from the definition above.

Solution: We see that

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{2z\Delta z + \Delta z^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} 2z + \Delta z \\ &= 2z \end{aligned}$$

which is a relief, because it agrees with what we expected from calculus for a single real variable!! ■

We will spend most our time considering functions that are differentiable for almost all points in the plane. We have some nomenclature to describe these functions:

Definition 2.2. If $f(z)$ is differentiable at $z = z_0$ and in a small neighborhood about z_0, then f is *analytic at* $z = z_0$.

Definition 2.3. If $f(z)$ is analytic for all z, then $f(z)$ is **entire**.

In fact, functions like z^n for n a positive integer, $\sin z$, $\cos z$ and e^z are entire functions, and their derivatives are what you would expect:

$$\frac{d}{dz}(e^z) = e^z \qquad \frac{d}{dz}(\sin z) = \cos z$$

$$\frac{d}{dz}(z^n) = nz^{n-1} \qquad \frac{d}{dz}(\cos z) = -\sin z$$

$$(n = 1, 2, \ldots)$$

Also, for example

$$\frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2} \leftrightarrow \frac{1}{z} \text{ is analytic everywhere except at } z = 0$$

Finally, the usual rules for differentiable function apply.

$$[f+g]' = f' + g' \qquad \& \qquad [fg]' = fg' + f'g$$

etc ...

We end with an example of how things can go wrong:

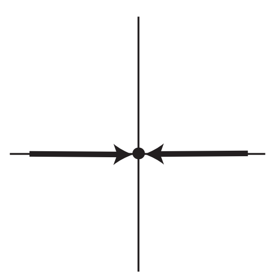
Example 2.5. A non-analytic function. Consider

$$f(z) = |z|^2$$

Compute $f'(z)$.

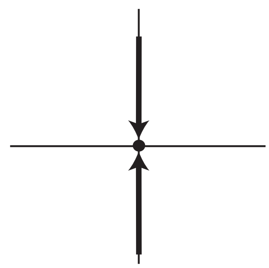
Solution: Remember, $f'(z) = \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$
Notice $\Delta z = 0$ means that Δz can approach 0 in any direction in the complex plane. The limit must be the same regardless of direction for the limit to exist.

(a) *Case 1:* approach 0 horizontally
Let $\Delta z = \Delta x \in \mathbb{R}$, $z = x + iy$



$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} &= \lim_{\Delta x \to 0} \frac{|x+iy+\Delta x|^2 - |x+iy|^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + y^2 - (x^2+y^2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\
&= 2x
\end{aligned}$$

(b) *Case 2:* approach 0 vertically
Let $\Delta z = i\Delta y, \;\; \Delta y \in \mathbb{R}, \;\; z = x + iy$



$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} &= \lim_{\Delta y \to 0} \frac{|x+iy+i\Delta y|^2 - |x+iy|^2}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{x^2 + (y+\Delta y)^2 - (x^2+y^2)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{y^2 + 2y\Delta y + (\Delta y)^2 - y^2}{i\Delta y} \\
&= \frac{2y}{i}
\end{aligned}$$

We just computed the derivative limit in two directions and got different answers (for $z \neq 0$), so $f(z)$ is not differentiable. It turns out that $f(z)$ is differentiable at $z = 0$ only, but showing that requires a little more work. We'll save that for another time.

■

Three

Analyticity and The Cauchy Riemann Equations

CHAPTER OUTLINE

- The Cauchy-Riemann Equations.
- Consequences of the Cauchy-Riemann (CR) equations.
- Determining region of analyticity.

3.1 CAUCHY-RIEMANN EQUATIONS

Let's say $f(z)$ is an analytic function,

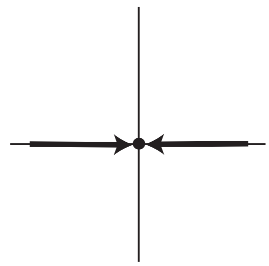
$$f(z) = u(x, y) + iv(x, y).$$

What can we say about the real and imaginary parts, $u(x, y)$ and $v(x, y)$? It turns out that they must satisfy two partial differential equations called the *Cauchy-Riemann Equations*. Let's derive them below.

3.1.1 Derivation of the Cauchy-Riemann Equations

If $f(z)$ is analytic at z_0, then f is differentiable at z_0 and the derivative limit exists and values match if we approach $\Delta z \to 0$ horizontally or vertically. Let's compute these two limits.

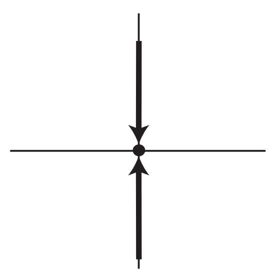
(a) *Case 1:* Let $\Delta z \to 0$ approach horizontally (parallel to the real axis). Let $z_0 = x + iy$ and $\Delta z = \Delta x$ where Δx is real, and let $\Delta x \to 0$.



Then, we compute

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \\ &= \lim_{\Delta x\to 0}\frac{(u(x+\Delta x,y)+iv(x+\Delta x,y))-(u(x,y)+iv(x,y))}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x}+i\frac{v(x+\Delta x,y)-v(x,y)}{\Delta x} \\ &= \frac{\partial u}{\partial x}+i\ \frac{\partial v}{\partial x} \end{aligned}$$

(b) *Case 2:* Let $\Delta z\to 0$ approach vertically (parallel to the imaginary axis).
Let $z_0=x+iy$ and $\Delta z=i\Delta y$ where Δy is real, and let $\Delta y\to 0$.



$$\begin{aligned} f'(z_0) &= \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \\ &= \lim_{\Delta y\to 0}\frac{(u(x,y+\Delta y)+iv(x,y+\Delta y))-(u(x,y)+iv(x,y))}{i\Delta y} \\ &= \lim_{\Delta y\to 0}\frac{u(x,y+\Delta y)-u(x,y)}{i\Delta y}+i\frac{v(x,y+\Delta y)-v(x,y)}{i\Delta y} \\ &= -i\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial y} \end{aligned}$$

If the limit exist, the answers in cases (a) and (b) must be the same,

$$f'(z_0) = \frac{\partial u}{\partial x} + i\ \frac{\partial v}{\partial x} = -i\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Equating the real and imaginary parts of each side, we get the **Cauchy-Riemann equations**,

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\ .}$$

It turns out that $f(z)$ is analytic in a region D, if and only if the Cauchy-Riemann equations hold in D. If $f(z)$ is analytic then the derivative can be written in terms of the partial derivatives of u and v with respect to x or y,

$$f'(z) = \frac{\partial u}{\partial x} + i\ \frac{\partial v}{\partial x} = -i\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

where the Cauchy-Riemann equation guarantees the expressions are equivalent.

3.1.2 Consequences of the Cauchy-Riemann (CR) equations

The Cauchy-Riemann equations imply a great deal of structure for the real and imaginary parts of an analytic function. We list some properties below:

(a) The real and imaginary parts of analytic functions are *harmonic*, that is they satisfy Laplace's equation. Remember from physics that the Laplace's equation for $\phi(x, y)$ is

$$\nabla^2 \phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

This equations relates to fluid dynamics, the wave equation, the diffusion equation, and the electrostatic potential to name a few places. We prove this using the CR equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Substituting into $\nabla^2 u$, we see that

$$\begin{aligned}
\nabla^2 u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) \\
&= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0.
\end{aligned}$$

where we have assumed that u and v have continuous second partial derivatives which guarantees that the mixed partials are equal.

(b) The CR equations show that level curves of $u(x,y)$ and $v(x,y)$ of an analytic function are mutually orthogonal. A normal vector to the u level curve is the gradient,

$$\nabla u = \langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \rangle.$$

Similarly, a normal vector to the v level curve is

$$\nabla v = \langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \rangle.$$

Taking the dot product of the two normal vectors yields

$$\begin{aligned}
\nabla u \cdot \nabla v &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \\
&= \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \\
&= 0
\end{aligned}$$

where we have used the CR equations to eliminate the partial derivatives of u. So the normals to the level curves are orthogonal and level curves are perpendicular.

(c) The real and imaginary parts of an analytic function are linked together by the CR equations; each determines the other up to an additive constant and one can "recover" the real part from the imaginary part and vica versa. This is best demonstrated by an example:

Exercise 3.1. For $f(z) = z^2$, the real and imaginary parts are

$$u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy$$

(a) Show that u and v satisfy the CR equations.

(b) Verify that the level curves of u and v are orthogonal.

(c) Suppose we only knew $u(x,y) = x^2 - y^2$, can we recover $f(z)$?

Solution:

(a) Evaluating the partial derivatives, we see that

$$u_x = 2x\ , \quad u_y = -2y\ , \qquad v_x = 2y\ , \quad v_y = 2x\ .$$

Clearly the Cauchy-Riemann equations, $u_x = v_y$ and $u_y = -v_x$, are satisfied.

(b) We take a slightly different tack from the proof above. Remember the gradient of a function is normal to its level curves. Consequently, the level curves are orthogonal if the gradients are perpendicular,

$$\nabla u \cdot \nabla v = u_x \cdot v_x + u_y \cdot v_y = (2x)(2y) + (-2y)(2x) = 0\ .$$

You can check that this is equivalent to the condition on the tangents above.

(c) If $u(x,y)$ is the real part of an analytic function, we can reconstruct $v(x,y)$, the imaginary part, using the Cauchy-Riemann equations.

$$\begin{aligned}
&u_x = 2x = v_y \implies v(x,y) = 2xy + \psi(x) \\
&u_y = -2y = -v_x = -2y - \psi'(x) \implies \psi'(x) = 0 \\
&\quad \implies \psi(x) = C \\
&\quad \implies v(x,y) = 2xy + C
\end{aligned}$$

Knowing u and v, we can even reconstruct the complex function that they represent.

$$\begin{aligned}
f(z) &= u(x,y) + iv(x,y) \\
&= \left(x^2 - y^2\right) + i\,(2xy + C) \\
&= (x+iy)^2 + iC \\
&= z^2 + iC
\end{aligned}$$

where C is a real constant.

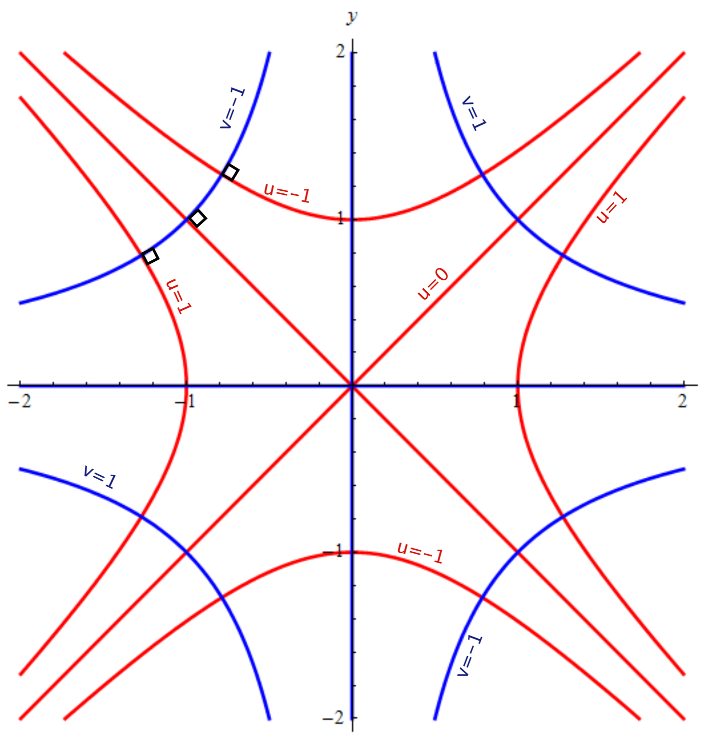


Figure 3.1: The level curves of $u(x,y)$ and $v(x,y)$ in the problem above. Note they are perpendicular

Exercise 3.2. Consider $u(x,y) = 3x^2y + Ay^3$. For what value of A is $u(x,y)$ the real part of an analytic function $f(z)$? For this value of A find $f(z)$.
Solution: If $u(x,y)$ is the real part of a harmonic function, then it is harmonic. That is

$$0 = u_{xx} + u_{yy} = (3x^2y + Ay^3)_{xx} + (3x^2y + Ay^3)_{yy} = 6y + 6Ay = 0$$

from which we determine that $\boxed{A = -1.}$
Now, from the Cauchy-Riemann equations, if $u(x,y) = 3x^2y - y^3$ then

$$\begin{aligned}
&u_x = 6xy = v_y \implies v(x,y) = 3xy^2 + \psi(x) \\
&u_y = 3x^2 - 3y^2 = -v_x = -3y^2 - \psi'(x) \implies \psi'(x) = -3x^2 \\
&\quad \implies \psi(x) = -x^3 + C \\
&\quad \implies v(x,y) = 3xy^2 - x^3 + C
\end{aligned}$$

Knowing u and v, we can even reconstruct the complex function that they represent.

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) \\ &= \left(3x^2y - y^3\right) + i\left(3xy^2 - x^3 + C\right) \\ &= -i(x+iy)^3 + iC \\ &= -iz^3 + iC \end{aligned}$$

where C is a real constant.

3.1.3 Determining region of analyticity

One way to determining where a function is differentiable is to use the Cauchy-Riemann equations. If the CR equations are true and f is continuous, then f is differentiable. If this holds in a region D, then f is analytic in D. However, that is time consuming. However, one can build up a library of known analytic functions which allows one to determine the region of analyticity by inspection.

Guidelines for determining analyticity:

- Identify pieces of the function that whose regions of analyticity you know. (For example, polynomials, sine, cosine, and exponential functions are *entire*.)
- Remember that products and compositions of analytic functions are analytic also. So $f(z) = z^2 \sin z$ and $g(z) = \sin(z^2)$ are entire also.
- Identify problem points – division by zero, or discontinuities of the function, etc ...
- Watch out for branch cuts. (Multiple-valued functions, like roots and log, will require a branch cut to be single-valued.)
- Complex conjugate function and magnitude operation typically make the function not analytic everywhere, so $f(z) = |z|^2 = z\bar{z}$ and $g(z) = \sin(\bar{z})$ are not analytic anywhere.

Example 3.1. Determine where these functions are analytic.

(a) The function $a(z) = \sin z + \frac{1}{z-1}$.

Solution: By looking at each piece of this function, we can determine analyticity. Note $\sin z$ is entire (analytic everywhere), but $\frac{1}{z-1}$ is not defined at $z = 1 \Rightarrow$ not differentiable at $z = 1$.

So $a(z)$ is analytic everywhere except at $z = 1$.

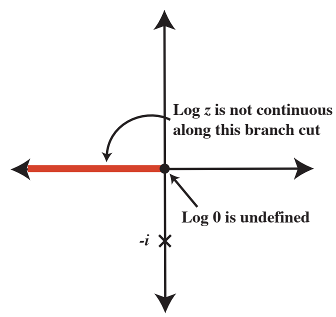


Figure 3.2: Determining where $d(z) = \sin\left(\frac{1}{z+i}\right)\text{Log}(z)$ is analytic.

(b) The function $b(z) = \dfrac{\sin(z^2)}{z^2+4}$.

Solution: The denominator of this fraction is zero at $z = \pm 2i$.

So $b(z)$ is analytic everywhere except $z = \pm 2i$.

(c) The function $c(z) = \csc z = \dfrac{1}{\sin z}$.

Solution: Note $c(z)$ is undefined where $\sin z = 0$, which is $z = n\pi,\ \ n \in \mathbb{Z}$.

So $c(z)$ is analytic everywhere $z \neq n\pi,\ \ n \in \mathbb{Z}$.

(d) The function $d(z) = \sin\left(\dfrac{1}{z+i}\right)\text{Log}(z)$.

Solution: Note $d(z)$ is not defined at $z = -i$ and $\text{Log}(z)$ has a branch cut along the negative real axis (see the figure above).

So $d(z)$ is analytic everywhere except at $z = -i$ and along negative real axis.

■

Part II

Contour Integration and the Residue Theorem

Four

Complex Integration

CHAPTER OUTLINE

- Complex integration via Parameterization.
- The fundamental integration theorems.
- The Cauchy's Integral Formula.

4.1 COMPLEX INTEGRATION VIA PARAMETERIZATION

You have seen line integrals in multivariable calculus and the physical concept of work, which is force integrated over a displacement, in mechanics. Integrating a function along a curve in the complex plane can be accomplished in a fashion analogous to a line integral, namely by parameterizing the contour of integration. Let us do a few examples.

Example 4.1. Compute the integral

$$I = \int |z|^2 \, dz.$$

along the curves:

(a) C: The quarter circle with radius 2, traversed from 2 to $2i$.

(b) D: The line segment from 2 to $2i$.

Solution: The basic idea here is that first you parametrize the curve for z as a function of the parameter, then you substitute in and calculate the integral. So:

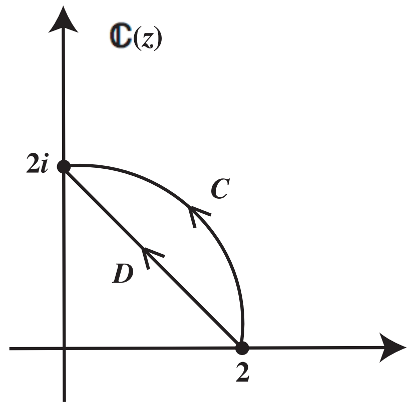


Figure 4.1: Two curves for integration in example 4.1.

(a) One parameterization of C is $z = 2e^{i\theta}$ where θ goes from 0 to $\frac{\pi}{2}$. Along C, $z = 2e^{i\theta}$ and on the curve parameterized by θ,

$$dz = \frac{dz}{d\theta}\, d\theta$$

so $dz = 2ie^{i\theta}d\theta$ for our particular example. Substituting into the integral, we see that

$$\begin{aligned}\int_C |z|^2\, dz &= \int_0^{\pi/2} |2e^{i\theta}|^2\ \ 2e^{i\theta}\, i\, d\theta \\ &= \int_0^{\pi/2} 4 \cdot 2 \cdot e^{i\theta} \cdot i\, d\theta = [8e^{i\theta}]_0^{\pi/2} \\ &= 8e^{i\pi/2} - 8e^{i\cdot 0} \\ &= 8i - 8\end{aligned}$$

(b) First we need to parameterize the line. It is useful to observe that a line between two points z_0 and z_1 in the complex plane can be parametrized as

$$z(s) = z_0 + (z_1 - z_0)s \qquad 0 \le s \le 1.$$

So a parameterization of D is $z = 2 + (2i - 2)s$ where s increases from 0 to 1. Along D, $dz = (2i - 2)ds$ and

$$\begin{aligned}\int_D |z|^2 dz &= \int_0^1 |2 + (2i-2)s|^2\,(2i-2)ds \\ &= \int_0^1 \left[(2-2s)^2 + (2s)^2\right](2i-2)ds \\ &= (2i-2)\int_0^1 [4 - 8s + 8s^2]ds \\ &= (2i-2)\left[4 - 4 + \frac{8}{3}\right] = \frac{16}{3}i - \frac{16}{3}\end{aligned}$$

Note the answer is different for the two contours above ■

Example 4.2. Now compute the integral

$$I = \int z^2\,dz$$

along the same two curves as in the example above.
Solution: We can use the same parametrizations as above:

(a) Along C, $z = 2e^{i\theta}$ and $dz = 2ie^{i\theta}d\theta$ so,

$$\begin{aligned}\int_C z^2\,dz &= \int_0^{\pi/2} (2e^{i\theta})^2\;\; 2e^{i\theta}\,i\,d\theta \\ &= 8i\int_0^{\pi/2} e^{3i\theta}\,d\theta \\ &= \frac{8i}{3i}[e^{3i\theta}]_0^{\pi/2} \\ &= \frac{8}{3}\left[e^{3i\pi/2} - e^{i\cdot 0}\right] = \frac{8}{3}(-i-1) \\ &= -\frac{8}{3}(1+i)\end{aligned}$$

(b) Along D, $\quad dz = (2i - 2)ds.$

$$
\begin{aligned}
\int_D z^2 dz &= \int_0^1 (2 + (2i-2)s)^2 \, (2i-2) ds \\
&= (2i-2) \int_0^1 \left[(2-2s) + (2is)]^2 \right] ds \\
&= (2i-2) \int_0^1 (2-2s)^2 + 2(2-2s)(2is) + (2is)^2 \, ds \\
&= (2i-2) \int_0^1 (4-8s) + 8i(s-s^2) ds \\
&= (2i-2) \left[(4s-4s^2) + 8i \left(\frac{s^2}{2} - \frac{s^3}{3} \right) \right]_0^1 \\
&= (2i-2) \left[(4-4) + 8i \left(\frac{1}{2} - \frac{1}{3} \right) \right] = (2i-2)(8i)\frac{1}{6} \\
&= -\frac{8}{3}(1+i)
\end{aligned}
$$

■

Note the when we integrate a non-analytic function, the answers are different but when we integrate an analytic function the answers are the same. In fact, this is an example of the *path independence* of the integrals of analytic functions. We'll learn more about this soon.

4.2 The Fundamental Integration Theorems

Let $f(z)$ be analytic in a simply-connected (basically, no holes) region of the complex plane. The following statements are true.

I. Cauchy-Goursat Integral Theorem:

Theorem 4.1. *For any closed, simple (doesn't cross itself) path C inside the region of analyticity,*

$$\oint_C f(z) dz = 0$$

So, for example

$$\oint_C e^{e^{\sin z}} dz = 0$$

for any closed path C. This theorem is often called *Cauchy's Theorem*.

II. Path Independence Property:

Theorem 4.2. *For any two paths C_1 and C_2 inside the region of analyticity with the same start and end points,*

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

that is the integral is independent of the path taken between the two points.

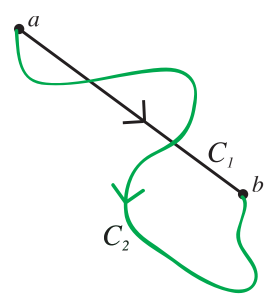


Figure 4.2: Two paths with the same start and end points.

III. Fundamental Theorem of Calculus for Analytic Functions:

Theorem 4.3. *If $f(z) = F'(z)$ then*

$$\int_a^b f(z)dz = F(b) - F(a)$$

where the integral is taken along any contour connecting a and b in the region of analyticity.

We will now outline the proof of these three theorems. We will begin by proving the Fundamental Theorem of Calculus.
Proof: We need to remember some facts about line integrals, in particular:

$$\int_{(x_0,y_0)}^{(x_1,y_1)} \nabla W \cdot d\vec{x} = \int W_x dx + W_y dy = \int dW = W|_{(x_1,y_1)} - W|_{(x_0,y_0)}$$

which is basically the Work-Energy theorem. where W is the *potential* then $-\nabla W$ is the associated force; the line integral is the *work* done along the curve which is just the change in the potential.

Also, remember the Cauchy-Riemann equations; if $F(z)$ is an analytic function and $f(z) = F'(z)$ is its derivative then

$$f(z) = F'(z) = U_x + i\,V_x = V_y - i\,U_y$$

Next we write $dz = dx + i\,dy$. Substituting into the contour integral yields

$$\begin{aligned}\int_a^b f(z)dz &= \int_a^b (U_x + i\,V_x)(dx + i\,dy) \\ &= \int_a^b U_x dx - V_x dy + i\int_a^b V_x dx + U_x dy \\ &= \int_a^b U_x dx + U_y dy + i\int_a^b V_x dx + V_y dx = \int_a^b d(U + i\,V) \\ &= U + i\,V|_{z=a}^{z=b} = F(b) - F(a).\end{aligned}$$

Where the answer is independent of the path of integration!![1]

This proves the Fundamental Theorem of Calculus for Analytic Functions and since we haven't specified the path of integration, it proves Path Independence property of these integrals also. To prove the Cauchy-Goursat Theorem, simply note that if the points a and b are the same, that is, the path C is closed, that $F(b) = F(a)$ and the integral vanishes.

Example 4.3. Compute the integral

$$I = \int_0^{i\pi} \cos z dz.$$

Since $\cos z$ is analytic everywhere, the path of integration does not matter. We can compute

$$\begin{aligned}\int_0^{i\pi} \cos z\; dz = \sin z|_0^{i\pi} = \sin i\pi - 0 &= \frac{e^{i(i\pi)} - e^{-i(i\pi)}}{2i} \\ &= i\frac{(e^{\pi} - e^{-\pi})}{2} = i\,\sinh \pi\ .\end{aligned}$$

■

[1]Some people find this confusing because we are using differentials ($dz = dx + i\,dy$); we could parameterize the path by writing $z(t) = x(t) + i\,y(t)$ and then write $dz = \frac{dz}{dt} = \left[\frac{dx}{dt} + i\frac{dy}{dt}\right] dt$ and then the result follows from the chain rule, an exercise we leave for the reader.

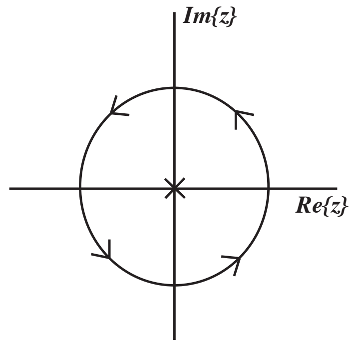


Figure 4.3: The contour of integration in example 4.4, namely the unit circle $|z| = 1$ traversed in a counter-clockwise sense.

Example 4.4. Evaluate the integral

$$\oint_C \frac{1}{z}\,dz,$$

where C is a unit circle traversed in a counter-clockwise sense.

Solution: We parameterize the unit circle

$$z = e^{i\theta} \quad 0 \leq \theta \leq 2\pi \qquad \Rightarrow \qquad dz = ie^{i\theta}dz$$

and substitute to obtain

$$\begin{aligned}\oint_C \frac{1}{z}\,dz &= \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}}d\theta \\ &= \int_0^{2\pi} id\theta = 2\pi i.\end{aligned}$$

A good question to ask is why doesn't the integral vanish? The answer is because the singularity at the origin is in the interior of C. Somehow the integral *smells the singularity,* ■

This is an example of the *Cauchy Integral Formula* which allows us to "count" the poles in the interior of a contour via an integral. Cauchy's Theorem has a wide range of uses; one is that it allows us to consider integrals over complicated curves.

The next example is very important for understanding the integral of analytic functions around closed contours.

Example 4.5. Calculate $I = \oint_C (z-a)^n dz$ for any integer n, where C is a circle of radius ρ about a (counterclockwise).

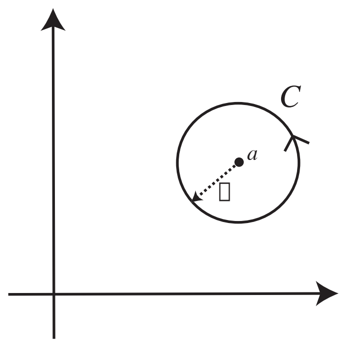


Figure 4.4: The contour for Example 4.5.

Solution: One way to evaluate the integral is to parameterize C by setting $z = a + \rho e^{i\theta}$ where θ increases from 0 to 2π. Then $dz = \rho i e^{i\theta} d\theta$ and we can reduce the problem to an integral in θ.

$$I = \oint_C (z-a)^n dz = \int_0^{2\pi} \left(a + \rho e^{i\theta} - a\right)^n \rho i e^{i\theta} d\theta = \int_0^{2\pi} \rho^{n+1} i e^{i(n+1)\theta} d\theta$$

This integral breaks into two cases:
If $n \neq -1$, then

$$\begin{aligned} I &= \left[\frac{\rho^{n+1} i e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} \\ &= \frac{\rho^{n+1}}{n+1}\left[\underbrace{e^{i(n+1)2\pi}}_{=1,\ \text{since}\ n\in\mathbb{Z}} - e^0\right] = 0 \end{aligned}$$

If $n = -1$, then

$$I = \int_0^{2\pi} \rho^0 i e^0 d\theta = [i\theta]_0^{2\pi} = 2\pi i$$

This yields our final answer:

$$\oint_C (z-a)^n dz = \begin{cases} 0, & \text{if } n \in \mathbb{Z}, \ \ n \neq -1 \\ 2\pi i, & \text{if } n \in \mathbb{Z}, \ \ n = -1 \end{cases}$$

■

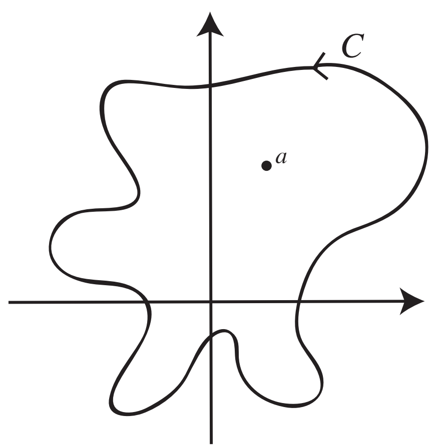
Note that if n is a positive integer, that $f(z) = (z-a)^n$ is analytic and the fact that the integral vanishes follows from the Cauchy-Goursat Theorem

4.3 Cauchy's Integral Formula

Let's revisit the warm-up problem; we will compute

$$\oint_C (z-a)^n dz, \quad n \in \mathbb{Z},$$

where now C is a simple (doesn't intersect itself) closed curve containing $z = a$, traversed counter-clockwise.



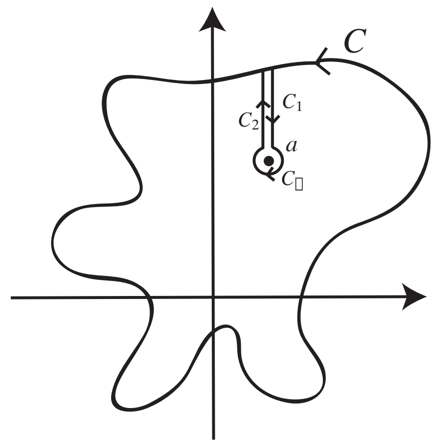
A simple closed curve, C, containing $z = a$.

We'll show that the answer is the same as in our warm-up problem.

Let's do some surgery on C. Consider

$$D = C \bigcup C_1 \bigcup C_\rho \bigcup C_2,$$

where C_ρ is the circle centered at $z = a$ of radius ρ traversed clockwise.



The interior of D no longer contains $z = a$ and by *Cauchy's Theorem:*

$$\oint_D (z-a)^n dz = 0$$

as the interior of D is simply-connected and $(z-a)^n$ is analytic inside D. Note also that

$$0 = \oint_D (z-a)^n dz = \int_C + \int_{C_1} + \int_{C_\rho} + \int_{C_2} (z-a)^n dz$$

Now let C_1 & C_2 get so close they lie on top of each other. Then,

$$C_1 \to -C_2 \Rightarrow \int_{C_1} (z-a)^n dz \approx -\int_{C_2} (z-a)^n dz.$$

Also in the limit as C_1 approached C_2, C_ρ approached a circle of radius ρ traversed *clockwise*. As such, the answer is the negative of the answer of the example computed at the beginning of the chapter,

$$\oint_{C_\rho} (z-a)^n dz = \begin{cases} 0, & \text{if } n \in \mathbb{Z},\ \ n \neq -1 \\ -2\pi i, & \text{if } n \in \mathbb{Z},\ \ n = -1 \end{cases}$$

So, as the integrals along C_1 and C_2 cancel, we see that

$$\oint_C (z-a)^n dz = -\oint_{C_\rho} (z-a)^n dz = \begin{cases} 0, & \text{if } n \in \mathbb{Z},\ \ n \neq -1 \\ 2\pi i, & \text{if } n \in \mathbb{Z},\ \ n = -1 \end{cases}$$

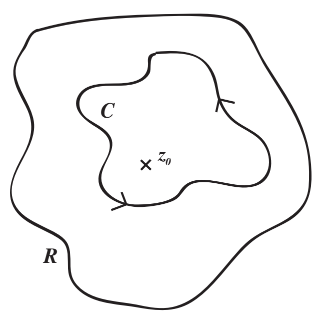
Consequently the integral around any closed contour C that encloses $z = a$ is the same. Cauchy's Theorem basically allows us to *deform* C into a circle of radius ρ, a much simpler curve for which we know the answer. Since C was quite arbitrary, one can see the power of Cauchy's Theorem.

Note: Cauchy's Theorem says that the integral around a simple closed curve of a function analytic in the curve's interior vanishes. The converse is not true. For instance, look at the warm-up exercise, for $n = -2, -3, -4, \ldots$.

This previous example motivates a big result in complex integration:

Theorem 4.4 (Cauchy's Integral Formula)**.** *Let $f(z)$ be analytic in a simply-connected region, R and let C be a simple, closed, positively-oriented (counterclockwise) curve in the region. Then for any point z_0 inside C,*

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz.$$



Proof: Let

$$I = \oint_C \frac{f(z)}{z - z_0} dz = \underbrace{\oint_C \frac{f(z) - f(z_0)}{z - z_0} dz}_{I_1} + \underbrace{\oint_C \frac{f(z_0)}{z - z_0} dz}_{I_2}$$

For I_1, we claim the function

$$F(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0 \\ f'(z_0) & \text{if } z = z_0 \end{cases}$$

is analytic. This follows from L'Hôpital's Rule,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0),$$

and some details about continuity which we have omitted (trust me, it works - if you have any questions look at any complex variables textbook). So I_1 vanishes.

For I_2, the integrand $\dfrac{f(z_0)}{z-z_0}$ is analytic except at $z = z_0$.
Consequently, we can deform the contour to a circle of radius ρ centered at $z = z_0$.
Now $dz = z_0 + \rho e^{i\theta} d\theta \quad 0 \le \theta \le 2\pi$

$$\begin{aligned} I_2 &= \oint_C \frac{f(z_0)}{z-z_0} dz = f(z_0) \oint_C \frac{dz}{z-z_0} \\ &= \int_0^{2\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}} d\theta = 2\pi i\, f(z_0). \end{aligned}$$

As a bonus, we can derive a contour integral for the higher derivatives of f at z_0.

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{\overbrace{f(z_0+\Delta z)}^{\text{apply CIF}} - \overbrace{f(z_0)}^{\text{apply CIF}}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[\frac{1}{2\pi i} \oint_C \frac{f(z)}{z-(z_0+\Delta z)} dz - \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz \right] \\ &= \frac{1}{2\pi i} \oint_C f(z) \lim_{\Delta z \to 0} \left[\frac{1}{z-(z_0+\Delta z)} - \frac{1}{z-z_0} \right] \frac{1}{\Delta z} \\ &= \frac{1}{2\pi i} \oint_C f(z) \frac{d}{dz_0} \left[\frac{1}{z-z_0} \right] dz \\ &= \frac{1}{2\pi i} \oint_C f(z) \frac{1}{(z-z_0)^2} dz \end{aligned}$$

So,

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} dz \tag{4.1}$$

and inductively one can show that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

where

$$f^{(n)}(z_0) = \frac{d^n}{dz^n} (f(z))|_{z=z_0} .$$

From this generalization, we can see that when a function is analytic at $z = z_0$, it is actually infinitely differentiable there. This follows from the fact that if $f(z)$ is bounded near z_0 (from continuity), these integrals exist and therefore the derivatives of the function are bounded. A little more hard work shows that a function analytic at $z = z_0$ also has a convergent power series at this point.

Let us examine this point in a bit more detail.

Five

Taylor's Theorem and Classifying Isolated Singularities

CHAPTER OUTLINE OF LECTURE

- Classifying Isolated Singularities
  - Taylor series
  - Isolated singularities
  - Laurent series

In this section we will generalize Taylor's series

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2!}f''(z_0)(z - z_0)^2 + \dots$$

from a function of a real variable to a function of a complex variable. We will also define it's generalization to a *Laurent series* which includes negative powers.

5.1 TAYLOR SERIES

Another big result from complex analysis that we will not prove is:

Theorem 5.1 (Taylor's Theorem). *$f(z)$ is analytic at $z = z_0$ if and only if it has a convergent Taylor series about $z = z_0$,*

$$\begin{aligned} f(z) &= f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2!} f''(z_0)(z - z_0)^2 + \ldots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \end{aligned}$$

Furthermore, the radius of convergence of the series is the distance from z_0 to the nearest singularity (any place where the function is not analytic).

We can apply this theorem to some classical examples.

Example 5.1. The Taylor series for e^z, $\sin z$, and $\cos z$ about $z = 0$ are

$$\begin{aligned} e^z &= 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \ldots \\ \sin z &= z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \ldots \\ \cos z &= 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \ldots \end{aligned}$$

Note e^z, $\sin z$, and $\cos z$ are analytic at $z = 0$, so they have convergent power series expansions for all z. The above functions converge for all z because these functions are entire. ■

Example 5.2. A geometric series. Consider

$$f(z) = \frac{1}{1 + z^2}.$$

The Taylor series expansion for $f(z)$ about $z = 0$ is

$$\frac{1}{1 + z^2} = 1 - z^2 + z^4 - z^6 + \ldots = \sum_{n=0}^{\infty} z^{2n}(-1)^n$$

where we have used the formula for the expansion of a geometric series,

$$1 + A + A^2 + A^3 + \ldots = \frac{1}{1 - A}$$

We can ask what the radius of convergence of this series is. The *ratio test* can be used to compute the radius of convergence.

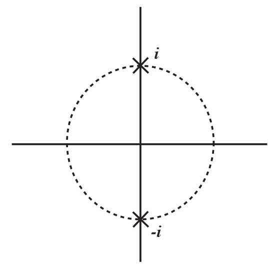


Figure 5.1: Singularities of $f(z) = \frac{1}{1+z^2}$. The Taylor series of $f(z)$ at $z = 0$ converges for $|z| < 1$ because this is the largest disc around $z = 0$ for which $f(z)$ is analytic.

Ratio Test:

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ the series diverges.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ the series converges.

Apply the ratio test, we see that

$$\lim_{n\to\infty} \left|\frac{z^{2n+2}(-1)^{n+1}}{z^{2n}(-1)^n}\right| = \lim_{n\to\infty} |z^2| = |z^2| < 1$$

The region $|z^2| < 1$ is the interior of the unit disc. The function

$$f(z) = \frac{1}{1+z^2}$$

has singularities at $\pm i$. Consequently the unit disc is also the largest circle at the origin in which the function is analytic, as predicted by Taylor's Theorem above. ■

Exercise 5.1. What is the power series expansion for $f(z) = \frac{1}{z}$ about $z = 1$?

Solution: One solution method is to use Taylor's formula; note that

$$f'(z) = -\frac{1}{z^2} \quad f''(z) = \frac{2}{z^3} \quad f'''(z) = -\frac{3!}{z^4} \quad \ldots \quad f^{(n)}(z) = (-1)^n \frac{n!}{z^{n+1}}$$

from which we see that

$$\begin{aligned} f(z) &= f(1) + f'(1)(z-1) + \frac{1}{2!}f''(1)(z-1)^2 + \frac{1}{3!}f'''(1)(z-1)^3 + \dots \\ &= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \dots \end{aligned}$$

Another method uses the formula for the sum of a geometric series:

$$\frac{1}{1-A} = 1 + A + A^2 + A^3 + \dots$$

which converges for $|A| < 1$. Using this we again see

$$\frac{1}{z} = \frac{1}{1+(z-1)} = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \dots$$

which converges for $|z-1| < 1$.

5.2 Isolated singularities

Analytic functions have very nice properties (infinitely differentiable, Cauchy-Riemann equations hold, etc.). But it turns out that the singularities (places where a function is not analytic) of complex functions are also really useful and contain lots of information.

Definition 5.1. A singularity of $f(z)$ at $z = z_0$ is *isolated* if $f(z)$ is analytic in a punctured disc centered at $z = z_0$; i.e., analytic in $0 < |z - z_0| < \epsilon$, for some $\epsilon > 0$.

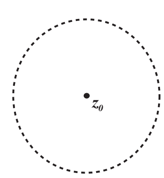


Figure 5.2: An isolated singularity at $z = z_0$.

Let's classify some singularities!

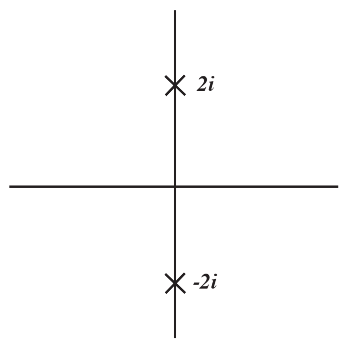


Figure 5.3: The function $f(z) = \frac{1}{z^2+4}$ has isolated singularities at $z = \pm 2i$.

(a) The function

$$f(z) = \frac{1}{z^2 + 4}$$

is analytic everywhere except for two isolated singgulaitries at $z = \pm 2i$.

(b) The function $g(z) = \text{Log}(z)$ is analytic everywhere except along the negative real axis. This branch cut is not isolated.

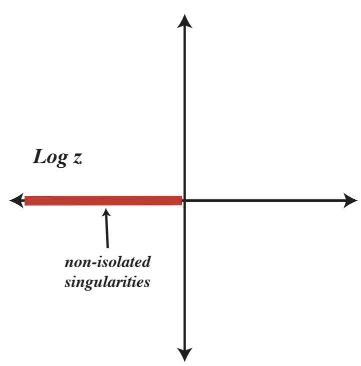


Figure 5.4: The function $g(z) = \text{Log}(z)$ has non-isolated singularities.

Big idea: There are only three kinds of isolated singularities.

Suppose $f(z)$ has an isolated singularity at $z = z_0$. The singularity must be:

(a) A **removable singularity**, which mean $\lim_{z \to z_0} f(z)$ exists.

(b) A **pole**, which means $\lim_{z \to z_0} |f(z)| = \infty$, but for some positive integer n and non-zero constant c, $\lim_{z \to z_0} (z - z_0)^n f(z) = c$. The number n is called the *order* of the pole.

(c) An **essential singularity**, if neither (a) or (b).

Example 5.3. The function

$$f(z) = \frac{1}{z}$$

has a pole at $z = 0$. To see this note that

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{1}{z}$$

does not exist because we get $+\infty$ if $z \to 0$ along the real positive axis, and we get $-\infty$ if $z \to 0$ along the negative real axis. But,

$$\lim_{z \to 0} z f(z) = \lim_{z \to 0} z \cdot \frac{1}{z} = 1$$

Therefore $f(z)$ has a *pole of order 1* at $z = 0$. ■

Example 5.4. The function

$$g(z) = \frac{z - 1}{z^2 - 1}$$

has isolated singularities at $z = \pm 1$. Note that

$$\lim_{z \to 1} g(z) = \lim_{z \to 1} \frac{z - 1}{(z - 1)(z + 1)} = \frac{1}{2}$$

so we can *remove* the singularity at $z = 1$ by defining

$$g(z) = \frac{1}{z + 1}$$

which only changes the value at $z = 1$ from undefined to $g(1) = 1/2$. Note that at $z = -1$

$$\lim_{z \to -1} (z+1)g(z) = \lim_{z \to -1} \frac{(z+1)(z-1)}{z^2-1} = \lim_{z \to -1} \frac{z^2-1}{z^2-1} = 1$$

which means $z = -1$ is a *pole of order one.* ■

In general, functions with a *removable singularity* at some point $z = z_0$ can be redefined at that point so the singularity goes away; that is the redefined function is now analytic at z_0.

Example 5.5. $h(z) = \dfrac{\sin z}{z}$ has a possible singularity at $z = 0$. But

$$\lim_{z \to 0} h(z) = \lim_{z \to 0} \frac{\sin z}{z} \overset{\text{l'H}}{=} \lim_{z \to 0} \frac{\cos z}{1} = 1$$

so we can define

$$h(z) = \begin{cases} 1 & \text{if } z = 0 \\ \dfrac{\sin z}{z} & \text{if } z \neq 0 \end{cases}$$

to clarify that the function is well-behaved at $z = 0$. Thus, this is another example of a removable singularity. ■

Example 5.6. The function $q(z) = e^{1/z}$ has an essential singularity at $z = 0$. To see this, we show that the function

$$\lim_{z \to 0} z^n f(z) = \lim_{z \to 0} z^n e^{1/z}$$

does not exist for n a non-negative integer. Consider

$$\begin{aligned}
\lim_{z\to 0}\left|z^n \exp\left(\frac{1}{z}\right)\right| &= \lim_{z\to 0}|z|^n\left|\exp\left(\frac{1}{z}\right)\right| \\
&= \lim_{r\to 0^+} r^n\left|\exp\left(\frac{1}{re^{i\theta}}\right)\right| \qquad \text{let } z = re^{i\theta} \\
&= \lim_{r\to 0^+} r^n\left|\exp\left(\frac{1}{r}e^{-i\theta}\right)\right| \\
&= \lim_{r\to 0^+} r^n\left|\exp\left(\frac{1}{r}\cos(\theta) - i\frac{1}{r}\sin(\theta)\right)\right| \\
&= \lim_{r\to 0^+} r^n\left|\exp\left(\frac{1}{r}\cos(\theta)\right)\underbrace{\exp\left(-\frac{i}{r}\sin(\theta)\right)}_{|e^{i\alpha}|=1 \text{ when } \alpha \text{ is real}}\right| \\
&= \lim_{r\to 0^+} r^n\left|\exp\left(\frac{1}{r}\cos\theta\right)\right| \\
&= \lim_{r\to 0^+} r^n\exp\left(\frac{\cos\theta}{r}\right) \\
&= \begin{cases} +\infty & \text{if } \cos\theta > 0 \\ 0 & \text{if } \cos\theta < 0 \end{cases}
\end{aligned}$$

so the limit doesn't exist and the isolated singularity must be essential. ■

5.3 Laurent series

The Laurent series generalizes the Taylor series by allowing for negative exponents.

$$\text{Taylor Series: } \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{Laurent Series: } \sum_{n=-k}^{\infty} a_n(z-z_0)^n$$

where $-k$ is a negative integer or possibly negative infinity. Functions with isolated singularities always have a Laurent Series.

Theorem 5.2 (Laurent Series)**.** *Suppose $f(z)$ is analytic in the punctured disc $0 < |z - z_0| < R$; that is $f(z)$ has an isolated singularity at $z = z_0$. Then $f(z)$ has a Laurent series at $z = z_0$,*

$$f(z) = \sum_{n=-k}^{\infty} a_n(z-z_0)^n$$

where $a_{-k} \neq 0$, which converges in the punctured disc. Moreover,

- *If $k = 0$ then $f(z)$ has a removable singularity,*
- *if k is a positive integer then $f(z)$ has a pole of order k, and*
- *if $k = -\infty$ then $f(z)$ has an essential singularity at $z = z_0$.*

Exercise 5.2. Suppose we have a function $f(z)$ with a pole at $z = z_0$. Previously we defined the *order* of a pole as a positive integer n such that $\lim_{z\to z_0}(z - z_0)^n f(z) = c$ for some non-zero constant c. Alternatively, we could define the order of a pole as the least nonnegative integer k such that

$$f(z) = \sum_{n=-k}^{\infty} a_n (z - z_0)^n$$

Show that these definitions are equivalent.

In general, the best way to find the Laurent series is to use the techniques we developed for Taylor series.

Exercise 5.3. Find the Laurent series for

$$h(z) = \frac{\sin z}{z}$$

at $z = 0$ and show it has a removable singularity.

Solution: Consider the Taylor expansion of $\sin z$,

$$\begin{aligned} &= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right] \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots \end{aligned}$$

which is valid for $z \neq 0$. We clearly see $\lim_{z\to 0} h(z) = 1$ is well-defined. Since there are no terms with negative exponents then this is a removable singularity.

Exercise 5.4. Show

$$f(z) = \frac{1}{z}$$

has a pole at $z = 0$.

Solution: The Laurent series expansion for $f(z)$ at $z = 0$ is just

$$f(z) = \frac{1}{z}$$

Since it has one term with a negative exponent of -1, it is *a pole of order one*.

Exercise 5.5. Show $q(z) = e^{1/z}$ has an essential singularity at $z = 0$.

Solution: we use the Taylor series expansion for the exponential function,

$$= 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

Since there are an infinite number of terms with negative exponents, it an *essential singularity*.

Exercise 5.6. Show that $a(z) = \dfrac{\cos z}{z^2}$ has an isolated singularity at $z = 0$.

Solution: The Taylor series expansion of $\cos z$ is useful here,

$$\begin{aligned} \frac{\cos z}{z^2} &= \frac{1}{z^2}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right] \\ &= \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \dots \end{aligned}$$

Since there are a finite number of terms with negative exponents, this is a pole, specifically a *a pole of order two*.

Example 5.7. Consider

$$b(z) = \frac{\sin z}{z^2(z - \pi)}$$

(a) Find locations of isolated singularities.

(b) Find the first three non-zero terms of the Laurent expansion for the function about each point.

(c) Determine what type of singularities the function has.

Solution:

(a) Note that $b(z)$ is analytic everywhere except possibly at $z = 0$ and $z = \pi$.

(b) <u>For $z = 0$</u>:

$$\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \dots$$

$$\frac{1}{z - \pi} = -\frac{1}{\pi} \cdot \frac{1}{1 - \frac{z}{\pi}} = -\frac{1}{\pi}\left[1 + \frac{z}{\pi} + \frac{z^2}{\pi^2} + \frac{z^3}{\pi^3} + \dots\right]$$

Combining these we see

$$\begin{aligned}\frac{\sin z}{z^2(z-\pi)} &= \frac{1}{z^2}\Big(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \dots\Big)\Big(-\frac{1}{\pi}\Big[1 + \frac{z}{\pi} + \frac{z^2}{\pi^2} + \dots\Big]\Big)\\ &= -\frac{1}{\pi}\left[\frac{1}{z} + \frac{1}{\pi} + z\left(\frac{1}{\pi^2} - \frac{1}{3!}\right) + \dots\right]\\ &= -\frac{1}{\pi}\frac{1}{z} - \frac{1}{\pi^2} + \left[\frac{1}{6\pi} - \frac{1}{\pi^3}\right] z \cdots\end{aligned}$$

<u>For $z = \pi$</u>:
Note that

$$\begin{aligned}\sin z &= \sin[(z-\pi)+\pi]\\ &= -\sin(z-\pi)\\ &= -(z-\pi) + \frac{1}{3!}(z-\pi)^3 - \frac{1}{5!}(z-\pi)^5 + \dots\end{aligned}$$

We use a trick to expand $1/z^2$ at $z = \pi$. First we compute that

$$\begin{aligned}\frac{1}{z} &= \frac{1}{\pi + (z-\pi)}\\ &= \frac{1}{\pi}\cdot\frac{1}{1 + \frac{(z-\pi)}{\pi}}\\ &= \frac{1}{\pi}\left[1 - \frac{z-\pi}{\pi} + \left(\frac{z-\pi}{\pi}\right)^2 - \left(\frac{z-\pi}{\pi}\right)^3 + \dots\right]\end{aligned}$$

Then, as

$$-\frac{d}{dz}\left(\frac{1}{z}\right) = \frac{1}{z^2}$$

we see that

$$\begin{aligned}\frac{1}{z^2} &= -\frac{d}{dz}\left(\frac{1}{z}\right)\\ &= -\frac{d}{dz}\left[\frac{1}{\pi} - \frac{z-\pi}{\pi^2} + \frac{(z-\pi)^2}{\pi^3} - \frac{(z-\pi)^3}{\pi^4} + \dots\right]\\ &= \left[\frac{1}{\pi^2} - \frac{2(z-\pi)}{\pi^2} + \frac{3(z-\pi)^2}{\pi^4} \cdots\right]\end{aligned}$$

Combining these we see

$$\begin{aligned}\frac{\sin z}{z^2(z-\pi)} &= \frac{1}{z-\pi}\left[-(z-\pi)+\frac{1}{3!}(z-\pi)^3\cdots\right]\cdot\left[\frac{1}{\pi^2}-\frac{2(z-\pi)}{\pi^2}+\frac{3(z-\pi)^2}{\pi^4}\cdots\right]\\ &= -\frac{1}{\pi^2}+\frac{2}{\pi^3}(z-\pi)+\left[\frac{1}{6\pi^2}-\frac{3}{\pi^4}\right](z-\pi)^2\cdots\end{aligned}$$

(c) For $z = 0$, we have one term with a negative exponent, so it is a pole of order 1. For $z = \pi$, there are no terms with negative exponents, so it is a removable singularity, and

$$\lim_{z\to\pi} b(z) = -\frac{1}{\pi^2}.$$

■

In the next lecture, we will show how to use the Laurent series to evaluate closed contour integrals for functions with only isolated singularities.

Six

The Residue Calculus

CHAPTER OUTLINE

- Evaluating Residues
- The Residue Theorem

The residue calculus is a tool for evaluating integrals around closed contour of functions which are analytic except for isolated singularities. One finds that the integral is a sum of contributions, called *residues*, from each singularity.

6.1 EVALUATING RESIDUES

We begin by defining the *residue* of an isolated singularity.

Definition 6.1. If $f(z)$ has an isolated singularity at $z = z_0$ then the *residue* of $f(z)$ is the coefficient of the $(z - z_0)^{-1}$ term in its Laurent series expansion at that point. Thai is, if the Laurent series at $z = z_0$ is

$$\sum_{n=-k}^{\infty} a_n (z - z_0)^n$$

then the residue of $f(z)$ at $z = z_0$ is

$$\text{Res}[f(z); z = z_0] \equiv a_{-1}$$

Let's do some exercises in identifying singularities and calculating residues.

Example 6.1. Identify the singularities and calculate the residues of the following functions:

(a) The function $a(z) = z^{-10}$ at $z = 0$.

Solution: This function is differentiable everywhere except at $z = 0$, where it is not defined. Therefore, it is analytic everywhere except at $z = 0$ where it has an isolated singularity. Note that

$$a(z) = \frac{1}{z^{10}} = \underbrace{\frac{1}{z^{10}} + \frac{0}{z^9} + \frac{0}{z^8} + \ldots + \frac{0}{z} + 0 + 0z + 0z^2 + \ldots}_{\text{series about } z=0}$$

If we look at the coefficient of the term with z^{-1}, we can see that the residue for the function is

$$\text{Res}\left[\frac{1}{z^{10}}; z = 0\right] = 0.$$

(b) The function $b(z) = \frac{1}{(z-2)}$ at $z = 2$.

Solution: the function $b(z)$ is analytic everywhere except at $z = 2$ it has a simple pole (i. e. a pole of order 1). It's Laurent series again has only one term and

$$\text{Res}\left[\frac{1}{z-2}; z = 2\right] = 1 .$$

(c) The function $c(z) = \sin(1/z)$ at $z = 0$.

Solution: Note $\sin(z)$ is entire, so $c(z)$ is analytic everywhere except at $z = 0$ where it has an essential singularity. Since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$$

we see

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} \ldots .$$

This function has an essential singularity at $z = 0$ because it has an infinite number of terms with negative exponents. The coefficient of the term with z^{-1} is 1, so the residue is

$$\text{Res}\left[\sin\left(\frac{1}{z}\right); z = 0\right] = 1$$

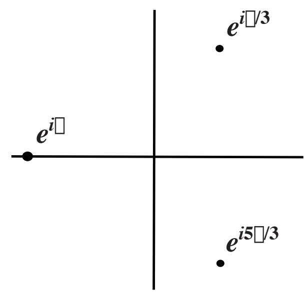


Figure 6.1: The singularities of $d(z) = \frac{1}{z^3+1}$ are three simple poles at $z = -1, e^{-i\pi/3}$, and $e^{i\pi/3}$.

(d) The function $d(z) = \frac{1}{z^3+1}$ has three isolated singularities. Find the residues at each singularity.

Solution: This function is analytic everywhere except where the denominator $z^3 + 1$ is zero. We begin by noting that

$$z^3 + 1 = 0 \Rightarrow z^3 = -1 = \cdots e^{-3i\pi}, e^{-i\pi}, e^{i\pi}, e^{3i\pi} \cdots$$

So that

$$z = \cdots e^{-i\pi}, e^{-i\pi/3}, e^{i\pi/3}, e^{i\pi} \cdots$$

from which we quickly deduce that there are three zeroes at $z = -1, e^{-i\pi/3}$, and $e^{i\pi/3}$. We can now factor the denominator as

$$z^3 + 1 = (z+1)(z - e^{-i\pi/3})(z - e^{i\pi/3})$$

and in fact, we will see that $d(z)$ has three simple poles at these locations.

Let's say we want the residue at $z = e^{i\pi/3}$. To analyze the function at $z = e^{i\pi/3}$, we note that

$$\frac{1}{z^3+1} = \frac{1}{(z - e^{i\pi/3}) \cdot (z - e^{-i\pi/3})(z+1)}$$

Let's define

$$R(z) = \frac{1}{(z - e^{-i\pi/3})(z+1)},$$

then

$$\frac{1}{z^3+1} = \frac{R(z)}{z-e^{i\pi/3}}$$

Observe that $R(z)$ is analytic at $z = e^{i\pi/3}$, so it has a Taylor series expansion there.

$$R(z) = R\left(e^{i\pi/3}\right) + R'\left(e^{i\pi/3}\right)\left(z-e^{i\pi/3}\right) + \frac{1}{2!}R''\left(e^{i\pi/3}\right)\left(z-e^{i\pi/3}\right)^2 + \dots$$

and $d(z)$ can now be expanded as

$$d(z) = \frac{R(z)}{z-e^{i\pi/3}} + R'\left(e^{i\pi/3}\right) + \frac{1}{2!}R''\left(e^{i\pi/3}\right)\left(z-e^{i\pi/3}\right) + \dots$$

And we now see that there is a simple pole (that is a pole of order one) at $z = e^{i\pi/3}$ and the associated residue is given by

$$\begin{aligned}
\text{Res}\left[d(z); z = e^{i\pi/3}\right] &= R\left(e^{i\pi/3}\right) \\
&= \frac{1}{\left(e^{i\pi/3}-e^{-i\pi/3}\right)\left(e^{i\pi/3}+1\right)} \\
&= \frac{1}{e^{i\pi/6}\left(e^{i\pi/3}-e^{-i\pi/3}\right)\left(e^{i\pi/6}+e^{-i\pi/6}\right)} \\
&= \frac{e^{-i\pi/6}}{(2i\sin(\pi/3))\,(2\cos(\pi/6))} \\
&= \frac{-ie^{-i\pi/6}}{\left(2\cdot(\sqrt{3}/2)\right)\left(2\cdot(\sqrt{3}/2)\right)} \\
&= \frac{-i}{3}e^{-i\pi/6} \\
&= \frac{-i}{3}[\cos(\pi/6) - i\sin(\pi/6)] \\
&= \frac{-i}{3}\left[\frac{\sqrt{3}}{2} - i\frac{1}{2}\right] \\
&= -\frac{1}{6} - i\frac{\sqrt{3}}{6}
\end{aligned}$$

where we used the identities

$$e^{iz} = \cos z + i\sin z \qquad e^{iz}+e^{-iz} = 2i\cos z \qquad e^{iz}-e^{-iz} = 2i\sin z.$$

■

Example 6.2. For the function

$$f(z) = \frac{\cos z}{z^2 + z^4}$$

locate the isolated singularities, characterize them and calculate their residues.

Solution: Note that $\cos z$ is an entire function, and the denominator can be factored as $z^2(z^2+1)$, so there are isolated singularities at $z = 0, \pm i$.

(a) At $z = 0$, the Laurent Series for $f(z)$ looks like

$$f(z) = \frac{1}{z^2}\left(\frac{\cos z}{1+z^2}\right) = \frac{1}{z^2}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right)\left(1 - z^2 + z^4 - \dots\right)$$
$$= \frac{1}{z^2} + \frac{0}{z} + \dots$$

So $f(z)$ has a 2^{nd} order pole at $z = 0$ & the residue there is 0.

(b) At $z = \pm i$, Laurent series for $f(z)$ looks like

$$f(z) = \frac{1}{z \mp i} \cdot \underbrace{\frac{\cos z}{z^2(z \pm i)}}_{\text{analytic @ } z=\pm i}$$

So $f(z)$ has a 1^{st}-order pole ("simple pole") at $z = \pm i$,

$$\text{Res} = \left[\frac{\cos z}{z^2(z \pm i)}\right]_{z=\pm i} = \frac{\cos \pm i}{(-1)(\pm 2i)} = \frac{\cosh 1}{\mp 2i}$$

■

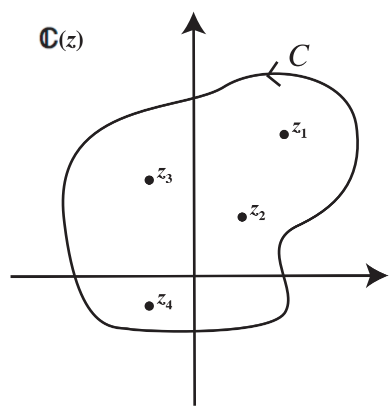
6.2 The Residue Theorem

The residue theorem allows us to calculate the integral around a closed curve of a function that is analytic except at a finite number of isolated singularities.

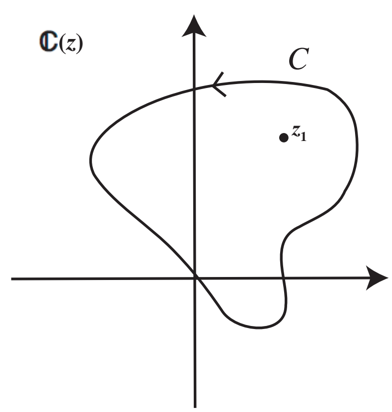
Theorem 6.1 (Residue Theorem)**.** *Let $f(z)$ be analytic in and on a simple, closed, positively-oriented curve C except for possibly finitely many isolated singularities inside C. Then,*

$$\oint_C f(z)dz = 2\pi i \sum_{n=1}^{N} Res[f; z = z_n],$$

where $z_1, z_2 \cdots z_N$ are the locations of the isolated singularities inside C.



Sketchy Proof: Suppose $f(z)$ is analytic on C except for one isolated singularity inside C, called z_1.



Expand $f(z)$ in a Laurent series about $z = z_1$.

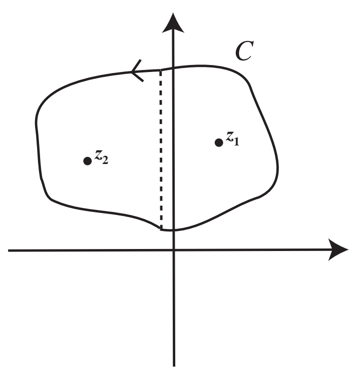
$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_1)^k$$

Then, as

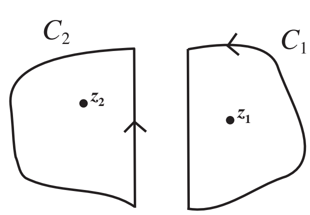
$$\oint_C (z - z_1)^k dz = \begin{cases} 0 & \text{if } k \in \mathbb{Z},\ k \neq -1 \\ 2\pi i & \text{if } k \in \mathbb{Z},\ k = -1 \end{cases}$$

we see that

$$\begin{aligned} \oint_C f(z)dz &= \oint_C \sum_{k=-\infty}^{\infty} a_k (z - z_1)^k dz \\ &= \sum_{k=-\infty}^{\infty} a_k \oint_C (z - z_1)^k dz \\ &= 2\pi i\ a_{-1} \\ &= 2\pi i\ \text{Res}[f; z = z_1] \end{aligned}$$



But what do we do if we have two singularities? We can break the contour into two closed loops, one which contains each singularity. Note the “up" and “down" portion of the contours cancel.



Now

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz = \text{Res}[f(z); z_1] + \text{Res}[f(z); z_2]$$

We will leave it as a challenge to you to think about how to generalize this to a functions with n singularities.

So to reiterate and present the theorem in a new way,

Residue Theorem: $\oint_C f(z)dz = 2\pi i \cdot \sum$ residues of f inside C

Example 6.3. Evaluate

$$(a) \quad I = \oint_C \frac{dz}{2z^2 + 5z + 2} \qquad (b) \quad J = \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$$

where C is the unit circle traversed in a counter-clockwise sense.

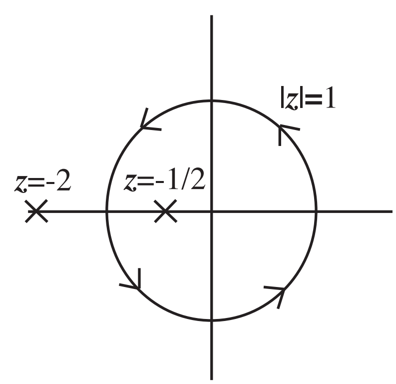


Figure 6.2: The contour of integration for Example 6.3; note one singularity is in the circle and the other is outside.

Solution: We factor the denominator

$$(2z^2 + 5z + 2) = (2z + 1)(z + 2)$$

so there are poles at $z = -\frac{1}{2}$ and $z = -2$. By the Residue Theorem

$$I = \oint_C \frac{dz}{2z^2 + 5z + 2} = 2\pi i \operatorname{Res}\left[\frac{1}{2z^2 + 5z + 2}; z = -\frac{1}{2}\right]$$

Because the pole at $z = -2$ is outside the contour of integration. But

$$\begin{aligned}
\frac{1}{2z^2 + 5z + 2} &= \frac{1}{(2z + 1)} \frac{1}{(z + 2)} \\
&= \frac{1}{2} \frac{1}{(z + \frac{1}{2})} \frac{1}{[(z + \frac{1}{2}) + \frac{3}{2}]} \\
&= \frac{1}{2} \frac{1}{(z + \frac{1}{2})} \left[\frac{2}{3} \cdot \frac{1}{1 + \frac{2}{3}\left(z + \frac{1}{2}\right)}\right] \\
&= \frac{1}{3} \frac{1}{(z + \frac{1}{2})} \left[1 - \frac{2}{3}\left(z + \frac{1}{2}\right) + \dots\right]
\end{aligned}$$

So

$$\operatorname{Res}\left[\frac{1}{2z^2 + 5z + 2}; z = -\frac{1}{2}\right] = \frac{1}{3}$$

and

$$I = \oint_C \frac{dz}{2z^2 + 5z + 2} = \frac{2\pi i}{3}$$

For part (b), let $z = e^{i\theta}$ where $0 \leq \theta \leq 2\pi$, so $dz = ie^{i\theta} d\theta$.

$$\begin{aligned}
I &= \int_0^{2\pi} \frac{ie^{i\theta} d\theta}{2\left(e^{i\theta}\right)^2 + 5e^{i\theta} + 2} \\
&= i \int_0^{2\pi} \frac{d\theta}{2\left(e^{i\theta}\right) + 5 + 2e^{-i\theta}} \\
&= i \int_0^{2\pi} \frac{d\theta}{5 + 2\left(e^{i\theta} + e^{-i\theta}\right)} \\
&= i \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}
\end{aligned}$$

So we see

$$I = iJ = \frac{2\pi i}{3} \Rightarrow J = \frac{2\pi}{3}$$

■

We demonstrate how to evaluate many more integrals in the next chapter.

Seven

Evaluation of Integrals

7.1 Evaluation of Integrals

Remember from the previous lecture:
Residue Theorem: (in a nutshell)

$$\boxed{\oint_C f(z)dz = 2\pi i \cdot \sum \text{residues of } f \text{ inside } C}$$

We will now use this to do some examples.

Example 7.1. Compute the integrals:

$$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\, dx\ ,$$
$$J = \int_{0}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx\ .$$

Solution: The integral I can be evaluated by relating it to an integral over a closed contour, where the closure is done through a semi-circular contour at"infinity." Define the function

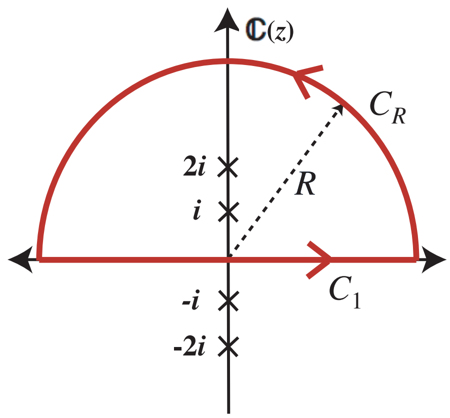
$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$$

and consider the integral

$$\oint_C f(z)\, dz$$

where C is a closed semi-circular contour in the upper half-plane.

Define the contour C,



where $C = C_1 \bigcup C_R$, C_1 is a portion of the real axis and C_R is a semi-circle of radius R.

Note that $f(z)$ has singularities at $\pm i, \pm 2i$ but only the ones at i and $2i$ are inside C. By the residue theorem,

$$\oint_C \frac{z^2}{(z^2+1)(z^2+4)} dz = 2\pi i \cdot (\text{Res}\,[f(z); z=i] + \text{Res}\,[f(z); z=2i])$$

The denominator $(z^2+1)(z^2+4) = (z-i)(z+i)(z-2i)(z+2i)$ has simple zeroes at $\pm i, \pm 2i$, so $f(z)$ has simple poles (poles of order one) at the same points. Evaluating the residues

$$\begin{aligned}
\text{Res}\,[f(z); z=i] &= \lim_{z\to i} \ (z-i)f(z) \\
&= \lim_{z\to i} \ \frac{z^2(z-i)}{(z-i)(z+i)(z^2+4)} \\
&= \lim_{z\to i} \ \frac{z^2}{(z+i)(z^2+4)} \\
&= \frac{(i)^2}{(i+i)((i)^2+4)} \\
&= \frac{-1}{(2i)(3)} \\
&= \frac{i}{6}
\end{aligned}$$

Likewise,

$$\begin{aligned}\text{Res}\left[\frac{z^2}{(z^2+1)(z^2+4)};\ z=2i\right] &= \lim_{z\to 2i}\ (z-2i)\cdot\frac{z^2}{(z^2+1)(z+2i)(z-2i)} \\ &= \left[\frac{z^2}{(z^2+1)(z+2i)}\right]_{z=2i} \\ &= -\frac{i}{3}\end{aligned}$$

$$\begin{aligned}\therefore \oint_C \frac{z^2}{(z^2+1)(z^2+4)}dz &= 2\pi i\cdot(\text{Res}\,[f(z);z=i]+\text{Res}\,[f(z);z=2i]) \\ &= 2\pi i\left(\frac{i}{6}-\frac{i}{3}\right)=\frac{\pi}{3}\end{aligned}$$

Now we need to relate this to I.

$$\oint_C = \int_{C_1} + \int_{C_R} \qquad \text{as } R\to\infty$$

Along C_1, parameterize as $z=x$, x goes from $-R$ to R.

$$\int_{C_1}\frac{z^2}{(z^2+1)(z^2+4)}\,dz = \int_{-R}^{R}\frac{x^2}{(x^2+1)(x^2+4)}\,dx$$

and as R tends to infinity, we see that

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{(x^2+1)(x^2+4)}\,dx = I$$

Along C_R we wish to show that the integral goes to zero as $R\to\infty$. That is

$$\boxed{\text{Claim:}\qquad \lim_{R\to\infty}\int_{C_R}\frac{z^2}{(z^2+1)(z^2+4)}\,dz=0}$$

we will establish this result below; for the moment, let's continue with our calculation.

Summarizing, we have established with the residue theorem that

$$\begin{aligned}\frac{\pi}{3} &= \oint_C \frac{z^2}{(z^2+1)(z^2+4)}\,dz \\ &= \int_{C_1}+\int_{C_R}\frac{z^2}{(z^2+1)(z^2+4)}\,dz\end{aligned}$$

and also

$$\lim_{R\to\infty} \int_{C_1} + \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\, dz = I + 0 \, .$$

Since the value of the integral around C is independent of R, we conclude that

$$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{\pi}{3} \, .$$

Notice this integrand is an even function, so

$$J = \frac{1}{2} \int_{0}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\, dx = \frac{1}{2} I = \frac{\pi}{6} \, .$$

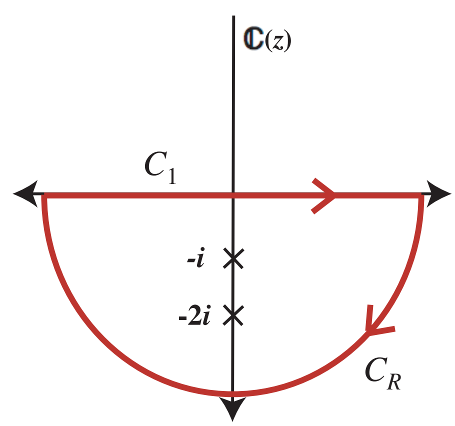
Salient features of this residue example:

(a) To use the residue calculus to evaluate a real integral, first think of a related integral over a closed path in the complex plane (because the Residue Theorem only applies to closed-path complex integrals).

(b) Evaluate the complex integral using the Residue Theorem. In other words, calculate residues for all the isolated singularities inside the closed path.

(c) Figure out how the real integral and the complex integral relate to each other. Often, you will have to use some sort of bounding argument to show that certain parts of the complex integral are negligible.

Note: In this problem, we could have used a different contour,



and we would get the same answer. The only difference in the method would be that we use the poles at $-i$ and $-2i$, and we take into consideration that this path is negatively oriented. ■ We now will

establish the result above that the integral along C_R vanishes as $R \to \infty$.

7.1.1 Establishing Bounds on Integrals

In the last example, we made a claim that:

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$$

for a particular function

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)} \, .$$

Here we will show that this is true for a large class of function $f(z)$, including the one above. The first result we will establish is

Lemma 7.1. *For a complex function, $f(z)$, and a contour C*

$$\left| \int_C f(z) dz \right| \le \max_{z\in C} |f(z)| \cdot [length\ of\ C]$$

assuming the integrals exist.

Proof: The triangle inequality for integrals states that

$$\left| \int_C f(z) dz \right| \le \int_C |f(z)|\, dz$$

which is basically the continuous version of the statement that: *The sum of the absolute values is less than the absolute value of the sum.* Also, clearly

$$|f(z)| \le \max_{z\in C} |f(z)|$$

for z on C. Combining these two results yields

$$\left| \int_C f(z) dz \right| \le \int_C |f(z)|\, dz \le \int_C \max_{z\in C} |f(z)|\, dz = \max_{z\in C} |f(z)| \cdot [\text{length of } C]$$

where the last statement follows because the integrand is constant.

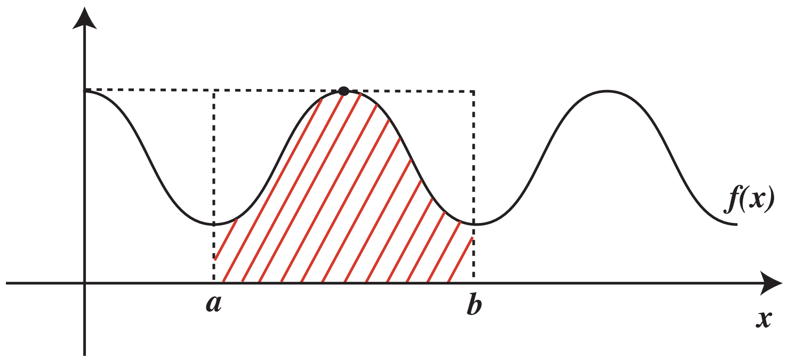


Figure 7.1: The real version of our lemma states that $\left|\int_a^b f(x)dx\right| \leq \max_{a\leq x\leq b}|f|\cdot|b-a|$ which is easy to see from the above picture.

Finally, we can use this to prove the theorem:

Theorem 7.1. *Suppose C_R is an arc of a circle of a circle of radius R centered at the origin and spanning an angle Δ and $f(z)$ is a continuous and differentiable function such that*

$$\lim_{R\to\infty} R|f(Re^{i\theta})| = 0$$

for any angle θ in the arc. Then

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0.$$

Proof: We use the Lemma above to bound the integral. Clearly

$$\left|\int_{C_R} f(z)\,dz\right| \leq \max_{z\in C}|f(z)|\cdot R\Delta$$

where Δ is the angle spanned by the arc c_R, so $R\Delta$ is its length. Now, if $f(z)$ is continuous and the arc is closed (that is contains its endpoints, then the

$$\lim_{R\to\infty} R|f(Re^{i\theta})| = 0$$

guarantees

$$\lim_{R\to\infty} R\cdot\max_{z\in C}|f(Re^{i\theta})| = 0.$$

For the analysts among you this follows from the fact that a differentiable function on a closed interval is uniformly continuous. Taking the limit as $R \to \infty$ in the inequality above proves the theorem.

In our example above we need to show that

$$\lim_{R\to\infty} R\left|\frac{z^2}{(z^2+1)(z^2+4)}\right| = \lim_{R\to\infty} \frac{R^3}{|R^4e^{4i\theta}+5R^2e^{2i\theta}+5|} = 0$$

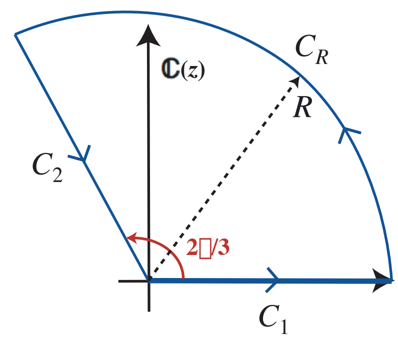
which is true as the R^4 dominates in the denominator at large R.

7.1.2 Two more examples

Example 7.2. Evaluate the integral:

$$A = \int_0^\infty \frac{1}{1+x^3}\,dx$$

Solution: It turns out one can evaluate this integral using standard means, but the residue calculus gives us a new way to evaluate this integral. Our first goal is relate the integral to one around a closed contour.



Let $C = C_1 \bigcup C_R \bigcup C_2$ and consider $B = \oint_C \frac{1}{z^3+1}dz$

The integrand $\frac{1}{z^3+1}$ has 3 poles at $e^{i\pi/3}, -1,\ e^{-i\pi/3}$ but only $e^{i\pi/3}$ is inside C. So, by the residue theorem

$$\begin{aligned} B = \oint_C \frac{1}{z^3+1}dz &= 2\pi i \cdot \text{Res}\left[\frac{1}{z^3+1}; e^{i\pi/3}\right] \\ &= 2\pi i \cdot \lim_{z\to e^{i\pi/3}} \frac{z-e^{i\pi/3}}{z^3+1} \\ &\overset{\text{l'H}}{=} 2\pi i \cdot \frac{1}{3(e^{i\pi/3})^2} \\ &= \frac{2\pi i}{3} e^{-2\pi i/3}\,. \end{aligned}$$

We need to figure out how B is related to A. Note

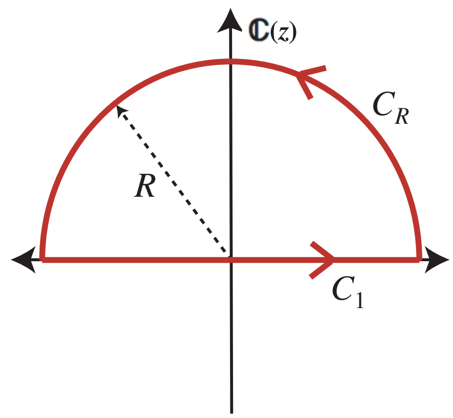
$$B = \oint_C \frac{1}{z^3+1}\,dz = \int_{C_1} \frac{1}{z^3+1}\,dz + \int_{C_R} \frac{1}{z^3+1}\,dz + \int_{C_2} \frac{1}{z^3+1}\,dz$$

We claim that as
$$\lim_{R\to\infty}\int_{C_R}\frac{1}{z^3+1}\,dz\to 0$$
which follows because
$$\lim_{R\to\infty} R\cdot\left|\frac{1}{(Re^{i\theta})^3+1}\right| = 0$$
as the R^3 dominates the denominator.
Parameterize C_1: Let $z=x$ where x goes from 0 to R. Then $dz=dx$ and
$$\int_{C_1}\frac{dz}{z^3+1}=\int_0^R\frac{dx}{x^3+1}\to A\quad\text{as } R\to\infty.$$
Parameterize C_2: Let $z=re^{i2\pi/3}$ so $dz=dr\;e^{i2\pi/3}$ and remember that r goes from R to 0, so
$$\begin{aligned}\int_{C_2}\frac{dz}{z^3+1} &= \int_R^0\frac{dr\,e^{2\pi i/3}}{\left(re^{i2\pi/3}\right)^3+1}\\ &= -e^{2\pi i/3}\int_0^R\frac{dr}{r^3e^{2\pi i}+1}\\ &= -e^{2\pi i/3}\int_0^R\frac{dr}{r^3+1}\to -e^{i2\pi/3}A \text{ as } R\to\infty\end{aligned}$$
Putting it all together, we see that
$$\begin{aligned}\frac{2\pi i}{3}e^{-2\pi i/3}=B &= \int_{C_1}+\int_{C_R}+\int_{C_2}\\ &\to A+0-e^{i2\pi/3}A\quad\text{as } R\to\infty\end{aligned}$$
So,
$$\begin{aligned}\frac{2\pi i}{3}e^{-2\pi i/3} &= A\left(1-e^{2\pi i/3}\right)\\ A &= \frac{2\pi i}{3}\frac{e^{-2\pi i/3}}{1-e^{2\pi i/3}}\\ &= \frac{2\pi i}{3}\frac{e^{-2\pi i/3}}{e^{\pi i/3}\left(e^{-i\pi/3}-e^{i\pi/3}\right)}\\ &= \frac{2\pi i}{3}\frac{e^{-i\pi}}{-2i\sin(\pi/3)}\\ &= \frac{2\pi i}{3}(-1)\frac{1}{-i\sqrt{3}}\\ &= \frac{2\pi\sqrt{3}}{9}\;.\end{aligned}$$

Therefore
$$\int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi\sqrt{3}}{9}$$
So from the residue theorem, we can calculate integrals by hand that we normally would not be able to do. ■

Example 7.3. Compute $Q = \displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}dx \quad (a>0)$

The first thing to try is $\displaystyle\oint_C \frac{\cos z}{z^2+a^2}dz$ for $C = C_1 \cup C_R$.



But this won't work because $\displaystyle\int_{C_R} \frac{\cos z}{z^2+a^2}dz \nrightarrow 0$ as $R \to \infty$.

$$\begin{aligned} |\cos z| = |\cos(x+iy)| &= |\cos x \cosh y - i \sin x \sinh y| \\ &= \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} \\ &\to \infty \quad \text{as} \quad y \to \infty \end{aligned}$$

Trick: Add zero to Q:

$$\text{Let} \quad Q = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} + i \underbrace{\int_{-\infty}^{\infty} \frac{\sin x}{x^2+a^2}dx}_{=0,\text{ because integrand is odd}}$$
$$= \int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{x^2+a^2}dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}dx$$

So consider $\displaystyle\oint_C \frac{e^{iz}}{z^2+a^2}dz = \int_{C_1} + \int_{C_R}$ as $R \to \infty$, $\displaystyle\int_{C_1} \to Q$.

How does $\int_{C_R}$ behave as $R \to \infty$?

$$\left|\int_{C_R} \frac{e^{iz}}{z^2+a^2}dz\right| \leq \underset{z\in C_R}{max} \left|\frac{e^{iz}}{z^2+a^2}\right| \cdot \text{ length of } C_R$$

Along $C_R,\ z = Re^{i\theta}$ for $\theta \in [0, \pi]$

$z = R\cos\theta + iR\sin\theta$

$$\begin{aligned}\left|\frac{e^{iz}}{z^2+a^2}\right| &= \left|\frac{\exp(iR\cos\theta - R\sin\theta)}{z^2+a^2}\right| \\ &= \frac{|\exp(iR\cos\theta)|\,|\exp(-R\sin\theta)|}{|z^2+a^2|} = \frac{e^{-R\sin\theta}}{|z^2+a^2|}\end{aligned}$$

$|\exp(iR\cos\theta)| = 1$, because $R, \cos\theta$ are real numbers, and

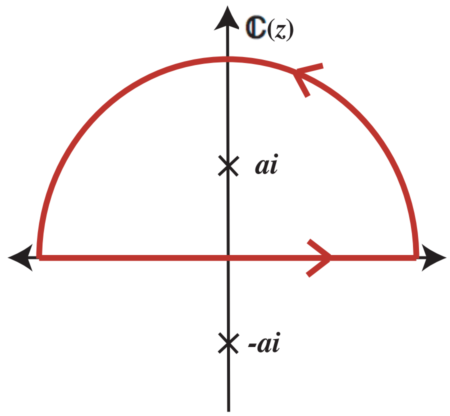
because $\sin\theta \geq 0$, $\dfrac{e^{-R\sin\theta}}{|z^2+a^2|} \to 0$ as $R \to \infty$.

so $\displaystyle\oint_C \frac{e^{iz}}{z^2+a^2}dz = \int_{C_1} + \int_{C_R}$
$\to Q + 0$ as $R \to \infty$.

Then, by the Residue Theorem,

$$Q = \oint_C \frac{e^{iz}}{z^2+a^2}dz = 2\pi i \sum \text{ Residues inside } C.$$

$\dfrac{e^{iz}}{z^2+a^2} = \dfrac{e^{iz}}{(z+ai)(z-ai)}$ has singularities at $z = \pm ai$.



So,

$$\begin{aligned}
Q &= 2\pi i \,\text{Res}\left(\frac{e^{iz}}{z^2+a^2}; z = ai\right) \\
&= 2\pi i \,\text{Res}\left(\underbrace{\frac{e^{iz}}{z+ai}}_{D(z)} \cdot \frac{1}{z-ai}; z = ai\right) \\
&= 2\pi i \left[\frac{e^{iz}}{z+ai}\right]_{z=ai} = 2\pi i \frac{e^{-a}}{2ai} \\
\therefore \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx &= \frac{\pi e^{-a}}{a} \qquad (a > 0)
\end{aligned}$$

■

Part III

Laplace Transforms

Eight

An Introduction to Laplace Transforms

CHAPTER OUTLINE

- Definition of the Laplace Transform
- Laplace Transforms of Derivatives
- First Shifting Theorem
- Inversion by Partial Fractions

The Laplace transform is a tool that is particularly useful for solving initial value problems for linear differential equations. It gives us tools for dealing with forcing that occurs as an impulse or that is switched on and off. The basic idea is that one looks at the differential equation in terms of a *transform variable*.

8.1 THE DEFINITION OF THE LAPLACE TRANSFORM

Definition 8.1. The Laplace transform of a function $f(t)$,

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)}$$

maps a function, $f(t)$, to a function of the *transform variable* s. By convention we will use lowercase letters to denote the origin function [$f(t)$] and uppercase to denote the transformed function [$F(s)$].

Note that the transform only depends the values of $f(t)$ for $t > 0$ and that $F(s)$ may only be defined for s sufficiently large. we will deal with this technical difficulties later; let's do some examples to get the hang of these transforms.

Example 8.1. Compute the following Laplace transforms of the following functions:

(a) The function $f(t) = 1$.
Solution:

$$\begin{aligned} \mathcal{L}\{1\} &= \int_0^\infty e^{-st} \cdot (1) dt = \int_0^\infty e^{-st} dt \\ &= -\frac{e^{-st}}{s}\bigg|_{t=0}^{t=\infty} \\ &= -\left[\frac{e^{-\infty}}{s} - \frac{e^0}{s}\right] = \frac{1}{s} \end{aligned}$$

which is valid for $s > 0$.

(b) The function $f(t) = t$.
Solution:

$$\begin{aligned} \mathcal{L}\{t\} &= \int_0^\infty e^{-st} \cdot (t) dt = \int_0^\infty t e^{-st} dt \\ &= -\frac{e^{-st}(st+1)}{s^2}\bigg|_{t=0}^{t=\infty} \\ &= -\left[\frac{e^{-\infty}}{s^2} - \frac{e^0}{s^2}\right] = \frac{1}{s^2} \end{aligned}$$

Note that again that this is defined when $s > 0$ for which

$$\lim_{t\to\infty} (st+1)e^{-st} = 0$$

because the exponential function decays much faster than the linear function grows.

(c) The function $f(t) = t^n$.
Solution:

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} \cdot (t^n) dt = \int_0^\infty t^n e^{-st} dt$$

Let $v = st$, to give

$$\mathcal{L}\{t^n\} = \frac{1}{s^{n+1}} \int_0^\infty v^n e^{-v} dv = \frac{n!}{s^{n+1}}$$

which again is valid for $s > 0$. we have used the fact that

$$\int_0^\infty v^n e^{-v}\, dv = n!$$

which can be proved via integration by parts and induction.

(d) The function $f(t) = e^{at}$ where a is a constant.
Solution:

$$\begin{aligned}
\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st} \cdot e^{at} dt \\
&= \int_0^\infty e^{-(s-a)t} dt = \frac{-1}{s-a}\, e^{-(s-a)t}\Big|_{t=0}^{t=\infty} \\
&= \frac{1}{-s+a}\left[e^{(-s+a)\infty} - e^{(-s+a)0}\right]
\end{aligned}$$

Only if $s > a$ does $e^{(-s+a)\infty}$ vanish, so we must restrict our answer

$$= \frac{1}{s-a} \qquad (\text{for } s > a)\ .$$

(e) The function $f(t) = \cos(at)$ where a is a constant.
Solution:

$$\begin{aligned}
\mathcal{L}\{\cos(at)\} &= \int_0^\infty e^{-st} \cdot \cos(at)\, dt \\
&= \text{Re}\left\{\int_0^\infty e^{-st} \cdot e^{iat}\, dt\right\} \\
&= \text{Re}\left\{\frac{1}{s-ia}\right\} = \text{Re}\left\{\frac{1}{s^2+a^2} \cdot (s+ia)\right\} \\
&= \frac{s}{s^2+a^2}
\end{aligned}$$

(f) The function $f(t) = \sin(at)$ where a is a constant.

Solution:

$$\begin{aligned}\mathcal{L}\{\sin(at)\} &= \int_0^\infty e^{-st} \cdot \sin(at)\, dt \\ &= \text{Im}\left\{\int_0^\infty e^{-st} \cdot e^{iat}\, dt\right\} \\ &= \text{Im}\left\{\frac{1}{s-ia}\right\} = \text{Im}\left\{\frac{1}{s^2+a^2} \cdot (s+ia)\right\} \\ &= \frac{a}{s^2+a^2}\end{aligned}$$

■

One can ask if every function $f(t)$ for $t > 0$ has a Laplace transform? The answer is no, but there is a very large class of functions for which the Laplace transform exist, namely those functions that grow no worse than exponentially.

Theorem 8.1. *Suppose that $f(t)$ is a piecewise continuous function such that*

$$|f(t)| \leq M e^{\beta t} \qquad \textit{for} \quad t > 0$$

for some real constants M and β. Then the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists in the half-plane Re$\{s\} > \beta$.

Remark. We've extended the Laplace transform $F(s)$ to a function of the complex variable s. In fact the much stronger statement that $F(s)$ is analytic in the half-plane R$e\{s\} > \beta$ is true!!

Proof. As the Laplace transform is defined as

$$\mathcal{L}\{f(t)\} \equiv \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

It suffices to show that the function $e^{-st}f(t)$ is absolutely integrable for R$e\{s\} > \beta$. Think of s as a complex variable, $s = p + iq$ for p and q real that

$$\begin{aligned}|F(s)| &\leq \int_0^\infty |e^{-st} f(t)|\, dt \\ &\leq M \int_0^\infty |e^{\beta t - (p+iq)t}|\, dt \\ &= M \int_0^\infty e^{(\beta - p)t}\, dt \\ &= \frac{M}{p-\beta} \text{ for } p > \beta.\end{aligned}$$

which shows that the integrand $e^{-st}f(t)$ is absolutely integrable in the right half-plane where Re$\{s\} = p > \beta$. If the function is absolutely integrable, then the function is integrable also (that is the infinite integral converges to a finite value) in this half plane also. □

A little more work can show that $F(s)$ is an analytic function in this region also, and in particular has no singularities in this region.

We can also derive properties of the Laplace transform that help expand the number of functions whose transforms we can find. The most important is *linearity*.

Example 8.2. *Linearity:* Show the Laplace Transform is a *linear operator*, that is

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}\ .$$

Solution: The proof follows directly from the definition,

$$\begin{aligned}\mathcal{L}\{af(t) + bg(t)\} &= \int_0^\infty e^{-st} \cdot [af(t) + bg(t)]dt \\ &= a\int_0^\infty e^{-st}f(t)\ dt + b\int_0^\infty e^{-st}g(t)\ dt \\ &= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}\ .\end{aligned}$$

Hence we've proved the linearity of the Laplace transform. ■

In practice one constructs a table of common Laplace transforms and properties of Laplace transforms and looks up transforms as opposed to re-evaluating them from definition.

8.2 Laplace transforms of derivatives

We can compute the Laplace transform of the derivative of a function in terms of the Laplace transform of the function.

Theorem 8.2. *Suppose the Laplace transform of $y(t)$ and $y'(t)$ exist. If $\mathcal{L}\{y(t)\} = \mathbb{Y}(s)$, then*

$$\mathcal{L}\{y'(t)\} = s\mathbb{Y}(s) - y(0).$$

Proof. We use integration by parts:

$$\mathcal{L}\{y'(t)\} = \int_0^\infty e^{-st}y'(t)\ dt.$$

Let

$$\begin{aligned} y(t) &= u & y'(t)dt &= du \\ e^{-st} &= v & -se^{-st}\,dt &= dv. \end{aligned}$$

So

$$\mathcal{L}\{y'(t)\} = y(t)e^{-st}\Big[_{t=0}^{\infty} - \int_0^{\infty} (-s)e^{-st}y(t)\,dt.$$

As $y(t)$ is integrable, we assume that $y(t)e^{-st} \to 0$ as $t \to \infty$ and the first term becomes $-y(0)$ the second term we identify as $s\mathbb{Y}(s)$ from which we conclude that

$$\mathcal{L}\{y'(t)\} = s\mathbb{Y}(s) - y(0)$$

as desired. □

This can be very useful, especially when it comes to ordinary differential equations.

Example 8.3. Use the derivative property of the Laplace transform to solve the following ODE

$$y' + y = 1, \qquad y(0) = 0.$$

Solution: We begin by transforming both sides of the differential equation,

$$\begin{aligned} \mathcal{L}\{y' + y = 1\} &\Rightarrow \mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{1\} \\ &\Rightarrow s\mathbb{Y} + \underbrace{y(0)}_{=0} + \mathbb{Y} = \frac{1}{s} \\ &\Rightarrow (s+1)\mathbb{Y} = \frac{1}{s} \\ &\Rightarrow \mathbb{Y} = \frac{1}{s(s+1)} \qquad \text{(apply partial fractions)} \\ &\Rightarrow \mathbb{Y} = \frac{1}{s} - \frac{1}{s+1} \end{aligned}$$

we now need to apply an inverse Laplace transform to get a function in terms of t. From our examples earlier, we can see that the inverse is

$$\begin{aligned} \mathcal{L}^{-1}\{\mathbb{Y}\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ y(t) &= 1 - e^{-t} \end{aligned}$$

So we can see that the Laplace transform is a very useful tool in evaluating ordinary differential equations. ■

The observant reader will ask if perhaps we have pulled a quick-one here. In particular, one should ask:

> *Is the Laplace transform of a function unique?*

The answer is yes, if we restrict the type of functions we are look at. For example if $f(t)$ and $g(t)$ are continuous and have the same Laplace transform, then $f(t) = g(t)$ for $t > 0$. We can even say that if $f(t)$ and $g(t)$ are both continuous at $t = t_0 > 0$ and have the same Laplace transform, then $f(t_0) = g(t_0)$ for $t > 0$.

Consequently, if we know the Laplace transform of a continuous function (like the solution to most differential equations), then we can match it up with the appropriate transform in our table and find the inverse to obtain a unique of t. We say this symbolically by writing $\mathcal{L}^{-1}\{\mathbb{Y}(s)\} = y(t)$.

8.2.1 Higher derivatives

What about second order derivatives? We can use the first derivative to derive the formula,

$$\begin{aligned}\mathcal{L}\{y''(t)\} &= \mathcal{L}\{(y'(t))'\} = s\mathcal{L}\{y'(t)\} - y'(0)\\ &= s[s\mathbb{Y}(s) - y(0)] - y'(0)\\ &= s^2\mathbb{Y}(s) - sy(0) - y'(0)\ .\end{aligned}$$

We can continue this process for higher-order derivatives,

$$\begin{aligned}\mathcal{L}\{y^{(n)}(t)\} &= \mathcal{L}\{\left(y^{(n-1)}(t)\right)'\} = s\mathcal{L}\{y^{(n-1)}(t)\} - y^{(n-1)}(0)\\ &= s[s\mathcal{L}\{\left(y^{(n-2)}(t)\right)'\} - y^{(n-1)}(0)] - y^{(n-2)}(0)\\ &\vdots\\ &= s^n\mathbb{Y}(s) - \sum_{k=1}^{n} s^{k-1} f^{(n-k)}(0)\ .\end{aligned}$$

So we can use the Laplace transform to evaluate any order derivative.

Example 8.4. Using the Laplace transform, solve the following ODE.

$$y'' + \omega^2 y = 0 \qquad y(0) = A \quad y'(0) = B$$

where A and B are constants.

$$
\begin{aligned}
\mathcal{L}\{y''(t)+\omega^2 y(t)=0\} &\to \mathcal{L}\{y''(t)\}+\omega^2\mathcal{L}\{y(t)\}=\mathcal{L}\{0\}\\
&\to s^2\mathbb{Y}(s)-s\underbrace{y(0)}_{=A}-\underbrace{y'(0)}_{=B}+\omega^2\mathbb{Y}(s)\\
&\to (s^2+\omega^2)\mathbb{Y}(s)-sA-B=0\\
&\to \mathbb{Y}(s)=A\frac{s}{s^2+\omega^2}+B\frac{1}{s^2+\omega^2}\\
&\to \mathbb{Y}(s)=A\frac{s}{s^2+\omega^2}+\frac{B}{\omega}\frac{\omega}{s^2+\omega^2}\\
\mathcal{L}^{-1}\{\mathbb{Y}(s)\} &= A\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\}+\frac{B}{\omega}\,\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\}\\
y(t) &= A\cos(\omega t)+\frac{B}{\omega}\sin(\omega t)
\end{aligned}
$$

■

Again, we are assuming the Laplace transform has a unique inverse. If we know a function in terms of s, then we can match it up with the appropriate transform in our table and move the opposite way to obtain a function in terms of t.

8.3 First Shifting Theorem

It turns out that the Laplace transform of an exponential times a functions shifts the transform by a constant amount.

Theorem 8.3. *Suppose* $\mathcal{L}\{f(t)\}=F(s)$, *then*

$$\mathcal{L}\{e^{-at}f(t)\}=F(s+a).$$

This is known as the *First Shifting Theorem*; we'll talk about the Second Shifting Theorem in the next chapter.

Proof. The theorem follows quickly from the definition of the Laplace transform,

$$\int_0^\infty e^{-st}e^{-at}f(t)dt=\int_0^\infty e^{-(s+a)t}dt=F(s+a)\ .$$

□

This observation proves useful for solving certain ODE problems.

Example 8.5. Solve the following ODE:

$$y'(t) + y(t) = e^{-bt} \qquad y(0) = 0$$

Make sure you consider both cases when $b \neq 1$ and $b = 1$.
Solution: Laplace transform both sides of the ODE

$$\mathcal{L}\{y'(t)\} + \mathcal{L}\{y(t)\} = s\mathbb{Y}(s) + \underbrace{y(0)}_{=0} + \mathbb{Y}(s) = \frac{1}{s+b}$$

Now solve for $\mathbb{Y}(s)$

$$\mathbb{Y}(s) = \frac{1}{(s+1)(s+b)} \qquad b \neq 1$$

If $b \neq 1$, we can use partial fractions

$$\begin{aligned} \mathbb{Y}(s) &= \frac{1}{b-1}\left[\frac{1}{s+1} - \frac{1}{s+b}\right] \\ \mathcal{L}^{-1}\{\mathbb{Y}(s)\} &= \frac{1}{b-1}\left[\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+b}\right\}\right] \\ y(t) &= \frac{1}{b-1}\left[e^{-t} - e^{-bt}\right] \end{aligned}$$

If $b = 1$, we use the first shifting theorem

$$\mathbb{Y}(s) = \frac{1}{(s+1)^2}$$

we can find the inverse transform by using the shifting theorem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} \\ &= e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \\ &= te^{-t} \end{aligned}$$

■

Example 8.6. Solve the following ODE:

$$y''(t) + 2y'(t) + 2y(t) = 0 \qquad y(0) = 1 \quad y'(0) = 0$$

Solution: Laplace Transform the ODE,

$$\begin{aligned} \mathcal{L}\{y''(t)\} + 2\mathcal{L}\{y'(t)\} + 2\mathcal{L}\{y(t)\} &= \mathcal{L}\{0\} \\ s^2\mathbb{Y}(s) - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0} + 2s\mathbb{Y}(s) - 2\underbrace{y(0)}_{=1} + 2Y(s) &= 0 \\ s^2\mathbb{Y}(s) - s + 2s\mathbb{Y}(s) - 2 + 2\mathbb{Y}(s) &= 0 \\ (s^2 + 2s + 2)\mathbb{Y}(s) &= s + 2 \end{aligned}$$

Solving for $\mathbb{Y}(s)$ yields

$$\begin{aligned} \mathbb{Y}(s) &= \frac{s+2}{s^2+2s+2} \\ &= \frac{s+2}{(s+1)^2+1} \\ &= \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} \end{aligned}$$

Again, we use the first shifting theorem,

$$\begin{aligned} &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} \\ &= e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ y(t) &= e^{-t}\cos t + e^{-t}\sin t \end{aligned}$$

■

8.4 Inversion by Partial Fractions

We can use some of our complex variable techniques to find partial fraction expansions.

Theorem 8.4. *Suppose $F(s) = q(s)/p(s)$ where p and q are polynomials, the degree of q is less than the degree of p, and p has distinct roots $s_1, s_2, \ldots, s_n$, then*

$$F(s) = \frac{q(s)}{p(s)} = \frac{a_1}{s-s_1} + \frac{a_2}{s-s_2} + \cdots + \frac{a_n}{s-s_n}$$

where $a_j = Res[F(s); s_j] = \lim_{s \to s_j} (s - s_j) \dfrac{q(s)}{p(s)}$.

Proof. For example, to compute the value of a_1, multiply by $s - s_1$

$$(s - s_1)F(s) = a_1 + a_2 \frac{(s - s_1)}{(s - s_2)} + \cdots + a_n \frac{(s - s_1)}{(s - s_n)}$$

as $s \to s_1$

$$\lim_{s \to s_1} (s - s_1)F(s) = \lim_{s \to s_1} \frac{q(s)}{p(s)} = a_1$$

and the result follows. □

Example 8.7. Compute the following inverse transforms using partial fractions:

(a) The function

$$\mathbb{Y}(s) = \frac{1}{(s+1)(s+2)}.$$

Solution: Using partial fractions, we see that

$$\frac{1}{(s+1)(s+2)} = \frac{a_1}{s+1} + \frac{a_2}{s+2}$$

and since a_1 and a_2 are the residues at $s = 1$ and 2 respectively,

$$a_1 = \frac{1}{-1+2} = 1 \quad a_2 = \frac{1}{-2+1} = -1$$

so

$$\mathcal{L}^{-1}\{\mathbb{Y}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} = e^{-t} - e^{-2t}$$

(b) The function

$$\mathbb{Y}(s) = \frac{1}{(s+1)(s^2+1)}.$$

Solution: Using partial fractions, we see that

$$\frac{1}{(s+1)(s^2+1)} = \frac{a_1}{s+2} + \frac{a_2}{s+i} + \frac{a_3}{s-i}$$

and again using residue methods,

$$\begin{aligned} a_1 &= \frac{1}{5} \\ a_2 &= \lim_{s\to -i} \frac{s+i}{(s+2)(s^2+1)} = \frac{1}{(2-i)(-2i)} = \frac{-1}{2+4i} \\ &= \frac{4i-2}{20} = -\frac{1}{10} + \frac{i}{5} \\ a_3 &= \lim_{s\to i} \frac{s-i}{(s+2)(s^2+1)} = -\frac{1}{10} - \frac{i}{5} \end{aligned}$$

so

$$\begin{aligned} \mathbb{Y}(s) &= \frac{1}{10}\left[\frac{2}{s+2} + \frac{-1+2i}{s+i} + \frac{-1-2i}{s-i}\right] \\ &= \frac{1}{10}\left[\frac{2}{s+2} + \frac{4-2s}{s^2+1}\right] \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}^{-1}\{\mathbb{Y}(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{5}\frac{1}{(s+2)} + \frac{1}{5}\frac{2-s}{(s^2+1)}\right\} \\ &= \frac{1}{5}e^{-2t} - \frac{1}{5}\cos t + \frac{2}{5}\sin t \end{aligned}$$

(c) The function

$$\mathbb{Y}(s) = \frac{1}{(s+1)^2}\frac{1}{(s+2)}$$

Solution: Again, using partial fractions, we see that

$$\frac{1}{(s+1)^2}\frac{1}{(s+2)} = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s+2} \ ,$$

and cross multiplying implies that

$$1 = A(s+2) + B(s+1)(s+2) + C(s+1)^2 \ .$$

Substituting $s=-2$ implies $C=1$ and substituting $s=-1$ implies $A=1$. Also, from the coefficient of s^2, we see that $B+C=0$ which implies $B=-1$. This yields the partial fraction expansion

$$\frac{1}{(s+1)^2(s+2)} = \frac{1}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s+2} \ .$$

Taking the inverse transform yields

$$\begin{aligned}\mathcal{L}^{-1}\{\mathbb{Y}(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s+2}\right\} \\ &= te^{-t} - e^{-t} + e^{-2t} \; .\end{aligned}$$

■

We end with an example where we use partial fractions to solve an initial value problem for inhomogeneous second-order ODE.

Example 8.8. Solve the ODE:

$$y''(t) + 2y'(t) = 8t \; , \qquad y(0) = y'(0) = 0 \; .$$

Solution: We transform the ODE to yield

$$\begin{aligned}\mathcal{L}\{y''(t)\} + 2\mathcal{L}\{y'(t)\} &= 8\mathcal{L}\{t\} \; , \\ s^2\mathbb{Y}(s) - sy(0) - y'(0) + 2s\mathbb{Y}(s) + 2y(0) &= \frac{8}{s^2} \; .\end{aligned}$$

Applying the initial conditions $y(0) = y'(0) = 0$ and solving for $\mathbb{Y}(s)$ we find that

$$\mathbb{Y}(s) = \frac{8}{s^3(s+2)} \; ,$$

which we will invert using partial fractions. The proper expansion is

$$\frac{8}{s^3(s+2)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+2}$$

and cross-multiplying yields

$$8 = A(s+2) + Bs(s+2) + Cs^2(s+2) + Ds^3 \; .$$

Substituting $s = -2$ yields $D = -1$, while $s = 0$ yields $A = 4$. Substituting these results and expanding yields

$$\begin{aligned}8 &= 4(s+2) + Bs(s+2) + Cs^2(s+2) - s^3 \; , \\ 8 &= 8 + (4+2B)s + (B+2C)s^2 + (C-1)s^3\end{aligned}$$

from which we quickly see that $B = -2$ and $C = 1$.

Putting it all together

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\mathbb{Y}(s)\right\} \ , \\ &= \mathcal{L}^{-1}\left\{\frac{4}{s^3} - \frac{2}{s^2} + \frac{1}{s} - \frac{1}{s+2}\right\} \ , \\ &= 2t^2 - 2t + 1 - e^{-2t} \ . \end{aligned}$$

So

$$\boxed{y(t) = 2t^2 - 2t + 1 - e^{-2t} \ .}$$

■

Nine

Heaviside Functions and the Second Shifting Theorem

CHAPTER OUTLINE

- Heaviside Functions
- Second Shifting Theorem
- An example of a DE with a switched input.

9.1 HEAVISIDE FUNCTIONS

Remember that the Heaviside function is defined as

$$H(t-t_0) = \begin{cases} 0 & t < t_0 \ , \\ 1 & t > t_0 \ . \end{cases}$$

The Heaviside function is essentially a switch; it turns on when $t = t_0$. We can compute its Laplace transform; since $H(t-t_0) = 0$ for $t < t_0$, we can just evaluate the integral from t_0 to ∞.

$$\mathcal{L}\{H(t-t_0)\} = \int_{t_0}^{\infty} e^{-st} dt = \left. \frac{e^{-st}}{s} \right|_{t=t_0}^{t=\infty} = \frac{e^{-st_0}}{s}, \qquad t_0 > 0 \ .$$

If we have a function $f(t)$ defined for $t \geq 0$, we can use the Heaviside function to create a new function whose start time is “delayed" to a time t_0,

Example 9.1. Suppose $f(t) = t^2$ for $t > 0$. Define a function that vanishes for $t < t_0$ and looks like $f(t)$ delayed to a start time of t_0.
Solution: Let

$$g(t) = \underbrace{H(t - t_0)f(t - t_0)}_{f(t) \text{ delayed until } t_0}$$

then

$$g(t) = \begin{cases} 0, & t < t_0 \\ (t - t_0)^2, & t > t_0 \end{cases}$$

■

We can compute the Laplace transform of this shifted function in terms of the Laplace transform of $f(t)$; this leads us to our second shifting theorem for Laplace Transforms.

9.2 Second Shifting Theorem

The first shifting theorem showed that multiplying a function by e^{-at} shifted the Laplace transform by a distance a in the transform variable s. The second shifting theorem shows that shifting the original function by a distance t_0 multiplies the Laplace transform by an exponential; the result has a nice sort of symmetry to it.

Theorem 9.1 (Second Shifting)**.** *If* $t_0 > 0$ *and* $\mathcal{L}\{y(t)\} = \mathbb{Y}(s)$ *then*

$$\mathcal{L}\{H(t - t_0)y(t - t_0)\} = e^{-st_0}\mathbb{Y}(s).$$

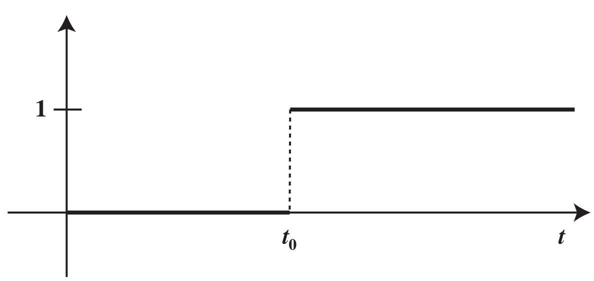


Figure 9.1: A graph of the Heaviside function, $H(t - t_0)$.

Proof. A simple calculation shows that:

$$\begin{aligned}
\mathcal{L}\{H(t-t_0)y(t-t_0)\} &= \int_{t_0}^{\infty} e^{-st} y(t-t_0)dt \\
t' = t - t_0, &\qquad dt' = dt \\
&= \int_0^{\infty} e^{-s(t'+t_0)} y(t')dt' \\
&= e^{-st_0} \int_0^{\infty} e^{-st'} y(t')dt' = e^{-st_0}\mathbb{Y}(s)
\end{aligned}$$

as advertised above. □

Example 9.2. Let us compute the Laplace transform of the shifted function in Example 1.

$$\mathcal{L}\{t^2\} = \frac{1}{s^2} \quad \Rightarrow \quad \mathcal{L}\{H(t-t_0)(t-t_0)^2\} = \frac{e^{-st_0}}{s^2}, \quad t_0 > 0.$$

■

Example 9.3. Compute the inverse Laplace transform of $\frac{e^{-s}}{s^2+4}$ We know that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$$

So by the shifting theorem

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+4}\right\} = \underbrace{\frac{1}{2}\sin 2(t-1)}_{f(t) \text{ delayed until } t=1} \cdot H(t-1)$$

■

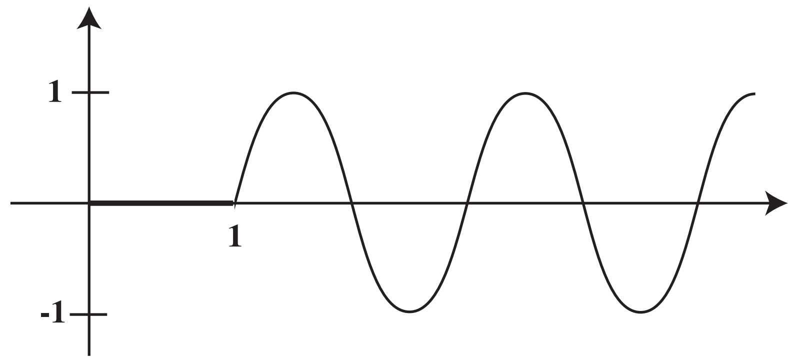


Figure 9.2: The sine wave $f(t)$ is "delayed" until $t = 1$

9.3 Response to a switched input

We can use the Heaviside function to describe an input that is switched on and off in a differential equation. This is one of the reasons the function is a favorite of electrical and systems engineers.

Exercise 9.1. Drug delivery
A patient receives morphine through an IV at the rate of $Q(t)$ ml/hr where

$$Q(t) = \begin{cases} 1 & 1 < t < 2 \\ 0 & 0 < t < 1 \text{ or } t > 2 \end{cases}$$

He excretes the drug at a rate proportional to the amount of morphine in the body. If the patient is initally drug-free, model the level of morphine in his body as a function of time.

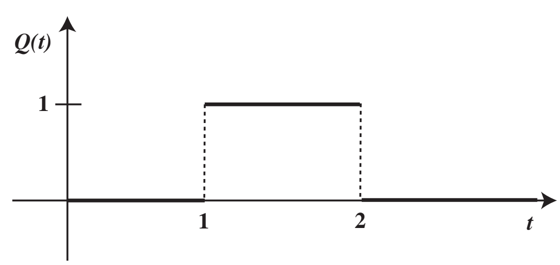


Figure 9.3: A graph of the morphine delivery rate, $Q(t)$.

Solution: Let $y(t) =$ amount of morphine in patient (ml) at time t. Then we can model the injection and excretion of morphine,

$$y'(t) = \underbrace{Q(t)}_{\text{input}} - \underbrace{ky(t)}_{\text{excretion}}, \qquad y(0) = 0$$

where k is the excretion rate in (hr)$^{-1}$ - remember that rate constants always have units of inverse time. Since the IV injection rate is $Q(t) = H(t-1) - H(t-2)$, we see that the governing equation can be written as

$$y' + ky = H(t-1) - H(t-2), \qquad y(0) = 0$$

which we can solve via a Laplace transform,

$$\begin{aligned}
\mathcal{L}\{y' + ky\} &= \mathcal{L}\{H(t-1) - H(t-2)\} \\
\mathcal{L}\{y'\} + k\mathcal{L}\{y\} &= \mathcal{L}\{H(t-1)\} - \mathcal{L}\{H(t-2)\} \\
s\mathbb{Y}(s) - \underbrace{y(0)}_{=0} + k\mathbb{Y}(s) &= \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \\
(s+k)\mathbb{Y}(s) &= \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \\
\mathbb{Y}(s) &= \frac{e^{-s}}{s(s+k)} - \frac{e^{-2s}}{s(s+k)}
\end{aligned}$$

We need to invert the transform to find $y(t)$, but first we need to decompose the function $1/s(s+k)$ via partial fractions,

$$\begin{aligned}
\frac{1}{s(s+k)} &= \frac{A}{s} + \frac{B}{s+k} \\
1 &= A(s+k) + Bs \\
Ak = 1 \quad \Rightarrow A &= \frac{1}{k} \qquad A = -B \quad \Rightarrow B = -\frac{1}{k} \\
\therefore \frac{1}{s(s+k)} &= \frac{1}{k}\left[\frac{1}{s} - \frac{1}{s+k}\right]
\end{aligned}$$

Now we can take the inverse transform pretty easily.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s(s+k)}\right\} &= \frac{1}{k}\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+k}\right\} \\
&= \frac{1}{k}\left(1 - e^{-kt}\right)
\end{aligned}$$

So

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+k)}\right\} = H(t-1)\left[\frac{1}{k}\left(1-e^{-k(t-1)}\right)\right]$$
$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+k)}\right\} = H(t-2)\left[\frac{1}{k}\left(1-e^{-k(t-2)}\right)\right]$$

and putting it all together yields

$$y(t) = H(t-1)\left(\frac{1-e^{-k(t-1)}}{k}\right) - H(t-2)\left(\frac{1-e^{-k(t-2)}}{k}\right) .$$

To graph $y(t)$, first graph $\left(1-e^{-kt}\right)/k$:

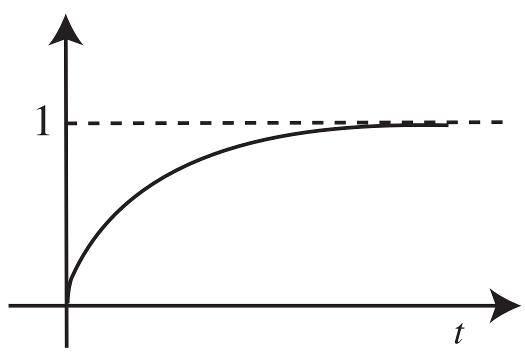
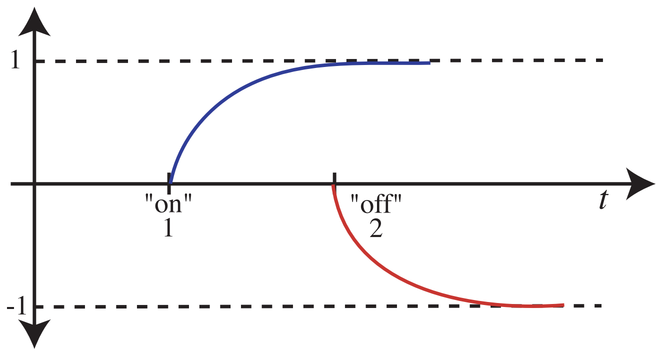


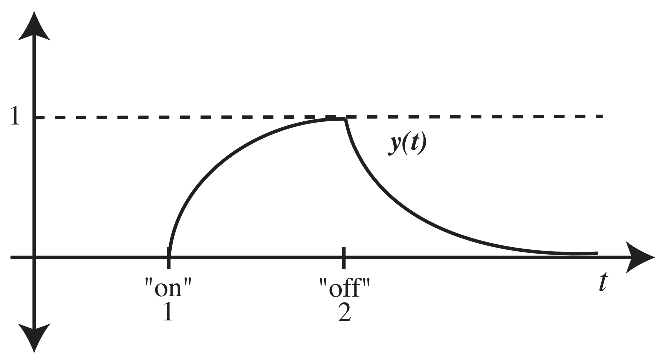
Figure 9.4: A graph of $\left(1-e^{-kt}\right)/k$ which would be the response to a constant unit input rate of morphine to the body.

This would be the response of the body to a constant drip, $Q(t) = 1$. The function approaches $1/k$ at large times which can be understood at the concentration where the flux in equals the rate excreted, that is $Q(t) = ky(t)$.

Now graph the function delayed one and two units in time:



The blue line is the graph of $H(t-1)[1-e^{-k(t-1)}]/k$ and the red line is the graph of $-H(t-2)[1-e^{-k(t-2)}]/k$. Now, graph the sum of the two functions: You can see that the function increases towards $1/k$ and then



decreases down to zero again.

Ten

Convolutions and Delta Functions

Chapter Outline

- Convolution Theorem
- Borel's Theorem
- Delta Functions
- Transfer Functions and Green's functions

10.1 Convolution Theorem

Suppose we wish to solve a differential equation of the form

$$y' + y = f(t) \qquad y(0) = 0 \qquad (f(t) \text{ is given})$$

If we apply a Laplace transform to the equation, we have

$$(s+1)\mathbb{Y}(s) = F(s)$$

where

$$\mathcal{L}\{y(t)\} = \mathbb{Y}(s), \qquad \mathcal{L}\{f(t)\} = F(s).$$

Solving for $\mathbb{Y}(s)$ we see that

$$\mathbb{Y}(s) = \underbrace{\frac{1}{s+1}}_{\text{transfer function}=G(s)} F(s)$$

or

$$y(s) = \mathcal{L}^{-1}\{G(s)F(s)\}$$

where $G(s)$ is the *transfer function* for this ODE; here $G(s) = \dfrac{1}{s+1}$.

It would be very useful to have a formula for the inverse transform of the product of two functions. In fact, such a formula exists and is usually known as Borel's Theorem.

Definition 10.1. Define the *convolution* of two functions, $f(t)$ and $g(t)$ defined for $t \geq 0$ as

$$f(t) * g(t) \equiv \int_0^t f(t-x)g(x)dx$$

Example 10.1. Compute the convolution $t * e^{-t}$.

Solution: We just use the definition of the convolution,

$$\begin{aligned} t * e^{-t} &= \int_0^t (t-x)e^{-x}dx = \int_0^t te^{-x} - xe^{-x}dx \\ &= (-te^{-x} + xe^{-x} + e^{-x})|_0^t \\ &= -te^{-t} + t + te^{-t} - 0 + e^{-t} - 1 \\ &= e^{-t} + t - 1 \end{aligned}$$

■

We now make an observation; note that

$$\mathcal{L}\{t\} = \frac{1}{s^2}\,, \qquad \mathcal{L}\{e^{-t}\} = \frac{1}{(s+1)}$$

and

$$\begin{aligned} \mathcal{L}\{t * e^{-t}\} &= \mathcal{L}\{e^{-t} + t - 1\} \\ &= \frac{1}{s^2} + \frac{1}{s+1} - \frac{1}{s} \\ &= \frac{1}{s^2}\frac{1}{(s+1)} \\ &= \mathcal{L}\{t\} \cdot \mathcal{L}\{e^{-t}\}\,. \end{aligned}$$

This is an example of:

Theorem 10.1 (Borel's Theorem)**.** *Suppose that*

$$w(t) = u(t) * v(t)$$

then

$$W(s) = U(s)V(s)$$

where

$$\mathcal{L}\{u(t)\} = U(s), \quad \mathcal{L}\{v(t)\} = V(s), \quad \mathcal{L}\{w(t)\} = W(s)\ .$$

In words: *The Laplace transform of a convolution is the product of the Laplace transforms*. We'll prove this in the next section, but for now let us use the theorem to derive some results.

First, back to our original example; we found that if we Laplace transformed the ODE

$$y' + y = f(t) \qquad y(0) = 0$$

that

$$\mathbb{Y}(s) = F(s) \cdot \frac{1}{s+1}\ ,$$

but now, we can evaluate the inverse transform.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{F(s) \cdot \frac{1}{s+1}\right\} \\ &= f(t) * e^{-t} \\ &= \int_0^t f(x)e^{-(t-x)}dx \end{aligned}$$

So

$$y(t) = e^{-t}\int_0^t e^x f(x)\, dx$$

which is exactly the result one can find using the method of integrating factors.

Here is another example:

Example 10.2. Show $f(t) * g(t) = g(t) * f(t)$.

Solution 1: Use the definition of the convolution.

$$\begin{aligned} f(t) * g(t) &= \int_0^t f(t-x)g(x)dx \qquad u = t - x \quad du = -dx \\ &= \int_{u=t}^{u=0} f(u)g(t-u)(-du) = \int_0^t f(u)g(t-u)du \\ &= g(t) * f(t) \end{aligned}$$

Solution 2: Use Borel's theorem.

$$\begin{aligned} f(t) * g(t) &= \mathcal{L}^{-1}\{F(s)G(s)\} \\ &= \mathcal{L}^{-1}\{G(s)F(s)\} \\ &= g(t) * f(t) \end{aligned}$$

which on the surface seems rather shallow, but since a convolution transforms to a product of Laplace transforms, and multiplication is commutative, $f(t) * g(t)$ must equal $g(t) * f(t)$ if Borel's theorem is true.

■

10.2 A proof of Borel's Theorem

We will now prove:

Theorem 10.2 (Borel's Theorem)**.** *Suppose that*

$$w(t) = u(t) * v(t)$$

then

$$W(s) = U(s)V(s)$$

where

$$\mathcal{L}\{u(t)\} = U(s), \quad \mathcal{L}\{v(t)\} = V(s), \quad \mathcal{L}\{w(t)\} = W(s)\ .$$

Proof: Remember double integrals,

$$\begin{aligned} W(s) = \mathcal{L}\{w(t)\} &= \int_0^\infty e^{-st} u(t) * v(t) dt \\ &= \int_0^\infty e^{-st} \int_0^t u(x)v(t-x)dx\ dt \end{aligned}$$

Exchange the order of integration

$$\begin{aligned} &= \int_0^\infty \int_x^\infty e^{-st} u(x)v(t-x)dt\ dx \\ &= \int_0^\infty u(x) \int_x^\infty e^{-st} v(t-x)dt\ dx \end{aligned}$$

Let $t' = t - x,\ dt' = dt$

$$\begin{aligned} &= \int_0^\infty u(x) \int_0^\infty e^{-s(t'+x)} v(t')dt'\ dx \\ &= \int_0^\infty u(x)e^{-xs} \left[\int_0^\infty e^{-st'} v(t')dt'\right] dx \\ &= U(s)V(s) \end{aligned}$$

which concludes the proof. The real power of this method is for second-order ODEs.

Example 10.3. Solve the following ODE using Laplace transforms and convolutions.

$$y'' + 3y' + 2y = f(t) \qquad y(0) = y'(0) = 0$$

and find the solution explicitly when $f(t) = 1$. ■

Solution: As usual, we Laplace transform and solve for $\mathbb{Y}(s)$

$$\begin{aligned}(s^2 + 3s + 2)\mathbb{Y}(s) &= F(s) \\ \mathbb{Y}(s) &= \underbrace{\frac{1}{s^2 + 3s + 2}}_{\text{transfer function}} F(s)\end{aligned}$$

Now, we need to find the inverse transform of the transfer function; using partial fractions

$$\frac{1}{s^2 + 3s + 2} = \frac{1}{(s+2)(s+1)} = \frac{1}{(s+1)} - \frac{1}{s+2}$$

so

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 3s + 2}\right\} = e^{-t} - e^{-2t} = g(t)$$

and we can write the solution as

$$y(t) = f(t) * g(t) = \int_0^t (e^{-x} - e^{-2x}) f(t - x) dt$$

which we could have found using Variation of Parameters.

We can solve the above convolution integral when $f(t) = 1$ to yield the explicit solution in this case,

$$\begin{aligned} y(t) &= \int_0^t e^{-t} - e^{-2t}\, dt\ , \\ &= -e^{-t} + \left.\frac{e^{-2t}}{2}\right|_0^t\ , \\ &= -e^{-t} + e^{-2t} + 1 - \frac{1}{2}\ . \end{aligned}$$

So

$$y(t) = e^{-2t} - e^{-t} + \frac{1}{2}$$

is the explicit solution.

10.3 AN INTRODUCTION TO THE δ-FUNCTION.

When modelling physical systems it is useful to be able to describe an *impulse*; a nearly instantaneous transfer of a finite amount of momentum. Examples of where this may occur is for a collision of two objects (a mallet striking a croquet ball), or the swallowing of a pill in which a certain amount of medicine is introduced into your body in a moment of time. The idea goes back to the nineteenth century and associated with names like Kirchoff (in the context of electrical circuits), however for physicists it is most strongly associated with Dirac and quantum mechanics.

10.3.1 How do we define a δ-function?

A δ-function models an impulsive forcing; basically the addition of a finite amount of energy to a system in an infinitesimal amount of time. It is defined as the limit of a sequence of functions. A set of functions, $\delta_\epsilon(t)$ is called a δ-sequence if:

(I) Positivity: The function $\delta_\epsilon(t) \geq 0$ for all t. Sometimes this condition is relaxed.

(II) Unit Mass:

$$\int_{-\infty}^{\infty} \delta_\epsilon(t)\, dt = 1.$$

(III) Vanishing Support: As ϵ tends to zero, the function becomes narrower and localized:

$$\lim_{\epsilon \to 0} \delta_\epsilon(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0. \end{cases}$$

Note the *s*upport of a function is the set on which the function is non-zero.

The limit of this sequence as $\epsilon \to 0$ is used to define the δ-function. We can define one such sequence using the Heaviside function:

$$\begin{aligned} \delta_\epsilon(t-T) &= \begin{cases} \frac{1}{\epsilon} & T - \frac{\epsilon}{2} \leq t \leq T + \frac{\epsilon}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{1}{\epsilon}\left[H\left(t - \left(T - \frac{\epsilon}{2}\right)\right) - H\left(t - \left(T + \frac{\epsilon}{2}\right)\right)\right] \end{aligned}$$

and now $\lim_{\epsilon\to 0}\delta_\epsilon(t-T)=\delta(t-T)$.
Note the following properties of the δ-function:

(i) Vanishing Support:

$$\delta(t-T)=\begin{cases}0 & t\neq T\\ \infty & t=T\end{cases}$$

(ii) Unit Mass: If we integrate over an interval containing the δ-function, we get its mass, namely unity,

$$\int_a^b \delta(t-T)\,dt=\begin{cases}1 & a<t<b\,,\\ 0 & t<a \text{ or } b<t\,.\end{cases}$$

where we have assumed that $a<b$. If the δ-function falls on the boundary of the integral of integration, traditionally one sets the integral to $\frac{1}{2}$, but this can be the source of confusion sometimes.

(iii) Relationship to the Heaviside Function: From the unit mass property we see that

$$\int_0^t \delta(t'-T)\,dt'=\begin{cases}1 & t>T\\ 0 & t<T\end{cases}\quad = H(t-T)$$

So we write

$$\delta(t-T)=\frac{dH(t-T)}{dt}$$

(iv) Sampling Property: The δ-function samples a function at a point when integrated against it,

$$\int_a^b \delta(t-T)f(t)\,dt=\begin{cases}f(T) & a<t<b\,,\\ 0 & t<a \text{ or } b<t\,.\end{cases}$$

To see this from the limit definition, for our δ-sequence written in terms of the Heaviside function, note that

$$\begin{aligned}\lim_{\epsilon\to 0}\int \delta_\epsilon(t-T)f(t)\,dt &= \lim_{\epsilon\to 0}\int_{T-\epsilon/2}^{T+\epsilon/2}\frac{f(t)}{\epsilon}\,dt\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{T-\epsilon/2}^{T+\epsilon/2} f(t)\,dt\\ &= f(T)\,.\end{aligned}$$

The last step follows from the fact that this integral is measuring the "average value" of $f(t)$ on the interval $T - \frac{\epsilon}{2} \leq t \leq T + \frac{\epsilon}{2}$; as the interval gets smaller, this average value must approach $f(T)$ if the function is continuous.

10.3.2 Laplace Transform of $\delta(t-T)$

If we wish to model impulsive forcing of differential equations, it is useful to be able to compute the Laplace transform of the δ-function. If we assume $T > 0$ then, we can compute the Laplace transform in three different ways:

(i) Using the sampling property:

$$\mathcal{L}\{\delta(t-T)\} = \int_0^\infty e^{-st}\delta(t-T)dt = e^{-sT}$$

(ii) Using the fact that $\delta(t-T)$ is the derivative of the Heaviside function $H(t-T)$:

$$\begin{aligned}\mathcal{L}\{\delta(t-T)\} &= \mathcal{L}\left\{\frac{dH}{dt}(t-T)\right\} \\ &= s\mathcal{L}\{H(t-T)\} - H(0-T) \\ &= s\frac{1}{s}e^{-st} - 0 = e^{-sT}\end{aligned}$$

(iii) Using the limit definition of the δ-function:

$$\begin{aligned}\mathcal{L}\{\delta(t-T)\} &= \lim_{\epsilon\to 0}\mathcal{L}\{\delta_\epsilon(t)\} \\ &= \lim_{\epsilon\to 0}\mathcal{L}\left\{\frac{1}{\epsilon}H\left(t-\left(T-\frac{\epsilon}{2}\right)\right) - \frac{1}{\epsilon}H\left(t-\left(T+\frac{\epsilon}{2}\right)\right)\right\} \\ &= \lim_{\epsilon\to 0}\frac{e^{-s(T-\epsilon/2)} - e^{-s(T+\epsilon/2)}}{s\epsilon} \\ &= \frac{e^{-sT}}{s}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(e^{s\epsilon/2} - e^{-s\epsilon/2}\right) = e^{-sT}\ .\end{aligned}$$

Let us now do an example demonstrating the response to an impulsive forcing:

Example 10.4. A croquet ball is hit at a time $t = T$ with an impulsive force $F_0\,\delta(t-T)$. It feels friction from the grass proportional to its velocity. Compute its velocity as a function of time.

Solution: We first model the problem as a DE; remembering Newton's Law, we see that:

$$Mv' + \alpha v = F_0\,\delta(t-T) \qquad v(0) = 0\ ,$$

or

$$v' + \frac{\alpha}{M} v = \frac{F_0}{M}\,\delta(t-T)\ .$$

A Laplace transform yields

$$\left(s + \frac{\alpha}{M}\right)\mathbb{V} = \frac{F_0}{M}\,e^{-sT}\ .$$

Solving for the transformed velocity, we find

$$\mathbb{V}(s) = \frac{F_0}{M}\frac{1}{\left(s+\frac{\alpha}{M}\right)}\,e^{-sT}\ .$$

An inverse transfer now yields the velocity function,

$$\boxed{v(t) = H(t-T)\frac{F_0}{M}\,e^{-\frac{\alpha}{M}(t-T)}}$$

which is graphed below.

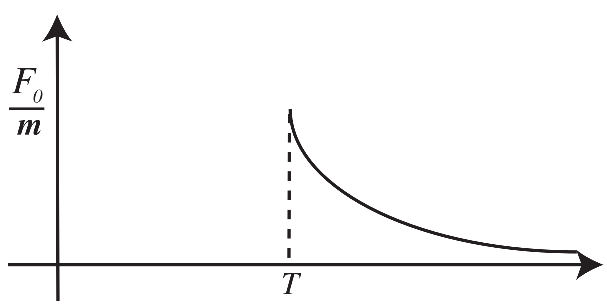


Figure 10.1: A graph of the velocity of the croquet ball; Note that the ball doesn't move before the mallet strike at $t = T$.

■

10.4 Transfer Functions and Green's Functions

We can use the response of a linear differential equation to an impulsive forcing to compute the response to an arbitrary forcing. Consider the Laplace transform of a δ-function applied just after $t = 0$.

$$\begin{aligned}\mathcal{L}\{\delta(t-0^+)\} &= \lim_{\mu\downarrow 0} \mathcal{L}\{\delta(t-\mu)\} \\ &= \lim_{\mu\downarrow 0} e^{-\mu s} \\ &= 1\end{aligned}$$

The *Green's function*, $g(t)$, is defined as the response of a linear differential equation to an impulse at $t = 0^+$. The Laplace transform of the Green's functions is the *transfer function*, $G(s)$.

Example 10.5. Compute the Green's function and transfer function associated with the differential operator

$$P[y] = y'' + 2y' + y.$$

Solution: The Green's function is defined as the response to an impulsive forcing at $t = 0^+$,

$$P[g] = g'' + 2g' + 2g = \delta(t - 0^+)$$

You may Laplace transform the equation to obtain,

$$\begin{aligned}(s^2 + 2s + 2)G(s) = 1 \\ G(s) = \frac{1}{(s+1)^2}\end{aligned}$$

where $G(s)$ is the transfer function. An inverse transform now yields

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}\ .$$

■

In general the solution to the forced differential equation

$$P[y] = f(t) \qquad\qquad (Ex : P[y] = y'' + 2y' + y)$$

is the convolution of the Green's function, which satisfies

$$P[g] = \delta(t - 0^+)$$

and the forcing function

$$y = g(t) * f(t) = \int_0^t f(t - x)g(x)dx$$

which may be best illustrated by an example.

Example 10.6. Solve for $y(t)$ where $t > 0$

$$y'' + 2y' + y = f(t) \qquad y(0) = 0$$

Solution: First we will solve for $\mathbb{Y}(s) = \mathcal{L}\{y(t)\}$,

$$\mathcal{L}\{y'' + 2y' + y = f(t)\} \Rightarrow [s^2 + 2s + 1]\mathbb{Y}(s) = F(s)$$

$$\mathbb{Y}(s) = \frac{1}{s^2 + 2s + 1}F(s)$$

Now we will use the Green's function; since

$$g'' + 2g' + g = \delta(t - 0^+)$$

$$s^2G + 2sG + G = 1$$

$$G(s) = \frac{1}{s^2 + 2s + 1}$$

we see that

$$\mathbb{Y}(s) = \frac{1}{s^2 + 2s + 1}F(s) = G(s)F(s)$$

that is the Laplace transform of the solution is the product of the transfer function and the Laplace transform of the forcing function. The Green's function is the inverse Laplace transform of the transfer function

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t} \ .$$

So we can see that the solution to the DE, $y(t)$, is just the inverse transform of the product, which by Borel's Theorm is the convolution of the Green's function with the forcing function

$$y(t) = \mathcal{L}^{-1}\{G(s)F(s)\} = g(t) * f(t) = \int_0^t f(x)g(t - x)dx$$

■

These ideas will appear again and again as we look at linear systems with arbitrary forcing.

Eleven

Bromwich's Integral & Inverse Laplace Transforms

OUTLINE OF CHAPTER

- The Laplace Transform: Properties and Preliminaries.
- Bromwich's Integrals for Inverse Laplace Transforms
- The Mellin Inversion Formula

11.1 THE LAPLACE TRANSFORM: PROPERTIES AND PRELIMINARIES.

In this Lecture we will discuss an integral transform for computing the inverse of the Laplace transform,

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

to do so, it is first useful to discuss some of the basic properties of the forward transform.

For this discussion, first let us extend the function being transformed to the real line. Let's define

$$\tilde{f}(t) = \begin{cases} f(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Note that this allows us to extend the Laplace transform to an integral over the real line also

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_{-\infty}^\infty e^{-st} \tilde{f}(t)\, dt$$

Second, let us restrict ourselves to functions that grow at worse exponentially. Suppose that

$$|f(t)| \leq Me^{\beta t} \qquad \text{for} \quad t > 0$$

for some real constants M and β. then we see that if we think of s as a complex variable, $s = p + iq$ for p and q real that

$$\begin{aligned} |F(s)| &\leq \int_0^\infty |e^{-st} f(t)| \, dt \\ &\leq M \int_0^\infty |e^{\beta t - (p+iq)t}| \, dt \\ &= M \int_0^\infty e^{(\beta - p)t} \, dt \\ &= \frac{M}{p - \beta} \qquad \text{for} \quad p > \beta. \end{aligned}$$

that is the Laplace transform, $F(s)$, exists in the right half-plane where $\mathrm{R}e\{s\} = p > \beta$. A little more work can show that $F(s)$ is an analytic function in this region also, and in particular has no singularities in this region.

11.2 BRONWICH'S INTEGRALS FOR INVERSE LAPLACE TRANSFORMS

To compute the inverse Laplace transform we need to define the Mellin Inversion Formula:

Definition 11.1. The Mellin Inversion Formula is defined in terms of the *Bronwich integral*,

$$f(t) = \mathcal{M}\{F(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) \, ds$$

where the contour of integration is the vertical line $\mathrm{R}e\{s\} = c$, and c is chosen to lie to the right of the singularities of $F(s)$.

Some notes:

- We showed above that the Laplace transform of functions that grow no worse than exponentially are analytic in a right half-plane, so one can always choose a large enough value of c so that the contour lies to the right of any singularities if $F(s)$ is the Laplace transform of a function.

- This integral is (usually) evaluated by closing the contour and using the residue theorem.
- We will argue below that this integral is in fact the inverse Laplace transform; that is that $\mathcal{M}\{F(s)\} = \mathcal{L}^{-1}\{F(s)\}$.

We will give a series of examples illustrating how to use the inverse transform, and then give an outline of a proof of the formula in the next section.

Example 11.1. Compute the inverse Laplace transform of

$$F(s) = \frac{1}{s+1}$$

via Bromwich's integral. ■

Solution: We wish to compute

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s+1}\, ds$$

where we take $c > -1$ so as to be to the right of the singularity at $s = -1$. We want to use the residue theorem to evaluate these integrals. We will do this by closing the contour in a semi-circle whose radius is tending to infinity in the left half plane when $t > 0$ and showing that the contribution from the semi-circle vanishes as the radius increases as long as the magnitude of F is bounded on these semi-circles. Similarly, when $t < 0$, we can close the contour in the right half-plane and in this case we find $f(t) = 0$.

For $t > 0$, we close the contour in the left half plane and apply the residue theorem to yield

$$\begin{aligned} f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)\, ds \qquad &= \frac{1}{2\pi i}\oint e^{zt}F(z)\, dz \\ = \frac{1}{2\pi i}\oint \frac{e^{zt}}{z+1}\, dz \\ = \text{Res}[\frac{e^{z}t}{z+1}; z=-1] \\ = e^{-t} \end{aligned}$$

so

$$f(t) = e^{-t} \qquad t > 0.$$

For $t < 0$, we will close the contour in the right half plane, but there are no singularities enclosed in the contour (remember, they are all to the left of the Bromwich coutour). Therefore $f(t)$ vanishes for $t < 0$. In summary,

$$f(t) = \begin{cases} e^{-t} & t > 0, \\ 0 & t < 0 \end{cases}$$

11.2.1 Closing the Bromwich contour

We wish to show that for $t > 0$ that we can close the Bromwich contour via a large semi-circle in the left half-plane that does not contribute to the integral in the limit of large radius. Consider

$$J(t) = \int_C e^{zt} F(z)\, dz \qquad\qquad t > 0$$

and let C be the contour $z = Re^{i\theta} + c$ for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Define $I(z)$ as the integrand of the above integral; then

$$I(z) = e^{R\cos\theta t + ct} e^{iR\sin\theta t} F(Re^{i\theta} + c)$$

and

$$\begin{aligned} |I(z)| &= e^{R\cos\theta t + ct} |F(Re^{i\theta} + c)| \;. \\ &\le M(R) e^{ct} e^{Rt\cos\theta} \end{aligned}$$

where $M(R)$ is the maximum value of $F(z)$ on the semi-circle.
Now

$$\begin{aligned} |J(t)| &\le M(R) e^{ct} \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}\, d\theta \\ &= 2M(R) e^{ct} \int_0^{\pi/2} e^{-Rt\sin\theta}\, d\theta \\ &= 2M(R) e^{ct} \int_0^1 \frac{e^{-Rtu}}{\sqrt{1-u^2}}\, du \\ &\le \frac{4M(R)e^{ct}}{Rt} \end{aligned}$$

which vanishes as R tends to infinity; note that the bound on the integral in the last step depends on t being positive. Therefore, the integral vanishes on the semi-circular contour in the left half plane.

11.3 More Examples of Inverse Transforms

Example 11.2. Compute the inverse Laplace transform of

$$F(s) = \frac{1}{s^n}$$

via Bromwich's integral.

Solution: We wish to compute

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s^n}\, ds$$

where we take $c > 0$ so as to be to the right of the singularity at $s = 0$. We want to use the residue theorem to evaluate these integrals. We will do this by closing the contour in a large semi-circle in the left half plane when $t > 0$ and showing that the contribution from the semi-circle vanishes as long as the magnitude of F is bounded on this semi-circle. Similarly, when $t < 0$, we can close the contour in the right half-plane and in this case we find $f(t) = 0$.

For $t > 0$, we close the contour in the left half plane and apply the residue theorem to yield

$$\begin{aligned} f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)\, ds \qquad &= \frac{1}{2\pi i}\oint e^{zt}F(z)\, dz \\ = \frac{1}{2\pi i}\oint \frac{e^{zt}}{z^n}\, dz \\ = \mathrm{Res}[\frac{e^{z}t}{z^n}; z = 0] \end{aligned}$$

Note

$$\frac{e^{zt}}{z^n} = \frac{1 + zt + (zt)^2/2! + \ldots}{z^n} = \frac{1}{z^n} + \frac{t}{z^{n-1}} + \frac{t^2/2!}{z^{n-2}} + \ldots \frac{t^{n-1}/(n-1)!}{z} \ldots$$

so

$$\mathrm{Res}\left[\frac{e^{zt}}{z^n}; z = 0\right] = \frac{t^{n-1}}{(n-1)!}.$$

and

$$f(t) = \frac{t^{n-1}}{(n-1)!} \qquad t > 0.$$

For $t < 0$, we will close the contour in the right half plane, but there are no singularities enclosed in the contour (remember, they are all to the left of the Bromwich coutour). Therefore $f(t)$ vanishes for $t < 0$. In summary,

$$f(t) = \begin{cases} \frac{t^{n-1}}{(n-1)!} & t > 0, \\ 0 & t < 0. \end{cases}$$

■

Example 11.3. Compute the inverse Laplace transform of

$$F(s) = \frac{1}{s^4 - 1}$$

via Bromwich's integral.

Solution: We wish to compute

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s^4 - 1}\, ds$$

where we take $c > 1$ so as to be to the right of the singularities.

We want to use the residue theorem to evaluate these integrals. We will do this by closing the contour in a large semi-circle in the left half plane when $t > 0$ and showing that the contribution from the semi-circle vanishes as long as the magnitude of F is bounded on this semi-circle. Similarly, when $t < 0$, we can close the contour in the right half-plane and in this case we find $f(t) = 0$. If

$$F(s) = \frac{1}{s^4 - 1}$$

then there are simple poles at the four zeroes of the denominator,

$$s^4 - 1 = (s-1)(s+1)(s+i)(s-i)$$

that is where $s = -1, 1, i, -i$.

We can use residue calculus is used to evalulate this integral. The vertical line $\text{Re}(z) = c$ must be closed either to the right or to the left. We will close the curve to the right or left using a semicircle of radius R (and letting $R \to \infty$), depending on the values of t.

To determine which curve to use, we analyze the integrand. Writing $s = x + iy$, the magnitude of the integrand becomes

$$\left|\frac{e^{st}}{s^4-1}\right| = \left|\frac{e^{(x+iy)t}}{s^4-1}\right| = \left|\frac{e^{xt}e^{iyt}}{s^4-1}\right| = \frac{e^{xt}}{|s^4-1|}.$$

Therefore, we see that if $t > 0$, we want to use the use the left semicircle (in which $x < 0$), so that the integrand is exponential small as Re$(z) \to -\infty$ and the integral over the semicircle vanishes as $R \to \infty$. We see that

$$\begin{aligned}
f(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s^4-1}\, ds \\
&= \text{Res}\left[\frac{e^{st}}{s^4-1}; s=1\right] + \text{Res}\left[\frac{e^{st}}{s^4-1}; s=-1\right] \\
&\qquad + \text{Res}\left[\frac{e^{st}}{s^4-1}; s=i\right] + \text{Res}\left[\frac{e^{st}}{s^4-1}; s=-i\right] \\
&= \lim_{s\to 1}\,(s-1)\frac{e^{st}}{s^4-1} + \lim_{s\to -1}\,(s+1)\frac{e^{st}}{s^4-1} + \lim_{s\to i}\,(s-i)\frac{e^{st}}{s^4-1} + \lim_{s\to -i}\,(s+i)\frac{e^{st}}{s^4-1} \\
&= \left.\frac{e^{st}}{4s^3}\right|_{s=1} + \left.\frac{e^{st}}{4s^3}\right|_{s=-1} + \left.\frac{e^{st}}{4s^3}\right|_{s=i} + \left.\frac{e^{st}}{4s^3}\right|_{s=-i} \\
&= \frac{e^t}{4} - \frac{e^{-t}}{4} - \frac{e^{it}}{4i} + \frac{e^{-it}}{4i} \\
&= \frac{1}{2}\sinh t - \frac{1}{2}\sin t \qquad \text{for} \qquad t > 0.
\end{aligned}$$

Likewise, if $t < 0$, then we want to the use the right semicircle. However, because the vertical line is chosen to the right of all singularities of $F(s)$, Cauchy's Theorem tells us that the integral is zero for $t < 0$.

To summarize

$$\boxed{f(t) = \begin{cases} \frac{1}{2}\sinh t - \frac{1}{2}\sin t & t > 0, \\ 0 & t \le 0. \end{cases}}$$

■

Example 11.4. Compute the inverse Laplace transform of

$$F(s) = \frac{e^{-\pi s}}{s^2+1}$$

via Bromwich's integral.

Solution: We wish to compute

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}e^{-\pi s}}{s^2+1}\, ds$$

where we take $c > 0$ so as to be to the right of the singularities at $s = \pm i$.

For what values of t should we use the left semicircle? (Remember that since the vertical line is chosen to the right of all singularities of $F(s)$, Cauchy's Theorem tells us that the integral is zero when we use the right semicircle.) Writing $s = x + iy$, the magnitude of the integrand becomes

$$\left|\frac{e^{s(t-\pi)}}{s^2+1}\right| = \left|\frac{e^{(x+iy)(t-\pi)}}{s^2+1}\right| = \left|\frac{e^{x(t-\pi)}e^{iy(t-\pi)}}{s^2+1}\right| = \frac{e^{x(t-\pi)}}{|s^2+1|}.$$

Therefore, we see that if $t > \pi$, we want to use the use the left semicircle (in which $x < 0$), so that the integral over the semicircle vanishes as $R \to 0$. For $t > \pi$,

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{s(t-\pi)}}{s^2+1} ds = \text{sum of all residues of } \frac{e^{s(t-\pi)}}{s^2+1} \\ &= \left.\frac{e^{s(t-\pi)}}{2s}\right|_{s=i} + \left.\frac{e^{s(t-\pi)}}{2s}\right|_{s=-i} \\ &= \frac{e^{i(t-\pi)}}{2i} - \frac{e^{-i(t-\pi)}}{2i} = \sin(t-\pi). \end{aligned}$$

For $t < \pi$, we must choose the right semicircle, so $f(t) = 0$.

To summarize,

$$f(t) = \sin(t-\pi)H(t-\pi).$$

■

11.4 Proof of the Mellin Inversion Formula

In this section, we will sketch a proof that the Laplace transform is indeed the inverse of the Mellin Inversion Formula.

Suppose $F(s)$ is known and we use the Mellin inversion formula to construct a function, $\tilde{f}(t) = \mathcal{M}\{F(s)\}$ – we don't know yet that it is really $f(t)$. Then

$$\tilde{F}(z) = \int_0^\infty e^{-zt}\tilde{f}(t)dt \qquad |\tilde{f}(t)| < \epsilon e^{\gamma x}$$

clearly exist for Re$\{z\} > \gamma$. We want to show $\tilde{F}(z) \equiv F(z)$.
From the Mellin inversion formula, then

$$\tilde{F}(z) = \int_0^\infty e^{-zt} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s)\, dsdt$$

Let's assume we can interchange the orders of integration; then

$$\begin{aligned} \tilde{F}(z) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s) \int_0^\infty e^{-(z-s)t} dtds \\ &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F(s)}{z-s} ds, \qquad Re(z) > \gamma \end{aligned}$$

or

$$\tilde{F}(z) = -\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F(s)}{s-z} ds$$

If $F(s) \to 0$ as $s \to \infty$ for $Re(z) \geq \gamma$ then we can close the contour to the

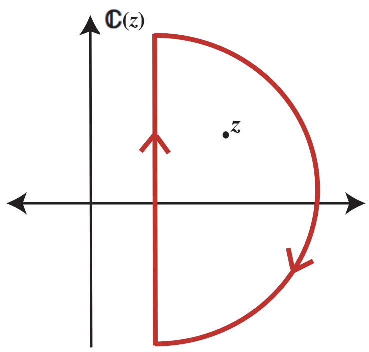


Figure 11.1: The contour for $\tilde{F}(z)$

right. We can also interpret the minus sign as saying that we should traverse the contour in a counter-clockwise sense. Consequently, we see that

$$\tilde{F}(z) = \frac{1}{2\pi i} \oint_C \frac{F(s)}{s-z} ds$$

or

$$\tilde{F}(z) \equiv F(z)$$

by the Cauchy integral formula.