A Primer of

Partial Differential Equations

Physical Models, Mathematical Insights, and Solution Techniques

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What is a Partial Differential Equation?

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What is a Partial Differential Equation?

1.1 What is a Partial Differential Equation?

You've probably seen an ordinary differential equation (ODE) before. For example, the pendulum equation

$$\frac{d^2\Theta}{dt^2} + \frac{g}{L}\sin\Theta = 0 \tag{1.1}$$

describes the angle $\Theta(t)$ a pendulum makes with the vertical as a function of time t. Here g (the acceleration due to gravity) and L (the length of the pendulum) are constants, t is the *independent variable*, and Θ is the *dependent variable*. This is an ODE because there is only one independent variable, t.

If there is more than one independent variable in a problem, we are likely to encounter a partial differential equation (PDE). A PDE relates the partial derivatives of a function of two or more independent variables. For example, Laplace's equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0, \tag{1.2}$$

relating the partial derivatives of a function $\Phi(x, y)$, arises in many places in mathematics and physics. For simplicity, we will use subscript notation for partial derivatives, so this equation can also be written $\Phi_{xx} + \Phi_{yy} = 0$.

A function is a *solution* to a PDE if it satisfies the equation in addition to any side conditions. Side conditions often consist of prescribed values the solution must take at an initial time, or at certain points in its domain.

A given PDE (with side conditions) could have no solution or many solutions, so the question of whether a solution *exists* and when it is *unique* is important to consider.

Exercise 1.1. Show that $\Phi_1 = x$ and $\Phi_2 = x^2 - y^2$ are solutions to Laplace's equation (1.2). Can you combine them to create a new solutions?

Exercise 1.2. Let c be a constant and f be any differentiable function. Show that $u(x,t) = f(x - ct)$ is a solution to the *transport equation*

$$u_t + cu_x = 0. \tag{1.3}$$

Find a solution $u(x,t)$ to the transport equation that satisfies the initial condition $u(x,0) = \sin x$.

Exercise 1.3. Show that

$$Z(x,y) = \ln\left(\frac{\sin(y)}{\sin(x)}\right)$$

is a solution to the *minimal surface equation*

$$(1 + Z_y^2)Z_{xx} - 2Z_xZ_yZ_{xy} + (1 + Z_x^2)Z_{yy} = 0 \tag{1.4}$$

in the region $0 < x < \pi$, $0 < y < \pi$. What happens on the boundary of this region? Suppose we consider a constant multiple of $Z(x,y)$ – is it still a solution of the PDE?

Exercise 1.4. Show that the function

$$h(x,t) = 2\alpha^2 \text{sech}\left(\alpha(x - 4\alpha^2 t)\right)$$

is a solution to the Korteweg-deVries (KdV) equation,

$$h_t + 6hh_x = h_{xxx}. \tag{1.5}$$

This solution is known as a *soliton* and can be thought of a model of a tsunami propagating in the ocean. A computer algebra system such as MAPLE or MATHEMATICA may be helpful with the algebra.

1.2 Classifying PDEs: Order, Linear vs. Nonlinear

When studying ODEs we classify them in an attempt to group similar equations which might share certain properties, such as methods of solution. We classify PDEs in a similar way. Two important properties for classification are order and linearity. The *order* of a differential equation is the highest derivative that appears in the equation. So, the transport equation (1.3) is first order, and Laplace's Equation (1.2) and the minimal surface equation (1.4) are both second order. Note that mixed derivatives also count, so for example $u_{xxy} = 0$ would be third order.

Much of this book will concentrate on *linear* PDE. These are the most commonly encountered PDE in mathematics and physics and enjoy the greatest availability of solution techniques. Roughly, a PDE is linear if all terms involving the dependent variables only appear to degree one. There is no restriction on how the independent variables can appear. To give a more precise definition of linear, we first define a *linear operator*.

Definition 1.6. A *linear operator* L is a mapping that takes functions u to functions $L[u]$ in such a way that for any two functions u_1 and u_2

$$L[c_1u_1 + c_2u_2] = c_1L[u_1] + c_2L[u_2]$$

for any constants c_1 and c_2. In words, *a linear operator respects linear combinations.*

For example, the Laplacian operator in two dimensions is defined as

$$L = \partial_{xx} + \partial_{yy}.$$

L is written using *operator notation*, which allows us to avoid naming the function L is operating on. We could also just write

$$L[\Phi] \equiv \Phi_{xx} + \Phi_{yy}.$$

A quick computation shows that $L[c_1\Phi + c_2\Psi] = c_1L[\Phi] + c_2L[\Psi]$, so L is a linear operator.

We can now define a *linear partial differential equation* as a PDE of the form $L[\Phi] = f$ where f is a known function of the independent variables. A *nonlinear* PDE as any equation that cannot be written in this form (i.e. a PDE that is not linear).

We should also define the *domain* and *codomain* (or *range*) of the operator. The domain is the space of functions which an operator acts on. So, for example, we might consider the domain of the Laplace operator as twice differentiable functions of x and y. The codomain is a space of functions that contains $L[u]$ where L is a linear operator and u is in the domain of L. For the Laplace operator we might consider the codomain to be just the set of functions of two variables, x and y. Note that for the purposes of this text, unless stated otherwise, we will assume that functions can be differentiate as many times as needed to do a particular calculation.

Exercise 1.5. Convince yourself that any linear combination of linear operators (with the same domain and codomain) is also a linear operator.

Example 1.1. Find the most general first-order linear PDE for $u(x,t)$.
Solution: Because the PDE is first-order, it can depend only on the independent variables (x,t), the dependent variable u and its first derivatives u_t and u_x. Also, for an operator $L[u]$ to be linear it must be linear in u, u_t and u_x. Therefore the most general first-order linear operator is

$$L[u] \equiv a(x,t)u_t + b(x,t)u_x + c(x,t)u$$

or in operator notation

$$L \equiv a(x,t)\partial_t + b(x,t)\partial_x + c(x,t)$$

where a, b and c are arbitrary functions. The most general first-order linear PDE for $u(x,t)$ is

$$a(x,t)u_t + b(x,t)u_x + c(x,t)u = f(x,t), \tag{1.7}$$

or in operator notation

$$L[u] = f(x,t).$$

Strictly speaking, we should also specify that a, b are not both identically zero for the equation to be first-order. ■

Exercise 1.6. Which of Laplace's equation (1.2), the convection equation (1.3), the minimal surface equation (1.4) and the Korteweg-deVries equation (1.5) are linear?

Exercise 1.7. Write down the most general constant coefficient linear second-order equation for $\Phi(x,y)$.

1.3 Homogeneous PDEs and Vector Spaces of Solutions

Linear equations can further be classified as *homogeneous*, which take the form $L[u] = 0$ and *inhomogeneous* which take the form $L[u] = f$ for some non-zero function of the independent variables, f. So the transport equation

$$u_t + cu_x = 0$$

is homogeneous, but its cousin, the general first-order linear PDE for $u(x,t)$, is inhomogeneous

$$a(x,t)u_t + b(x,t)u_x + c(x,t)u = f(x,t),$$

unless $f(x,t) = 0$.

By the definition if two solutions, say u_1 and u_2, satisfy a linear homogeneous PDE, that any linear combination of them

$$u = c_1u_1 + c_2u_2 \tag{1.8}$$

is also a solution because

$$L[u] = L[c_1u_1 + c_2u_2] = c_1L[u_1] + c_2L[u_2] = 0. \tag{1.9}$$

So, for example, since

$$\Phi_1 = x^2 - y^2 \qquad \Phi_2 = x$$

both satisfy Laplace's equation, $\Phi_{xx} + \Phi_{yy} = 0$, so does any linear combination of them

$$\Phi = c_1\Phi_1 + c_2\Phi_2 = c_1(x^2 - y^2) + c_2x.$$

This property is extremely useful for constructing solutions which satisfy certain initial conditions and boundary conditions. Mathematicians refer to a set of functions closed under addition and scalar multiplication as a *vector space*. The set of solutions to a linear homogeneous PDE form a vector space, a fact that is incredibly useful for constructing solutions that satisfy particular initial conditions or boundary conditions. Physicists, mathematicians and engineers often refer to the idea *superposition* of solutions, whereby a solution can be represented as a sum of simpler solutions; an example of this is a vibrating string that oscillates with some superposition of its fundamental frequency and its harmonics or overtones. The most common origin of superposed solutions in physics is the underlying linearity of the governing equations.

1.4 Challenge Problems for Lecture 1

Problem 1.1. Classify the follow differential equations as ODEs or PDEs, linear or nonlinear, and determine their order. For the linear equations, determine whether or not they are homogeneous.

(a) The *diffusion equation* for $h(x, t)$:

$$h_t = Dh_{xx}$$

(b) The *wave equation* for $w(x, t)$:

$$w_{tt} = c^2 w_{xx}$$

(c) The *thin film equation* for $h(x, t)$:

$$h_t = -(hh_{xxx})_x$$

(d) The *forced harmonic oscillator* for $y(t)$:

$$y_{tt} + \omega^2 y = F \cos(\Omega t)$$

(e) The *Poisson Equation* for the electric potential $\Phi(x, y, z)$:

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 4\pi\rho(x, y, z)$$

where $\rho(x, y, z)$ is a known charge density.

(f) *Burger's equation* for $h(x, t)$:

$$h_t + hh_x = \nu h_{xx}$$

Problem 1.2. Show that the helicoid

$$Z(x, y) = \tan^{-1}(y/x)$$

satisfies the minimal surface equation,

$$(1 + Z_y^2)Z_{xx} - 2Z_x Z_y Z_{xy} + (1 + Z_x^2)Z_{yy}$$

MAPLE or MATHEMATICA may be helpful with the algebra.

Two

The Diffusion Equation

2.1 An Introduction to Heat Flow

A classical example of the application of ordinary differential equations is Newton's Law of Cooling which, basically, answers the question *"How does a cup of coffee cool?"* Newton hypothesized that the rate at which the temperature, $U(t)$, changes is proportional to the difference with the ambient temperature, which we call $\bar{U}$,

$$\mathbb{DE}: \qquad \frac{dU}{dt} = -\kappa(U - \bar{U}). \tag{2.1}$$

Here κ is a positive rate constant (with units of inverse time) that measures how fast heat is lost from the coffee cup to the ambient environment. If we specify the initial temperature,

$$\mathbb{IC}: \qquad U(0) = U_0, \tag{2.2}$$

we can solve for the evolution of the temperature,

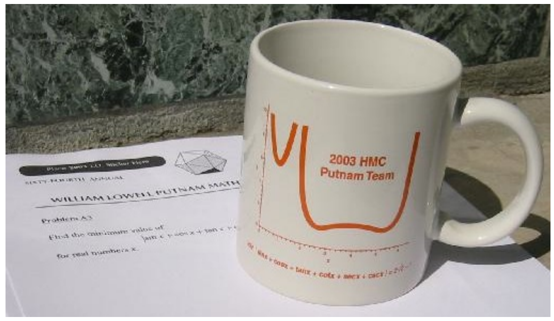
$$\boxed{U(t) = \bar{U} + (U_0 - \bar{U})e^{-\kappa t}.} \tag{2.3}$$

If we graph the temperature as a function of time, we see that it decays exponentially to the ambient temperature, $\bar{U}$, at a rate governed by κ.

When we derived Newton's Law of cooling we made several assumptions – most importantly that the temperature in the coffee cup did not vary with location. If we account for the variation of temperature with location, we can derive a PDE called the *heat equation* or, more generally, the *diffusion equation*. If the temperature, $U(x,t)$ is a function of a single spatial variable, x, we will show that it satisfies the diffusion equation,

$$U_t = DU_{xx},$$

(a)



(b)

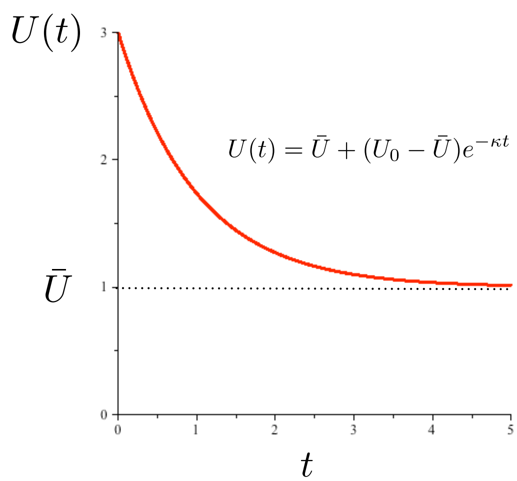


Figure 2.1: Newton's Law of Cooling. (a) A coffee cup (b) Coffee temperature as a function of time.

where D is a constant known as the thermal diffusivity. In higher dimensions, the equation can be written

$$U_t = D\nabla^2 U,$$

where ∇^2 is the *Laplacian*.

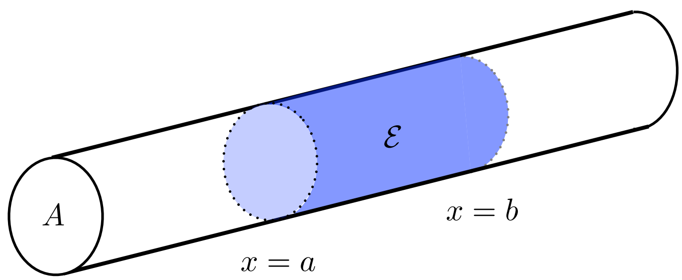


Figure 2.2: Thermal energy $\mathcal{E}$ in a metal bar of cross-sectional area A.

2.2 Derivation of the Diffusion Equation

The diffusion equation will be our second example of a conservation law; we can derive the equation by accounting for the flow of thermal energy. Suppose we consider a metal bar, with a uniform cross-sectional area, A, whose temperature, $U(x,t)$, is a function of time, t, and the position, x, along the bar (that is we assume the temperature is uniform in every cross-section).

Let the thermal energy in the region $a < x < b$ is given by

$$\mathcal{E} = \rho_0 c_m A \int_a^b U(x,t)\, dx \tag{2.4}$$

The important term in the integral is the temperature, $U(x,t)$, measured in degrees. The remaining constants are A, the cross-sectional area (with units of [(length)^2]); ρ_0, the density [mass/(length)^3]; and c_m, the specific heat capacity per unit mass [energy/(degree $\cdot$ mass)]. Note c_m is the amount of energy needed to raise one gram of a substance one degree - it is sometimes call just the specific heat (but you need to be careful to distinguish between specific heat of an object, specific heat per unit mass and specific heat per unit volume). Note that ρ_0 and c_m are physical properties of the material, while A is determined by the geometry.

We wish to equate the change in thermal energy to the heat flux out of the bar through the planes at $x = a$ and $x = b$. To do this we use *Fourier's heat law* which states that the flux density (with units of [energy/((length)$^2\cdot$time)]

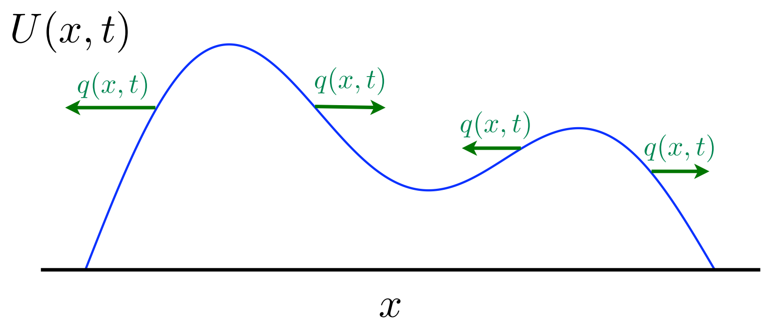


Figure 2.3: Heat flux is from hot to cold and proportional to the gradient.

of thermal energy, $q(x,t)$ is proportional to the temperature gradient,

$$q(x,t) = -kU_x, \tag{2.5}$$

where the negative sign reflects the fact that heat flows from hot to cold, just as in Newton's law of cooling, with a constant of proportionality, k, called the thermal conductivity [(energy)/(length·degrees·time)].

Now, the total flux of thermal energy into the *into* the region $a < x < b$ is given by

$$Q = A[q(a,t) - q(b,t)], \tag{2.6}$$

where we multiply by the area A to get the total flux through the cross-section.

By *conservation of energy*, the rate of change of the energy between a and b is given by the flux into the region,

$$\frac{d\mathcal{E}}{dt} = Q. \tag{2.7}$$

Once again we can rewrite the flux by a clever application of the fundamental theorem of calculus,

$$\begin{aligned} Q &= A[q(a,t) - q(b,t)] = -\,Aq(x,t)|_{x=a}^{x=b} & (2.8)\\ &= -A\int_a^b q_x\,dx. & (2.9) \end{aligned}$$

We now rewrite the conservation of energy equation as

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}\left[\rho c_m A \int_a^b U\, dx\right] = \int_a^b \rho c_m A U_t \, dx = Q = -A\int_a^b q_x dx, \tag{2.10}$$

or, rearranging

$$\int_a^b \rho c_m A\, U_t + A q_x \, dx = 0. \tag{2.11}$$

Since this is true for *every* interval $a < x < b$, the integrand must vanish identically. So

$$\rho c_m A U_t + A q_x = 0. \tag{2.12}$$

Substituting for the flux function $q(x,t) = -kU_x$ yields

$$\rho c_m A U_t - kA(U_x)_x = 0. \tag{2.13}$$

Rearranging the equation yields the diffusion equation,

$$\boxed{U_t = DU_{xx},} \tag{2.14}$$

where $D = k/(\rho c_m)$ is a constant with units of [(length)^2/time] called the *thermal diffusivity* which is determined by the physical properties of the metal bar.

Exercise 2.1. The thermal diffusivity is an example of a *diffusion constant*; verify the units of D and explain why they are consistent with the Diffusion equation 2.14. Find the value of D for an iron bar and an aluminum bar; can you explain physically the difference?

2.2.1 Initial conditions and Boundary Conditions

To complete the description of the problem, we need to supplement the diffusion equation with boundary conditions and initial conditions. Suppose we consider a bar of finite length L, occupying the region $0 < x < L$. At the boundaries of the metal bar we can specify a fixed temperature,

$$U(0,t) = U_0 \qquad U(L,t) = U_1, \tag{2.15}$$

which are usually referred to as *Dirichlet* boundary conditions. Alternatively, we could specify a heat flux,

$$q_0 = q(0,t) = -kU_x(0,t) \qquad q_1 = q(L,t) = -kU_x(L,t). \tag{2.16}$$

Specifying the gradient across the boundary is referred to as *Neumann* boundary conditions.

Finally, we also need to specify the initial temperature distribution,

$$U(x,0) = f(x) \qquad 0 < x < L. \tag{2.17}$$

We will demonstrate below that the solution to this problem (if it exists) is unique; later in this course we will solve this problem using the method of separation variables.

For completeness, we also comment here that the problem can be posed on the infinite line, $-\infty < x < \infty$ sometime called the *Cauchy* problem – in this case one usually replaces the boundary condition with the specification that the temperature remains bounded as we approach infinity,

$$\lim_{x \to \pm\infty} |U(x,t)| < C, \tag{2.18}$$

for some constant C. This condition may seem superfluous at first glance, but actually is necessary to stop heat from leaking in from infinity (speaking very, very informally an infinite source of heat infinitely far away can have a finite effect in a short amount of time). If you are interested in details, look for the examples of Tychonov in a PDEs text[1].

2.3 Examples of Solution to the Diffusion Equation

We can summarize the last section by restating a well-posed problem for the diffusion equation on the interval $0 < x < L$ with Dirichlet boundary conditions,

The Dirichlet Problem for the Diffusion Equation
(Inhomogeneous Boundary Conditions)

$$\begin{aligned}
\mathbb{DE}: &\quad U_t = DU_{xx} && 0 < x < L, t > 0\\
\mathbb{BC}: &\quad U(0,t) = U_0, \quad U(L,t) = U_1 && t > 0\\
\mathbb{IC}: &\quad U(x,0) = f(x) && 0 < x < L.
\end{aligned}$$

Solving the general problem will have to wait, but we can find some specific solutions to the problem using the ideas of *Separation of Variables*. For the moment, we will restrict ourselves to homogeneous boundary conditions,

[1]See, for example, T. W. Körner, "Fourier Analysis," Cambridge University Press, p. 338.

THE DIRICHLET PROBLEM FOR THE DIFFUSION EQUATION
(HOMOGENEOUS BOUNDARY CONDITIONS)

$$\begin{array}{lll} \mathbb{DE}: & U_t = DU_{xx} & 0 < x < L, t > 0 \\ \mathbb{BC}: & U(0,t) = 0, \quad U(L,t) = 0 & t > 0 \\ \mathbb{IC}: & U(x,0) = f(x) & 0 < x < L. \end{array}$$

If you want, you can skip the derivation for the moment and jump ahead to Exercise 1, if you don't mind the solution appearing *deus ex machina* (a fancy term for "out of thin air").

2.3.1 A Solution to the Homogeneous Dirichlet Problem

Let us look for solutions to the homogeneous Dirichlet problem of the form

$$U(x,t) = X(x)T(t) \tag{2.19}$$

we find from the differential equation ($\mathbb{DE}$) that

$$XT_t = DX_{xx}T \tag{2.20}$$

and dividing by XT we find

$$\frac{T_t}{DT} = \frac{X_{xx}}{X} = -\lambda. \tag{2.21}$$

where λ is to be determined. Now because T_t/DT is *only* a function of t and X_{xx}/X is *only* a function of x we know that λ must be independent of x and t respectively, and therefore must be a constant – consequently it is known as the *separation constant*. We can now solve the resulting ODE for $T(t)$

$$T_t = -\lambda DT \quad \Rightarrow \quad T(t) = e^{-\lambda Dt}, \tag{2.22}$$

or some constant multiple of it.

We now look for a solution for the $X(x)$ equation that also satisfies the homogeneous boundary conditions. From the boundary conditions (BC), we know that

$$U(0,t) = X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0 \tag{2.23}$$

$$U(L,t) = X(L)T(t) = 0 \quad \Rightarrow \quad X(L) = 0 \tag{2.24}$$

So finally we conclude that we are looking for solutions to the *Boundary Value Problem* for $X(x)$,

$$\boxed{X_{xx} + \lambda X = 0, \qquad X(0) = 0 \quad X(L) = 0.} \tag{2.25}$$

Solving the DE, we find that

$$X(x) = B\cos(\sqrt{\lambda}x) + C\sin(\sqrt{\lambda}x) \tag{2.26}$$

and applying the boundary conditions we see that $X(0) = 0$ implies that $B = 0$, and that

$$C\sin(\sqrt{\lambda}L) = 0. \tag{2.27}$$

Consequently, a non-trivial solution (that is a solution for which $X(x) \neq 0$) for $X(x)$ can be found if and only if

$$\boxed{\lambda = \lambda_n \equiv \left(\frac{n\pi}{L}\right)^2 \quad for \quad n = 1, 2, 3\ldots} \tag{2.28}$$

for which we find

$$\boxed{X(x) = X_n(x) \equiv \sin\left(\frac{n\pi x}{L}\right) \quad for \quad n = 1, 2, 3\ldots,} \tag{2.29}$$

or some constant multiple of it. These special values of λ are called *eigenvalues* and the associated functions, $X_n(x)$, are known as *eigenfunctions*.

Multiplying the solution for $X_n(x)$ and $T(t)$ together finally yields a solution for $U_n(x, t)$,

$$\boxed{U(x, t) = U_n(x, t) \equiv \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt} \, for \; n = 1, 2, 3\ldots.} \tag{2.30}$$

The method of separation of variables is very powerful – it will be one of our primary tools for finding solutions to PDE's in the coming lectures.

Exercise 2.2. Verify that

$$U_n(x, t) \equiv \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t} \quad for \quad n = 1, 2, 3\ldots,$$

satisfies the diffusion equation $U_t = DU_{xx}$ and the homogeneous boundary conditions $U(0, t) = U(L, t) = 0$. Explain why any linear combination of U_n,

$$U(x, t) = \sum_{n=1}^{\infty} a_n U_n(x, t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt}$$

also satisfies the diffusion equation and the homogeneous boundary condition. Does it worry you that this is an infinite sum? What initial condition, $U(x, 0)$, does this correspond to?

2.3.2 A Solution to the Cauchy Problem

We can also consider a solution to the Cauchy problem for the diffusion equation, which you hopefully remember is the problem posed on the entire real line,

THE CAUCHY PROBLEM FOR THE DIFFUSION EQUATION

$$\begin{aligned} \mathbb{DE}: &\quad U_t = DU_{xx} && -\infty < x < \infty, t > 0 \\ \mathbb{BC}: &\quad \lim_{x \to \pm\infty} |U(x,t)| < C && t > 0 \\ \mathbb{IC}: &\quad U(x,0) = f(x) && -\infty < x < \infty. \end{aligned}$$

While there are many clever derivation for the solution to this problem, for the moment I will simply give you the most important solution, usually called the *fundamental solution* or the *diffusion kernel*,

$$U(x,t) = G(x,t+\tau) \equiv \frac{1}{\sqrt{4\pi D(t+\tau)}} e^{-\frac{x^2}{4D(t+\tau)}}. \tag{2.31}$$

where τ is a constant (which we will assume is positive). This solution can be used to construct a general solution of the diffusion equation for an arbitrary initial condition, $f(x)$.

Exercise 2.3. Verify that

$$G(x,t+\tau) \equiv \frac{1}{\sqrt{4\pi D(t+\tau)}} e^{-\frac{x^2}{4D(t+\tau)}}.$$

satisfies the diffusion equation and the boundary conditions for the Cauchy problem when $\tau > 0$. Show that this solution corresponds to a Gaussian with time varying width and height. How does the Gaussian's width, height and area vary in time?

2.4 THE MAXIMUM PRINCIPLE

Looking at solutions to the heat equation, we note that they tend to average out maximums and minimums. We can develop some intuition for this by considering what the equation says. Basically, $U_t = DU_{xx}$ means: *The temperature is decreasing when the profile is convex down and the temperature is increasing when the profile is convex up.*

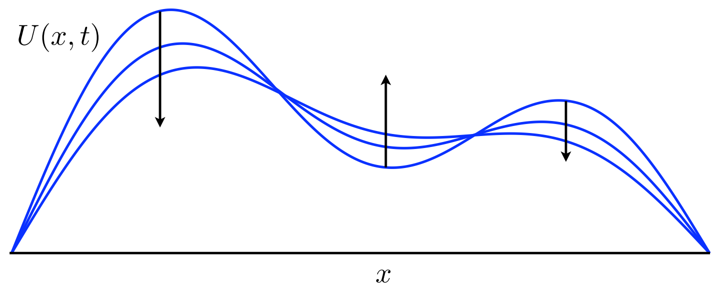


Figure 2.4: The heat equation interpreted graphically. The temperature is decreasing when the profile is convex down and the temperature is increasing when the profile is convex up.

From which we conclude that interior maximums in temperature are decreasing and interior minimums of temperature are increasing. This reasoning is not quite airtight (how to make it tighter is a good question to ponder). We can give a rigorous statement (without proof) of the maximum principle:

Theorem 2.1 (Maximum Principle for the Diffusion Equation)**.** *If $U(x,t)$ satisfies the Dirichlet problem for the diffusion equation in the semi-infinite strip $0 < x < L$, $0 < t$, then it assumes its maximum value (as a function of x and t) either initially (when $t = 0$) or on the lateral boundaries (where $x = 0$ or $x = l$).*

The same is also true of the minimum of $u(x,t)$. A proof can be found in most advanced PDE texts.

Exercise 2.4. Interpret the solutions we have found for the diffusion equation in terms of the maximum principle. Show examples where the maximum value of $u(x,t)$ occur in the initial condition and on the lateral boundaries.

2.5 Energy Dissipation and Uniqueness

By looking at what is normally known as energy for the diffusion equation, we can show that the solution for the Dirichlet problem is unique. Note this energy is a mathematical construct, not to be confused with the thermal energy discussed in the derivation of the diffusion equation.

First, suppose that $U(x,t)$ is a solution to the homogeneous Dirichlet problem,

$$\begin{array}{lccl} \mathbb{DE}: & U_t = DU_{xx} & & 0 < x < L, t > 0 \\ \mathbb{BC}: & U(0,t) = 0, & U(L,t) = 0 & t > 0 \\ \mathbb{IC}: & U(x,0) = f(x) & & 0 < x < L. \end{array}$$

Let's define, the energy,

$$W = \frac{1}{2}\int_0^L U^2\,dx, \tag{2.32}$$

which is a function of t dependent on the particular solution $U(x,t)$ (technically it is a function of t and a *functional* of $U(x,t)$). Note that $W \geq 0$ with $W = 0$ only for the trivial solution $U(x,t) = 0$.

If we differentiate the energy with respect to time, we find

$$\begin{aligned} \frac{dW}{dt} &= \int_0^L UU_t\,dx, \\ &= D\int_0^L UU_{xx}\,dx, \\ &= -\int_0^L (U_x)^2\,dx + UU_x\,\big|_{x=0}^{x=L}, \end{aligned}$$

where we have substituted the $\mathbb{DE}$ and used integration by parts. Now, applying the $\mathbb{BC}$'s, we fine that the boundary terms from the integration by parts vanish, so,

$$\frac{dW}{dt} = -\int_0^L (U_x)^2\,dx \leq 0$$

Now, we can conclude that W is decreasing (that is energy is dissipated!!) *unless* $U_x = 0$, that is to say that U is constant. As the only constant solution satisfying the boundary conditions is $U = 0$, we might be tempted to conclude that the solution always decays to this trivial state. This turns out to be true, although one must invest some analysis to show it rigorously.

A second conclusion one can reach is that if $f(x) = 0$, that $U(x,t) = 0$ for all $t > 0$. This follows quickly because $W = 0$ at $t = 0$, it is non-increasing and non-negative. While this seems like a trivial result, it has a very powerful consequence. Suppose we had two solutions to the non-homogeneous Dirichlet problem, call them V_1 and V_2. You should be able to convince yourself that there difference $U = V_1 - V_2$ satisfies the homogeneous Dirichlet problem with $f(x) = 0$. Consequently, we know that $U(x,t) = 0$ for all $t > 0$, which implies $V_1 = V_2$. From this we conclude that *The solution to the non-homogeneous Dirichlet problem is unique*, a powerful result indeed.

Exercise 2.5. Convince yourself the energy argument for uniqueness of solutions in the previous paragraph is correct. Show that a similar argument can be made for the Neumann problem.

2.6 Problems for Chapter 2

Problem 2.1. Consider the diffusion equation with homogeneous Neumann boundary conditions.

$$\begin{aligned}
\mathbb{DE}: &\quad U_t = DU_{xx} && 0 < x < L, t > 0,\\
\mathbb{BC}: &\quad U_x(0,t) = 0 \quad U_x(L,t) = 0 && t > 0,\\
\mathbb{IC}: &\quad U(x,0) = f(x) && 0 < x < L.
\end{aligned}$$

(a) Explain physically why this corresponds to the diffusion of heat in a metal bar with insulated ends. Make sure you understand what each of the equations corresponds to.

(b) Show that

$$\begin{aligned}
(i) &\quad U_0(x,t) = 1\\
(ii) &\quad U_n(x,t) = \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt} \qquad n = 1, 2, 3, \dots
\end{aligned}$$

satisfy both the diffusion equation (DE) and the homogeneous Neumann boundary conditions (BC).

(c) Write down a general solution as a linear combination of the solutions you found in part (b). What does this say about $f(x)$ if we assume that this solution also satisfies the initial condition (IC)?

Problem 2.2. In this problem, we will argue that for the homogeneous Neumann problem discussed in Problem 1, that the solution approaches a constant temperature, given by the average of the initial temperature.

(a) Suppose we define the total heat energy in the bar as

$$Q(t) = \int_0^L U(x,t)\, dx.$$

Show that Q is *conserved*, that is that it is independent of time (Hint: compute $\frac{dQ}{dt}$).

(b) Use the initial condition to compute Q in terms of $f(x)$.

(c) Modify the energy argument in the previous section show that the energy is decreasing unless $U(x,t)$ is constant. Use this to argue that $U(x,t)$ approaches a constant solution as $t \to \infty$.

(d) Finally, use parts (a) and (b) of the problem to show that there is only one possible constant solution for U that is consistent with the conservation of Q. Show that solution corresponds to the bar approaching the average temperature of the initial condition.

Problem 2.3. Previously we showed that the diffusion kernel,

$$U(x,t) = G(x,t+\tau) \equiv \frac{1}{\sqrt{4\pi D(t+\tau)}} e^{-\frac{x^2}{4D(t+\tau)}},$$

satisfies the diffusion equation with an initial condition

$$U(x,0) = G(x,\tau).$$

(a) Show that the total heat ,

$$Q(t) = \int_{-\infty}^{\infty} U(x,t)\, dx,$$

is conserved, and in fact $Q(t) = 1$, for the heat kernel.

(b) Show that as $\tau \to 0$, that $G(x,\tau) \to 0$ for $x \neq 0$ and that $G(0,\tau) \to \infty$.

(c) Explain why the solution $G(x,t)$ (i.e. with $\tau = 0$) corresponds to introducing a unit amount of heat concentrated at the origin when $t = 0$. This is called a *δ-function* initial condition.

Problem 2.4. A curious property of the diffusion equation is that both derivatives and integrals of solution satisfy the diffusion equation also. This can be used to generate new solutions from existing ones.

(a) Show that if $U(x,t)$ satisfies the diffusion equation then $\psi(x,t) = U_x$ satisfies the diffusion equation also (Hint: differentiate both sides of the diffusion equation with respect to x).

(b) Generate a new solution for the heat equation by differentiating the heat kernel with respect to x. Graph this solution (MAPLE may be useful here) – what does it look like?

(c) Show that if $\psi(x,t)$ satisfies the heat equation, then so does $U(x,t) = \int_{x_0}^{x} \psi(\xi,t)\, d\xi$.

(d) Show that

$$U(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4Dt}} e^{-z^2}\, dz$$

satisfies the heat equation, by showing its derivative is the heat kernel. Graph the solution at various times – what physical problem does this solution correspond to ?

Part II

Eigenfunction Expansions and Orthogonality

Three

A Motivational Example: Solving the Diffusion Equation via Separation of Variables

In Chapter 2, we derived the homogeneous Dirichlet problem for the diffusion equation.

THE DIRICHLET PROBLEM FOR THE DIFFUSION EQUATION
(HOMOGENEOUS BOUNDARY CONDITIONS)

$$\begin{aligned}
\mathbb{DE}: &\quad u_t = \kappa u_{xx} &\quad 0 < x < \pi, t > 0 &\qquad (3.1)\\
\mathbb{BC}: &\quad u(0,t) = 0, \quad u(\pi,t) = 0 &\quad t > 0 &\qquad (3.2)\\
\mathbb{IC}: &\quad u(x,0) = f(x) &\quad 0 < x < \pi. &\qquad (3.3)
\end{aligned}$$

This equation, also called the *heat equation*, governs the heat distribution in a finite metal bar of length π, where we keep the endpoints at a fixed temperature, in our case 0. The initial temperature at time $t = 0$ is given by $f(x)$. We saw a few solutions to this system, but we didn't have a systematic way of solving the problem given a particular $f(x)$.

In this section, we will present a method to solve the equation for (essentially) any initial $f(x)$. However, along the way we will make a few assumptions and take a few shortcuts that will hopefully leave the reader with the appreciation that a deeper structure (namely Fourier Series with a soupçon of Sturm-Liouville Theory) lurks beneath the surface. These ideas will be explored in the subsequent few chapters.

3.1 The Method of Separation of Variables

In 1807 Jean Baptiste Joseph Fourier caused a big stir when he managed to solve a problem of heat dispersion using what are now called Fourier series. We will use the method he developed to solve our homogeneous Dirichlet problem[].

When solving a differential equation, it is frequently advantageous to first look for special solutions that might be easier to find then the general case. Fourier's first step was to look for solutions in the special form

$$u(x,t) = X(x)T(t). \tag{3.4}$$

Plugging this form into the differential equation $u_t = \kappa u_{xx}$, we get

$$X(x)T'(t) = \kappa X''(x)T(t)$$

and dividing by $\kappa X(x)T(t)$ we find

$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)}.$$

Notice that the left hand side is a function of t alone, while the right is a function of x only. This implies that both sides must indeed be constant! We will call this constant $-\lambda$. It is known as the *separation constant*. The reason for the negative sign in front of the λ will be apparent shortly. Thus we have

$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} = -\lambda. \tag{3.5}$$

We can separate this equation into two equations, one involving only x, one involving only t:

$$\frac{T'(t)}{\kappa T(t)} = -\lambda,$$

and

$$\frac{X''(x)}{X(x)} = -\lambda.$$

We will now assume that λ is real and positive. This assumption could plausibly cause us to lose some solutions, but eventually we will show that it is the only case that yields non-trivial solutions.

Each of these equations is now an ordinary differential equation, and thus we can draw on the theory of ordinary differential equations to solve them. The first equation,

$$T'(t) = -\lambda\kappa T(t)$$

has the solution

$$T(t) = Ce^{-\lambda\kappa t}.$$

Before we solve the second-order ordinary differential equation in x, we will derive some boundary conditions for this equation by apply the separation of variable ansatz to the boundary conditions on the PDE. Note that

$$u(0,t) = X(0)T(t) = 0$$

which implies either $X(0) = 0$ or $T(t) = 0$. Note that if we choose $T(t) = 0$ that $u(x,t) = X(x)T(t) = 0$, which, while true, is just the trivial solution. Therefore we conclude that

$$u(0,t) = X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0$$

for us to find non-trivial solutions. Similarly

$$u(\pi,t) = X(\pi)T(t) = 0 \quad \Rightarrow \quad X(\pi) = 0$$

by analogous reasoning. We have now derived the boundary value problem,

$$\mathbb{DE}: \quad X''(x) + \lambda X(x) = 0 \qquad 0 < x < \pi \tag{3.6}$$

$$\mathbb{BC}: \quad X(0) = 0, \quad X(\pi) = 0. \tag{3.7}$$

this is our first example of a *Sturm-Liouville Eigenvalue Problem*; we will show that this problem only has non-trivial solutions for certain special values of λ called *eigenvalues*.

Solving (5.1) yields

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

where we have used the fact that $\lambda > 0$. Applying the first boundary condition yields

$$X(0) = 0 \quad \Rightarrow \quad A = 0$$

so now

$$X(x) = B\sin(\sqrt{\lambda}x).$$

Applying the second boundary condition implies

$$X(\pi) = 0 \quad \Rightarrow \quad B\sin(\sqrt{\lambda}\pi) = 0.$$

so either $B = 0$ which again yields the trivial solution or $\sin(\sqrt{\lambda}\pi) = 0$ which is true only when

$$\sqrt{\lambda} = n \qquad \text{for} \quad n = 1, 2, 3, ...$$

This now yields a countable set of eigenvalues (λ_n) and associated eigen-functions ($X_n(x)$),

$$\boxed{X_n(x) = \sin(nx) \qquad \lambda_n = n^2} \tag{3.8}$$

which satisfy the eigenvalue problem. We have essentially chosen $B = 1$ in the solution we found above – don't worry, we will bring back the arbitrary constant later.

We can associated with each eigenvalue a solution to the ODE for $T(t)$,

$$T_n(t) \equiv e^{-\lambda_n \kappa t} = e^{-\lambda_n \kappa t}$$

where we note that again we have chosen the multiplicative constant $C = 1$ and the index n recognizes that we restricting ourselves to the case when $\lambda = \lambda_n$.

To summarize, we now have a countable set of solutions which satisfy both the differential equation, and the boundary values, namely

$$u_n(x,t) \equiv X_n(x)T_n(t) = e^{-n^2\kappa t}\sin(nx) \qquad n = 1, 2, 3, ...$$

However, we know that the differential equation and boundary condition are homogeneous, the solutions form a vector space!! So the most general solution is a linear combination of the u_n's.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\kappa t}\sin(nx)} \tag{3.9}$$

where the b_n are arbitrary constants.

3.2 Solving the initial value problem

To summarize, so far we have found a general solution (9.7) that satisfies both the differential equation and the associated boundary conditions for the homogeneous Dirichlet problem for the diffusion equation. We still need to satisfy the initial condition, $u(x, 0) = f(x)$ which we will argue determines the arbitrary constants b_n.

Apply the initial condition to the general solution yields

$$u(x,0) = \sum_{n=1}^{\infty} b_n u_n(x,0) = \sum_{n=1}^{\infty} b_n X_n(x) = \sum_{n=1}^{\infty} b_n \sin(nx) = f(x). \tag{3.10}$$

This infinite sum of sine functions is called a *Fourier Sine Series*. The numbers b_n are called the *Fourier Coefficients* of $f(x)$. In general the question of whether and how a Fourier Sine Series converges is rather subtle. For the moment we will ignore the technical details and blithely compute a plausible answer. First, we consider an example where the answer is straightforward; namely when the initial condition is a finite linear combination of terms of the form $\sin(nx)$.

Example 3.1. Find the solution to

$$\begin{aligned}
\mathbb{DE}: &\quad u_t = \kappa u_{xx} && 0 < x < \pi, t > 0\\
\mathbb{BC}: &\quad u(0,t) = 0 \quad u(\pi,t) = 0 && t > 0\\
\mathbb{IC}: &\quad u(x,0) = 4\sin x + 3\sin 4x && 0 < x < \pi.
\end{aligned}$$

Solution: The solution can be found by inspection; looking at the initial condition (9.12) associated with the general solution (9.7),

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) = 4\sin x + 3\sin 4x$$

we see that choosing $b_1 = 4$ and $b_4 = 3$ and setting all the remaining terms to zero yields

$$\boxed{u(x,t) = 4e^{-\kappa t}\sin(x) + 3e^{-16\kappa t}\sin(4x).}$$

Note that as we showed in Chapter 2, this solution is the unique solution to this problem. ■

To solve the general problem we first introduce an *inner product*,

$$\langle u, v\rangle \equiv \int_0^{\pi} uv\, dx.$$

which takes as arguments two real functions (here u and v) on the $x \in [0,\pi]$ and associates with them a real number. We will define the characteristics of an inner product in detail in the next section, but for the moment we will concentrate on one amazing fact, namely that the functions X_n are *orthogonal* with respect to this inner product,

$$\begin{aligned}\langle X_n(x), X_m(x)\rangle &= \langle \sin(nx), \sin(mx)\rangle \\ &= \int_0^\pi \sin(nx)\sin(mx)\,dx \\ &= \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}\end{aligned}$$

or to summarize

ORTHOGONALITY CONDITION

$$\langle X_n(x), X_m(x)\rangle = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases} \tag{3.11}$$

where the diligent reader may check the evaluation of the integral above.

We can use this identity to *project* the infinite sum of functions in the Fourier Sine Series onto a single function and thus determine the value of the Fourier coefficients, b_n. To calculate the Fourier coefficients, we compute the inner product of $X_m(x)$ with the initial condition,

$$f(x) = \sum_{n=1}^{\infty} b_n X_n(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

This yields

$$\begin{aligned}\langle X_m(x), f(x)\rangle &= \left\langle X_n(x), \sum_{n=1}^{\infty} b_n X_n(x)\right\rangle \\ &= \sum_{n=1}^{\infty} b_n \langle X_m(x), X_n(x)\rangle \\ &= b_m \langle X_m(x), X_m(x)\rangle \\ &= \frac{\pi}{2} b_m\end{aligned}$$

where we have first used the fact that the inner product is linear in its arguments (you can check this yourself or wait for the discussion in the next chapter), then noted from the orthogonality condition that $\langle X_n(x), X_m(x)\rangle$ vanishes unless $n = m$.

Solving for b_m now yields a formula to calculate the Fourier coefficients,

$$\boxed{b_m = \frac{2}{\pi}\langle X_m(x), f(x)\rangle = \frac{2}{\pi}\int_0^\pi f(x)\sin(mx)\,dx \qquad m = 1, 2, 3, \ldots.}$$

The astute reader will notice that the calculation above exchanges an infinite sum and integral. Moreover, there is no guarantee that with this choice of b_m that the Fourier sine series converges (or if that it does that it converges to the function $f(x)$). Nonetheless, we will persevere with this plausible result for the moment and show, at least numerically, that the results make sense for an example problem.

Example 3.2. Find the solution to

$$\begin{array}{rcl} \mathbb{DE}: & u_t = \kappa u_{xx} & 0 < x < \pi, t > 0 \\ \mathbb{BC}: & u(0,t) = 0 \quad u(\pi,t) = 0 & t > 0 \\ \mathbb{IC}: & u(x,0) = x(\pi - x) & 0 < x < \pi. \end{array}$$

Solution: From the solution above, we see that

$$\begin{aligned} b_m &= \frac{2}{\pi}\int_0^\pi f(x)\sin(mx)\,dx \\ &= \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin(mx)\,dx \\ &= \begin{cases} 0 & n \quad \text{even} \\ \dfrac{8}{\pi n^3} & n \quad \text{odd} \end{cases} \end{aligned}$$

This suggests the full solution is

$$\boxed{u(x,t) = \frac{8}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{e^{-n^2\kappa t}}{n^3}\sin(nx)\ .}$$

■

It is easy to show that this sum converges absolutely, and it certainly satisfies the diffusion equation and the associated boundary condition. To satisfy the initial condition, we need to check that

$$u(x,0) = x(\pi - x) = \frac{8}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{e^{-n^2\kappa t}}{n^3}\sin(nx) \qquad \text{for} \quad 0 < x < \pi.$$

We graph the solution below:

INSERT GRAPH HERE.

3.3 Solution to the Dirichlet Problem: What Just Happened?

To summarize, we have suggested that the solution to

The Dirichlet Problem for the Diffusion Equation
(Homogeneous Boundary Conditions)

$$\begin{array}{rcl} \mathbb{DE}: & u_t = \kappa u_{xx} & 0 < x < \pi, t > 0 \\ \mathbb{BC}: & u(0,t) = 0, \quad u(\pi,t) = 0 & t > 0 \\ \mathbb{IC}: & u(x,0) = f(x) & 0 < x < \pi. \end{array}$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\kappa t} \sin(nx) \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx.$$

However, the derivation has left us with a number of questions:

- Are the eigenvalues always real? Are they always positive?
- Have we found all the eigenvalues and eigenfunctions?
- Why are the eigenfunctions orthogonal?
- When does the Fourier Sine Series for the initial condition converge?
- How should we define convergence here?

We will attempt to answer these questions (or at least indicate the answers) in the next few chapters. As we will see the answers are surprisingly deep and nuanced.

3.4 Problems for Chapter 3

Problem 3.1. Find the exact solution for Dirichlet problem with initial conditions:

(a) $f(x) = \sin(x) + 5\sin(2x) + 3\sin(3x)$.

(b) $f(x) = \sin^3 x$.

Your answer should only contain a finite set of functions.

Problem 3.2. Rework the solution to the homogeneous Dirichlet problem for a bar of length L instead of length π. That is, solve

$$\begin{array}{rcl} \mathbb{DE}: & U_t = \kappa U_{xx} & 0 < x < L, t > 0 \\ \mathbb{BC}: & U(0,t) = 0 \quad U(L,t) = 0 & t > 0 \\ \mathbb{IC}: & U(x,0) = f(x) & 0 < x < L. \end{array}$$

Four

Inner Products and Orthogonal Expansions

In previous section, we used an orthogonality condition to determine the coefficients of a Fourier Sine Series. While at first this may seem to be a clever trick introduced *deus ex machina* to solve a particular problem, it is actually an example of a much deeper idea that motivates the idea of separation of variables. To understand this, we first need to remember some ideas from linear algebra, namely the concepts of an *inner product* and an *orthogonal basis*. We will begin with an example from linear algebra and then generalize the machinery we developed for vectors to deal with functions.

4.1 Orthogonal Bases: An Example in $\mathbb{R}^3$

Consider the following example from linear algebra,

Example 4.1. Solve the linear system

$$\vec{v} = c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 = \vec{v} \tag{4.1}$$

where $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, and $\vec{v}$ are vectors in $\mathbb{R}^3$,

$$\vec{e}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Solution: The temptation here is rewrite the problem as a linear system

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

and just use gaussian elimination to solve for the unknowns. However the astute reader will notice something about the vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Remember that we say two vectors $\{\vec{p}, \vec{q}\}$ are perpendicular or *orthogonal* if their dot product is zero, $\vec{p} \cdot \vec{q} = 0$. In fact, the vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are an *orthogonal basis* because they satisfy the orthogonality condition

$$\vec{e}_i \cdot \vec{e}_j = 0 \qquad \text{if} \quad i \neq j$$

and because they span $\mathbb{R}^3$ (so they form a basis).

We can now solve for each c_i by *projecting* onto the vector $\vec{e}_i$. We can rewrite (4.1) using summation notation,

$$\vec{v} = \sum_{i=1}^{3} c_i \, \vec{e}_i$$

and take the dot product of each side with $\vec{e}_j$ to yield

$$\begin{aligned} \vec{e}_j \cdot \vec{v} &= \vec{e}_j \cdot \left(\sum_{i=1}^{3} c_i \, \vec{e}_i \right) \\ &= \sum_{i=1}^{3} c_i \, \vec{e}_j \cdot \vec{e}_i \\ &= c_j \, \vec{e}_j \cdot \vec{e}_j . \end{aligned}$$

solving for c_j yields

$$\boxed{c_j = \frac{\vec{e}_j \cdot \vec{v}}{\vec{e}_j \cdot \vec{e}_j} = \frac{\vec{e}_j \cdot \vec{v}}{|\vec{e}_j|^2}} \tag{4.2}$$

where we have used the fact that the dot product of a vector with itself is its length squared, $\vec{p} \cdot \vec{p} = |\vec{p}|^2$.

We leave the algebra as an exercise for the reader, but using this formula we discover that

$$c_1 = -\frac{1}{2} \qquad c_2 = 2 \qquad c_3 = -\frac{1}{2}$$

so

$$\boxed{\vec{v} = -\frac{1}{2}\vec{e}_1 + 2\vec{e}_2 - \frac{1}{2}\vec{e}_3 .}$$

■

The take-away lesson from this example is that one can expand a function in an orthogonal basis easily by project onto the basis elements. Let us now generalize this idea from vectors to functions.

4.2 Inner Products and Inner Product Spaces

In the previous section we introduced an inner product of two functions u and v on an interval $x \in [a, b]$,

$$\langle u, v\rangle \equiv \int_a^b uv\, dx. \tag{4.3}$$

More generally, a *real inner product* (later we will talk about complex inner products) is a function from two elements of a vector space to the real numbers that satisfies three properties

- **Symmetry:** The inner product is symmetric in its arguments; that is

  $$\langle u, v\rangle = \langle v, u\rangle$$

  for all elements u and v in the vector space.

- **Linearity:** The inner product is linear in each of its arguments; that is

  $$\langle u, \alpha v + \beta w\rangle = \alpha \langle u, v\rangle + \beta \langle u, w\rangle$$

  for all real numbers α and β and any u, v and w in the vector space. Note the symmetry properties implies that linearity in the second argument guarantees linearity in the first argument and vice versa.

- **Positive Definiteness:** We say an inner-product is positive definite if

  $$\langle u, u\rangle \geq 0$$

  with equality when $u = 0$ only[1]. This allows to define the *norm* of a vector,

  $$\|u\| \equiv \sqrt{\langle u, u\rangle}.$$

An *inner product space* is simply a vector space together with an inner product defined on that vector space.

Exercise 4.1. Show the dot product is an inner product for vectors in $\mathbb{R}^n$. Show the norm of a vector with this inner product is its length. Explain why $\mathbb{R}^n$ is an inner-product space.

[1]This assumes the function is continuous; if one chooses to venture into the intricate machinery of measure theory and allow for discontinuous functions one would write "$u = 0$ *almost everywhere*".

Exercise 4.2. Show that

$$\|a-b\|^2 = \|a\|^2 + \|b\|^2 - 2\langle a, b\rangle .$$

Explain how this is related to the law of cosines in $\mathbb{R}^n$.

One very important inner product space is the set of *square-integrable functions*. For our inner product (8.2) above, the set of functions u whose norm is finite,

$$\|u\| \equiv \sqrt{\int_a^b u^2\, dx},$$

is the vector space of square integrable functions, usually refereed to as $L^2[a,b]$. The L here refers to the french mathematician Henri Lebesgue and strictly speaking, the integral should be a *Lebesgue integral* which one generally learns about in a course on measure theory as opposed to the Riemann integral one learns about in calculus. We can safely ignore this nuance for the moment.

4.3 Orthogonal Sets

We can now define an *orthogonal set* of functions,

Definition 4.4. Given a set of non-zero elements $\{\vec{e}_i\}$ of a vector space, $\mathcal{V}$, and an inner product that acts on elements of a vector space, we say the set is an *orthogonal set* if $\vec{e}_i \cdot \vec{e}_j = 0$ if $i \neq j$.

We have already seen two examples of orthogonal sets; in Section (4.1) the vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

are an orthogonal set in $\mathbb{R}^3$ with respect to the dot product. We also used the orthogonality of the set

$$\{\sin(nx)\} \qquad \text{for} \quad n = 1, 2, 3 \cdots$$

in $L^2[0,\pi]$ to solve for the initial condition of the diffusion heat equation solution we found via separation of variables in Chapter (3) . The amazing thing is that orthogonal sets arise naturally when we apply separation of variables due to the ideas of Sturm-Liouville Theory which we discuss in the next chapter.

Exercise 4.3. Prove the elements of an orthogonal set are linearly independent.

Exercise 4.4. Verify that the following sets are orthogonal

(a) $\{\sin\left(\frac{n\pi x}{\ell}\right)\}$ for $n = 1, 2, 3 \cdots$ in $L^2[0, \ell]$ where $\ell > 0$.

(b) $\{1, x, 3x^2 - 2, 5x^3 - 3x\}$ in $L^2[-1, 1]$.

(c) $\{\cos(nx)\}$ for $n = 0, 1, 2, 3 \cdots$ in $L^2[0, \pi]$.

4.4 Orthogonal Expansions and Least Squares Minimization

A natural question to ask is what is the "best" expansion of a function in terms of a linear combination of an orthogonal set of functions; suppose we are trying to expand a function $f(x)$ in terms of an orthogonal set $\{e_i\}$

$$F \approx \sum_{n=1}^{N} a_n e_n. \tag{4.5}$$

where the a_i are to be determined. A logical choice is to set the projection of both sides of this expression onto the vector e_m equal to each other

$$\begin{aligned}
\langle e_m, F\rangle &\approx \left\langle e_m, \sum_{n=1}^{N} c_n e_n)\right\rangle \\
&\approx \sum_{n=1}^{N} c_n \langle e_m, e_n\rangle \\
&\approx c_m \langle e_m, e_m\rangle \\
&\approx c_m \|e_m\|^2
\end{aligned}$$

which suggests that

$$c_m = \frac{\langle e_m, F\rangle}{\|e_m\|^2}. \tag{4.6}$$

In fact this choice is the best approximation in the *least squares* sense. This is an important result, so we state it as a theorem

Theorem 4.1. *Suppose F is an element of the inner product space $\mathcal{V}$ and $\{e_i\}$ are an orthogonal set in $\mathcal{V}$. The approximation of F by a linear combination of the e_i's,*

$$S_N(x) = \sum_{n=1}^{N} a_n e_n,$$

that minimizes the error

$$E_N = \|F - S_N\|$$

is to choose $a_n = c_n$ *where*

$$c_n \equiv \frac{\langle e_n, F\rangle}{\|e_n\|^2}.$$

Proof. First we note that

$$\begin{aligned}
\|S_N\|^2 &= \langle S_N, S_N\rangle && (4.7)\\
&= \left\langle \sum_{n=1}^{N} a_n e_n, \sum_{m=1}^{N} a_m e_m \right\rangle && (4.8)\\
&= \sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_m \langle e_n, e_m\rangle && (4.9)\\
&= \sum_{n=1}^{N} (a_n)^2 \|e_n\|^2. && (4.10)
\end{aligned}$$

Now we expand the square of the error

$$\begin{aligned}
(E_N)^2 &= \|F - S_N\|^2\\
&= \|F\|^2 + \|S_N\|^2 - 2\langle F, S_N\rangle\\
&= \|F\|^2 + \sum_{n=1}^{N}(a_n)^2\|e_n\|^2 - 2\left\langle F, \sum_{n=1}^{N} a_n e_n\right\rangle \qquad \boxed{\text{Using (4.10)}}\\
&= \|F\|^2 + \sum_{n=1}^{N}(a_n)^2\|e_n\|^2 - 2\sum_{n=1}^{N} a_n \langle F, e_n\rangle\\
&= \|F\|^2 + \sum_{n=1}^{N}(a_n)^2\|e_n\|^2 - 2\sum_{n=1}^{N} a_n c_n \|e_n\|^2\\
&= \|F\|^2 + \sum_{n=1}^{N}[(a_n)^2 - 2a_n c_n]\|e_n\|^2\\
&= \|F\|^2 + \sum_{n=1}^{N}\left[(a_n - c_n)^2 - c_n^2\right]\|e_n\|^2
\end{aligned}$$

This expression is minimized when the terms proportional to $(a_n - c_n)^2$ vanish, that is when we choose $a_n = c_n$, which completes the proof. □

A nice corollary to this result is that with this choice of coefficients, the square of the error can be written as

$$(E_N)^2 = \|F\|^2 - \sum_{n=1}^{N}(c_n)^2\|e_n\|^2. \tag{4.11}$$

4.5 Bessel's Inequality and Parseval's Identity

Because the square of the error is positive, $(E_N)^2$ an immediate consequence of (4.12) is *Bessel's Inequality*,

$$\|F\|^2 \geq \sum_{n=1}^{N} (c_n)^2 \|e_n\|^2. \tag{4.12}$$

Bessel's inequality can be thought of as a measure of the goodness of the least squares approximation. We can now define when an *orthogonal set* is an *orthogonal basis*. The term basis employs that the set is complete; that is any element in the vector space can be written as a linear combination of the elements of the set. For a finite dimensional space, such as $\mathbb{R}^n$, this definition is clear. However, the space $L^2[a,b]$ is infinite dimensional, so one must be a little more careful. We say that an orthogonal set, $\{e_i\}$, is complete in L^2 if

$$\lim_{N\to\infty} E_N = 0.$$

In this case Bessel's Inequality becomes *Parseval's Identity*,

$$\|F\|^2 = \sum_{n=1}^{\infty} (c_n)^2 \|e_n\|^2. \tag{4.13}$$

which is true for any orthogonal basis in L^2. This can be a useful practical check of completeness and convergence of an expansion, as we will see in our discussion of Fourier series.

Exercise 4.5. In $\mathbb{R}^N$, Parseval's Identity states that

$$\|F\|^2 = \sum_{n=1}^{N} (c_n)^2 \|e_n\|^2.$$

Verify this identity for the $\mathbb{R}^3$ example in Section (4.1). Explain how this result relates to Pythagoras' Theorem in $\mathbb{R}^N$.

Five

Sturm-Liouville Eigenvalue Problems

In Chapter (3), we showed the boundary value problem

$$\mathbb{DE}: \quad X''(x) + \lambda X(x) = 0 \qquad 0 < x < \pi \tag{5.1}$$

$$\mathbb{BC}: \quad X(0) = 0, \quad X(\pi) = 0. \tag{5.2}$$

yielded a countable set of real positive eigenvalues $\{\lambda_n\}$ and associated eigenfunctions $\{X_n(x)\}$,

$$X_n(x) = \sin(nx) \qquad \lambda_n = n^2 \qquad n = 1, 2, 3, \cdots \tag{5.3}$$

that were orthogonal in the $L^2[0, \pi]$ inner-product,

$$\langle X_n(x), X_m(x) \rangle \equiv \int_0^\pi X_n(x) X_m(x)\, dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases} \tag{5.4}$$

In this section, we will show that the existence of this orthogonal set is not a coincidence, but a consequence of the form of this boundary value problem. This is an example of a *Sturm-Liouville Eigenvalue Problem* or SLEP; we will explore some of the properties associated with these problems below. We will concentrate on a class of problems associated with Fourier Series which we call the *Fourier Eigenvalue Problem*.

To understand SLEPs perhaps the best analogy is the matrix eigenvalue problem from Linear Algebra. Remember, if we have a real symmetric $N \times N$ matrix A, that the eigenvalue problem

$$A\vec{x} = \lambda \vec{x} \qquad \vec{x} \in \mathbb{R}^N$$

has the following properties:

- The matrix A has N real eigenvalues, each associated with a real eigenvector,
- Eigenvectors associated with different eigenvalues are orthogonal with respect to the dot product in $\mathbb{R}^N$,
- If the matrix is positive definite, the eigenvalues are positive.

We explore the analogous problem for SLEPs below.

5.1 The Fourier Eigenvalue Problem

In this Section we consider an example of a Sturm-Liouville Eigenvalue Problem associated with Fourier Series:

Definition 5.5 (Fourier Eigenvalue Problem)**.** Let $y(x)$ be a twice continuously differentiable function on the interval $a \leq x \leq b$ (i.e. $y(x) \in C^2[a, b]$) . Let $\mathcal{L}$ be a differential operator defined by $\mathcal{L}y \equiv -y''$. The *Fourier Eigenvalue Problem* is defined by the differential equation

$$\mathbb{DE}: \qquad \mathcal{L}y = \lambda y, \quad a < x < b. \tag{5.6}$$

Together with one of the boundary conditions

$$\begin{array}{llrll} \text{(D)} & \mathbb{BC}: & y(a) = 0, & y(b) = 0 & \text{Dirichlet} \\ \text{(N)} & \mathbb{BC}: & y'(a) = 0, & y'(b) = 0 & \text{Neumann} \\ \text{(P)} & \mathbb{BC}: & y(a) = y(b), & y'(a) = y'(b) & \text{Periodic} \end{array}$$

we call a constant λ and a non-zero function y that satisfy this problem an *eigenvalue/eigenfunction pair*.

We will prove a set of theorems about the eigenfunctions and eigenvalues of this problem, but first we need to introduce an appropriate vector space of real functions and an inner product. Define

$$\mathcal{U}_D \equiv \{y(x) \in C^2[a, b], y(a) = 0, y(b) = 0\}, \qquad \text{Dirichlet} \tag{5.7}$$

$$\mathcal{U}_N \equiv \{y(x) \in C^2[a, b], y'(a) = 0, y'(b) = 0\}, \qquad \text{Neumann} \tag{5.8}$$

$$\mathcal{U}_P \equiv \{y(x) \in C^2[a, b], y(a) = y(b), y'(a) = y'(b)\}. \qquad \text{Periodic} \tag{5.9}$$

The idea here is we only consider the set of functions that satisfy the boundary conditions.

Exercise 5.1. Show that $\mathcal{U}_D$, $\mathcal{U}_N$, and $\mathcal{U}_P$ are vector spaces.

If we associate the usual L^2 inner product

$$\langle u, v \rangle = \int_a^b uv \, dx.$$

with these spaces, we see that they are also real inner product spaces.

5.2 Self-adjoint Operators

We say a differential operator $\mathcal{P}$ acting on elements in an inner product space is *self-adjoint* if

$$\langle u, \mathcal{P}v \rangle = \langle \mathcal{P}u, v \rangle .$$

If we consider the above statement in the inner product space of $\mathbb{R}^N$ with the usual dot product, this is exactly the condition for $\mathcal{P}$ to be an $N \times N$ symmetric matrix. Let us apply this idea to the Fourier Eigenvalue Problem.

Theorem 5.1. *The differential operator $\mathcal{L}$ is self-adjoint in the inner product spaces $\mathcal{U}_D$, $\mathcal{U}_N$, or $\mathcal{U}_P$.*

Proof. The proof follows by integration by parts; suppose u and v are elements of one of the inner product spaces. Then

$$\begin{aligned}
\langle u, \mathcal{L}v \rangle &= -\int_a^b uv'' \, dx \\
&= \int_a^b u'v' \, dx - uv'|_{x=a}^{x=b} \\
&= -\int_a^b u''v \, dx + u'v - uv'|_{x=a}^{x=b} \\
&= \langle \mathcal{L}u, v \rangle + u'v - uv'|_{x=a}^{x=b} .
\end{aligned}$$

If u and v satisfy the Neumann or Dirichlet boundary conditions then $u'v - uv' = 0$ at each endpoint. Similarly if u and v satisfy the periodic boundary conditions, then $u'v - uv'$ is the same at $x = a$ and $x = b$ so $u'v - uv'|_{x=a}^{x=b}$ vanishes. Therefore

$$\langle u, \mathcal{L}v \rangle = \langle \mathcal{L}u, v \rangle ,$$

and we have shown the operator is self-adjoint. □

5.2.1 The Eigenvalues of a Self-adjoint Operator are Real

In a moment, we will show the eigenvalues of a self-adjoint operator are real. Perhaps the first question to ask is:

Why might we consider the possibility that the eigenvalues could be complex?

If we think about the matrix eigenvalue problem in $\mathbb{R}^N$, we know that for a general matrix A the eigenvalues satisfy the *characteristic polynomial*, $P(\lambda) = \det(A - \lambda I)$. This real polynomial may have roots that are real or that occur in complex conjugate pairs. Only for special classes of matrices, such as symmetric matrices, do we know *a priori* that the eigenvalues are real. To show the eigenvalues of a self-adjoint operator are real, we will use an argument that is analogous to that used to show the eigenvalues of a real symmetric matrix are real.

Theorem 5.2. *Suppose $\mathcal{L}$ is self-adjoint linear operator associated with the inner product space $\mathcal{U}$. Then the eigenvalue problem*

$$\mathcal{L}y = \lambda y$$

where

$$y = p + iq \qquad p, q \in \mathcal{U}$$

has only real eigenvalues, λ.

Remark. Note that we have temporarily expanded the eigenvalue problem to allow y to be a complex function with real part p and imaginary part q (we know p and q are real functions because they are in the real inner product space $\mathcal{U}$). Also, for the Fourier Eigenvalue Problem remember that the boundary conditions are hidden in the definition of the vector space.

Proof. First we need to be clear about how $\mathcal{L}$ acts on the complex function y; it is linear so

$$\mathcal{L}y = \mathcal{L}(p + iq) = \mathcal{L}p + i\mathcal{L}q.$$

next define the complex conjugate of y,

$$\bar{y} = p - iq.$$

We now use linearity to extend the definition of the real inner product to complex functions. Consider

$$\begin{aligned}
\langle \bar{y}, \mathcal{L}y \rangle &= \langle p - iq, \mathcal{L}p + i\mathcal{L}q \rangle \\
&= \langle p, \mathcal{L}p \rangle + \langle q, \mathcal{L}q \rangle + i(\langle p, \mathcal{L}q \rangle - \langle q, \mathcal{L}p \rangle) \\
&= \langle p, \mathcal{L}p \rangle + \langle q, \mathcal{L}q \rangle
\end{aligned}$$

where we have used the fact that the operator $\mathcal{L}$ is self-adjoint and the symmetry of the inner product to see that $\langle p, \mathcal{L}q\rangle = \langle q, \mathcal{L}p\rangle$. From the eigenvalue problem and linearity we also know

$$\begin{aligned}\langle \bar{y}, \mathcal{L}y\rangle &= \langle \bar{y}, \lambda y\rangle \\ &= \lambda \langle \bar{y}, y\rangle \\ &= \lambda \langle p - iq, p + iq)\rangle \\ &= \lambda \left[\langle p, p\rangle + \langle q, q\rangle + i(\langle p, q\rangle - \langle q, p\rangle)\right] \\ &= \lambda(\|p\|^2 + \|q\|^2)\end{aligned}$$

Now, solving for λ from these expressions yields

$$\lambda = \frac{\langle p, \mathcal{L}p\rangle + \langle q, \mathcal{L}q\rangle}{\|p\|^2 + \|q\|^2}.$$

The quotient on the righthand side is real, therefore λ is real. As a side note, if we now rewrite the eigenvalue problem,

$$\mathcal{L}y = \lambda y \Rightarrow \mathcal{L}(p + iq) = \lambda(p + iq),$$

and equate real and imaginary parts,

$$\mathcal{L}p = \lambda p, \qquad \mathcal{L}q = \lambda q,$$

we see that both the real and imaginary parts of y are eigenfunctions; that is, the real eigenvalue λ can be associated with a real eigenfunction (either p and/or q which can't both be zero). □

Exercise 5.2. Use this proof to convince yourself that a real $N \times N$ symmetric matrix has real eigenvalues.

5.2.2 Orthogonality of Eigenfunctions

Eigenfunctions associated with self-adjoint operators inherit a natural orthogonality from the inner product space.

Theorem 5.3. *Suppose $\mathcal{L}$ is self-adjoint linear operator associated with the inner product space $\mathcal{U}$. If y_n and y_m are eigenfunctions with distinct associated eigenvalues $\lambda_n \neq \lambda_m$ for the eigenvalue problem*

$$\mathcal{L}y = \lambda y$$

then the eigenfunctions are orthogonal, that is

$$\langle y_m, y_n\rangle = 0.$$

Proof. From the self-adjointness of $\mathcal{L}$ we see that

$$\langle y_m, \mathcal{L}y_n \rangle = \langle \mathcal{L}y_m, y_n \rangle$$

and from the eigenvalue problem and linearity this implies

$$\lambda_n \langle y_m, y_n \rangle = \lambda_m \langle y_m, y_n \rangle .$$

Rearranging yields

$$(\lambda_n - \lambda_m) \langle y_m, y_n \rangle = 0.$$

As $\lambda_n \neq \lambda_m$ we conclude $\langle y_m, y_n \rangle = 0$, that is the eigenfunctions are orthogonal. □

The fact that the eigenvalues are real and the eigenfunctions are orthogonal depended solely on the operator $\mathcal{L}$ being self-adjoint. If we specialize to the Fourier Eigenvalue Problem we can also deduce some results about the sign of the eigenvalues.

5.3 Solving the Fourier Eigenvalue Problem

In this Section we will solve the Fourier Eigenvalue Problem for the three different boundary conditions (Dirichlet, Neumann and Periodic). First however we can prove a theorem about the non-negativity of the eigenvalues which will save us some work.

Theorem 5.4. *Suppose y and λ are an eigenvalue/eigenfunction pair for the Fourier Eigenvalue Problem. Then $\lambda \geq 0$. Moreover, If $\lambda = 0$ is an eigenvalue then then the associated eigenfunction is constant.*

Proof. We know that

$$\langle y_n, \mathcal{L}y_n \rangle = \lambda \langle y_n, y_n \rangle = \lambda \|y\|^2$$

however, using integration by parts, we also know that

$$\begin{aligned} \langle y_n, \mathcal{L}y_n \rangle &= -\int_a^b yy''\, dx, \\ &= \int_a^b y'y'\, dx - yy'\big|_{x=a}^{x=b}, \\ &= \|y'\|^2, \end{aligned}$$

where we have used the fact that the boundary term yy' vanishes at each end point for the Dirichlet and Neumann problems, and the endpoint contributions cancel for the periodic problem. Now, solving for λ from these expressions yields

$$\lambda = \frac{\|y'|^2}{\|y\|^2}\,.$$

Clearly, the right-hand side is non-negative. Moreover, if $\lambda = 0$ then $y' = 0$ for $a < x < b$, that is if y is constant. So we conclude that $\lambda \geq 0$ and if $\lambda = 0$, then y is constant. □

We can now quickly solve the Fourier Eigenvalue Problem for the three boundary conditions

5.3.1 The Fourier Eigenvalue Problem: Dirichlet Boundary Conditions

We essentially solved this problem already in Chapter 3. For simplicity we will solve the problem on the interval $0 < x < \ell$. We wish to find non-zero, twice differentiable eigenfunctions $y_n(x)$ and associated eigenfunctions λ_n solving

$$\begin{aligned} \mathbb{DE}: &\qquad -y'' = \lambda y, \quad 0 < x < \ell, \\ \mathbb{BC}: &\qquad y(0) = 0, \quad y(\ell) = 0. \end{aligned}$$

we know that λ is real and non-negative from Theorem 5.4. If $\lambda = 0$, y is constant, but as $y(0) = 0$ this yields only the trivial solution $y(x) = 0$. If $\lambda > 0$, then

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Applying the first boundary condition yields

$$y(0) = 0 \quad \Rightarrow \quad A = 0$$

so now

$$y(x) = B\sin(\sqrt{\lambda}x).$$

Applying the second boundary condition implies

$$y(\ell) = 0 \quad \Rightarrow \quad B\sin(\sqrt{\lambda}\ell) = 0.$$

so either $B = 0$ which again yields the trivial solution or $\sin(\sqrt{\lambda}\pi) = 0$ which is true only when

$$\sqrt{\lambda} = \frac{n\pi}{\ell} \qquad \text{for} \quad n = 1, 2, 3, \ldots.$$

This now yields a countable set of eigenvalues and associated eigenfunctions,

$$\boxed{y_n(x) = \sin\left(\frac{n\pi x}{\ell}\right) \qquad \lambda_n = \frac{n^2\pi^2}{\ell^2} \qquad \text{for} \quad n = 1, 2, 3, \ldots.} \tag{5.10}$$

which satisfy the eigenvalue problem. We have set the arbitrary constant $B = 1$ in the solution we found above – don't worry, when we look for a solution or construct a Fourier Series, it is a linear combination of the elements of this orthogonal set so we will bring back the arbitrary constant later. In the next Chapter we will show that the orthogonal set generated here is associated with the *Fourier Sine Series*.

5.3.2 The Fourier Eigenvalue Problem: Neumann Boundary Conditions

Again, for simplicity we will solve the problem on the interval $0 < x < \ell$. We wish to find non-zero, twice differentiable eigenfunctions $y_n(x)$ and associated eigenfunctions λ_n solving

$$\begin{aligned} \mathbb{DE}: \qquad & -y'' = \lambda y, \quad 0 < x < \ell, \\ \mathbb{BC}: \qquad & y'(0) = 0, \quad y'(\ell) = 0. \end{aligned}$$

we know that λ is real and non-negative from Theorem 5.4. If $\lambda = 0$, y is a constant, which satisfies both the DE and the Neumann boundary conditions. This yields

$$y_0(x) = \frac{1}{2} \qquad \lambda_0 = 0$$

where we have chosen the value $y_0 = \frac{1}{2}$ to agree with a convention that arises in the definition of a Fourier Series. If $\lambda > 0$, then

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

and

$$y'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

Applying the first boundary condition yields

$$y'(0) = 0 \quad \Rightarrow \quad B = 0$$

so now

$$y(x) = A\cos(\sqrt{\lambda}x).$$

Applying the second boundary condition implies

$$y'(\ell) = 0 \quad \Rightarrow \quad -A\sqrt{\lambda}\sin(\sqrt{\lambda}\ell) = 0.$$

so either $A = 0$ which again yields the trivial solution or $\sin(\sqrt{\lambda}\pi) = 0$ which is true only when

$$\sqrt{\lambda} = \frac{n\pi}{\ell} \qquad \text{for} \quad n = 1, 2, 3, \ldots.$$

Putting this together with the constant solution now yields a countable set of eigenvalues and associated eigenfunctions,

$$\boxed{\begin{array}{lll} y_0(x) = \dfrac{1}{2} & \lambda_0 = 0 & \\ y_n(x) = \cos\left(\dfrac{n\pi x}{\ell}\right) & \lambda_n = \dfrac{n^2\pi^2}{\ell^2} & \text{for} \quad n = 1, 2, 3, \ldots. \end{array}} \tag{5.11}$$

which satisfy the eigenvalue problem. In the next section will show that the orthogonal set generated here is associated with the *Fourier Cosine Series*.

5.3.3 The Fourier Eigenvalue Problem: Periodic Boundary Conditions

We solve the periodic problem on the interval $-\ell < x < \ell$. We wish to find non-zero, twice differentiable eigenfunctions $y_n(x)$ and associated eigenfunctions λ_n solving

$$\mathbb{DE}: \qquad -y'' = \lambda y, \quad -\ell < x < \ell, \tag{5.12}$$

$$\mathbb{BC}: \qquad y(-\ell) = y(\ell), \quad y'(-\ell) = y'(\ell). \tag{5.13}$$

we know that λ is real and non-negative from Theorem 5.4. If $\lambda = 0$, y is a constant, which satisfies both the DE and the periodic boundary conditions. This yields

$$y_0(x) = \frac{1}{2} \qquad \lambda_0 = 0$$

where again we have chosen the value $y_0 = \frac{1}{2}$ to agree with a convention that arises in the definition of a Fourier Series. If $\lambda > 0$, then

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

and

$$y'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

Applying the first boundary condition yields

$$y(-\ell) = y(\ell) \quad \Rightarrow \quad A\cos(\sqrt{\lambda}\ell) - B\sin(\sqrt{\lambda}\ell) = A\cos(\sqrt{\lambda}\ell) + B\sin(\sqrt{\lambda}\ell)$$

which implies

$$2B\sin(\sqrt{\lambda}\ell) = 0$$

Applying the second boundary condition implies

$$y'(-\ell) = y'(\ell) \quad \Rightarrow \quad A\sqrt{\lambda}\sin(\sqrt{\lambda}\ell) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\ell) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\ell) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\ell).$$

which implies

$$2A\sqrt{\lambda}\sin(\sqrt{\lambda}\ell) = 0$$

If A and B are not both zero and remember that $\lambda > 0$, we see that the eigenvalue condition reduces to

$$\sin(\sqrt{\lambda}\ell) = 0$$

or

$$\sqrt{\lambda} = \frac{n\pi}{\ell} \qquad \text{for} \quad n = 1, 2, 3, \ldots.$$

Note that in this case there are two linear independent eigenfunctions $\cos\left(\frac{n\pi x}{\ell}\right)$ and $\sin\left(\frac{n\pi x}{\ell}\right)$ associated with the eigenvalue. Putting this together with the constant solution now yields a countable set of eigenvalues and associated eigenfunctions,

$$\boxed{\begin{aligned} y_0(x) &= \frac{1}{2} & \lambda_0 &= 0 & & \\ y_n^c(x) &= \cos\left(\frac{n\pi x}{\ell}\right) & \lambda_n &= \frac{n^2\pi^2}{\ell^2} & \text{for} \quad & n = 1, 2, 3, \ldots. \\ y_n^s(x) &= \sin\left(\frac{n\pi x}{\ell}\right) & \lambda_n &= \frac{n^2\pi^2}{\ell^2} & \text{for} \quad & n = 1, 2, 3, \ldots. \end{aligned}} \tag{5.14}$$

which satisfy the eigenvalue problem. Strictly speaking, we need to check the orthogonal of the eigenfunctions $y_n^c(x)$ and $y_n^s(x)$ because they are associated with the same eigenvalue. In practice,

$$\langle y_n^c(x), y_n^s(x)\rangle \equiv \int_{-\ell}^{\ell} \cos\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right) dx = 0$$

is guaranteed as one of the functions is odd and the other is even. In the next section will show that the orthogonal set generated here is associated with the general *Fourier Series*.

Part III

Fourier Series

Six

Fourier Series I: Definitions and Least Squares Approximation

A Fourier Series approximates a periodic function by a sum of sines and cosines of the same period. From the oscillation of a guitar string to the compression of images, an idea Joseph Fourier employed to approximate the solution to the heat equation has grown into a valuable tool in many disciplines include math, science and engineering. Below we develop the basic tools need to expand a function into a Fourier Series. In the next two chapters we study the nature of convergence for these expansions.

6.1 FOURIER SERIES: THE BASICS

In the previous Chapter we used Sturm-Liouville Theory to generate three sets of orthogonal functions that will be the bases for studying Fourier Series. For periodic functions on the interval $x \in [-\ell, \ell]$ we showed the set of functions

$$\left\{\frac{1}{2},\ \cos\left(\frac{n\pi x}{\ell}\right),\ \sin\left(\frac{n\pi x}{\ell}\right)\right\}$$

are orthogonal in the L^2 inner-product. This allows us to propose the *Fourier Series* expansion for a function $f(x)$,

$$\mathbb{FS}[f(x)] \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right). \tag{6.1}$$

Note that as the Fourier Series is the sum of periodic functions with period 2ℓ, it also is 2ℓ-periodic. We know that the choice of coefficients

$$\begin{aligned} a_0 &= \frac{\left\langle \frac{1}{2}, f(x)\right\rangle}{\left\|\frac{1}{2}\right\|^2} = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx, \\ a_n &= \frac{\left\langle \cos\left(\frac{n\pi x}{\ell}\right), f(x)\right\rangle}{\left\|\cos\left(\frac{n\pi x}{\ell}\right)\right\|^2} = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx, \\ b_n &= \frac{\left\langle \sin\left(\frac{n\pi x}{\ell}\right), f(x)\right\rangle}{\left\|\sin\left(\frac{n\pi x}{\ell}\right)\right\|^2} = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx, \end{aligned}$$

will minimize the L^2-error between the function and the Fourier Series approximation. We can also now explain the curious choice of a constant term of $\frac{1}{2}$. This choice means that the formula for a_n is exactly the same as the formula for a_0 when $n = 0$.

Example 6.1. Find the Fourier series for

$$g(x) = x^2 \qquad -\pi \leq x \leq \pi\ .$$

Graph some partial sums of the series and examine their convergence.
Solution: Note that we take $\ell = \pi$ in the formulae above, because the interval is $x \in [-\pi, \pi]$. The sine terms in the Fourier series will vanish,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(nx)\ dx = 0,$$

which is easily seen because $g(x)$ is an even function and $\sin(nx)$ is odd. So the Fourier series for $g(x)$ will only involve cosines,

$$\mathbb{FS}[g(x)] = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \tag{6.2}$$

The coefficients are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)\ dx.$$

We see that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\ \cos xt\,dx = \frac{1}{\pi}\left[\frac{2x\ \cos nx}{n^2} + \frac{(n^2x^2-2)\sin nx}{n^3}\right]_{x=-\pi}^{\pi} = \frac{4(-1)^n}{n^2},$$

when $n = 1, 2, 3, \cdots$. When $n = 0$,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dt = \frac{2\pi^2}{3}.$$

Therefore,

$$\mathbb{FS}[g(x)] = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx.$$

We can define the *partial sums* of this series,

$$S_N(x) = \frac{\pi^2}{3} + \sum_{n=1}^{N} \frac{4(-1)^n}{n^2}\cos nx.$$

which we graph below, for $N = 1, 2, 4, 8$ and 16 terms along with $g(x) = x^2$. Note that the partial sums appear to converge quickly and uniformly to $g(x) = x^2$.

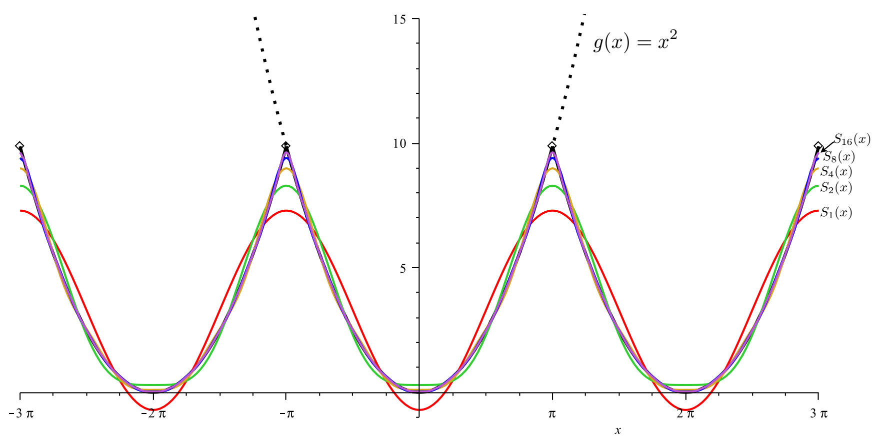


Figure 6.1: The partial sums, $S_N(x)$, for $N = 1, 2, 4, 8$ and 16. Note the partial sums converge to the function $g(x) = x^2$ (dotted) in the range $-\pi < x < \pi$. Outside this range the Fourier series converges to the 2π-periodic extension of $g(x)$ which is indicated in black.

■

The Fourier Series (6.1) is 2ℓ-periodic and although it appears to converge to $g(x)$ on the interval $x \in [-\pi, \pi]$, it periodically repeats to either side of this function. We call this function the *periodic extension* of f.

Definition 6.3. Suppose a function, $f(x)$, is defined on an interval $x \in [-\ell, \ell)$. We define the *periodic extension* of $f(x)$, $\tilde{f}(x)$ to be the unique function such that $\tilde{f}(x) = f(x)$ for $x \in [-\ell, \ell)$ and $\tilde{f}(x + 2\ell) = \tilde{f}(x)$.

The first condition assures agreement on the *basic period*, $x \in [-\ell, \ell)$, and the second condition assures the function is 2ℓ-periodic.

We know that from Theorem 4.1 that the Fourier series is the best approximation of a periodic function in the least squares sense on the interval $x \in [-\ell, \ell]$. We want to say something like:

Slogan 1. The Fourier Series

$$\mathbb{FS}[f(x)] \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

with coefficients

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right)\, dx \qquad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right)\, dx,$$

converges to $\tilde{f}(x)$, the periodic extension of $f(x)$ from $x \in [-\ell, \ell)$.

The difficulty here is the meaning of the word *converges*. We will investigate some details of the convergence of Fourier series in the next two chapters.

6.2 The Fourier Sine Series

Similarly, for functions on the interval $x \in [0, \ell]$ which vanish at the endpoint we showed the set of functions

$$\left\{\sin\left(\frac{n\pi x}{\ell}\right)\right\}$$

are orthogonal in the L^2 inner-product. This allows us to propose the *Fourier Sine Series* expansion for a function $f(x)$,

$$\mathbb{FSS}[f(x)] \equiv \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right). \tag{6.4}$$

as a natural expansion for functions that vanish at the endpoints of the interval $[0, \ell]$. We know that the choice of coefficients

$$b_n = \frac{\left\langle \sin\left(\frac{n\pi x}{\ell}\right), f(x)\right\rangle}{\left\|\sin\left(\frac{n\pi x}{\ell}\right)\right\|^2} = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\, dx,$$

will minimize the L^2-error between the function and the Fourier Sine Series approximation.

Example 6.2. Expand

$$f(x) = x(\pi - x) \qquad 0 \leq x \leq \pi$$

in a Fourier Sine Series. Graph some partial sums of the series and examine their convergence.
Solution: We recognize as the initial condition problem from Example 3.2. We take $\ell = \pi$ in the formulae above, because the interval is $x \in [0, \pi]$. Consequently,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin(nx)\,dx \\ &= \begin{cases} 0 & n \quad \text{even} \\ \dfrac{8}{\pi n^3} & n \quad \text{odd} \end{cases} \end{aligned}$$

So the full series is

$$\mathbb{FSS}[f(x)] = \frac{8}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{\sin(nx)}{n^3}\ .$$

■

If we extend the Fourier Sine Series (6.4) beyond the interval $x \in [0, \ell]$ we expect it to converge to an odd 2ℓ-periodic function because the basis elements $\left\{\sin\left(\frac{n\pi x}{\ell}\right)\right\}$ are odd and 2ℓ-periodic. Let's be a bit more specific about this.

Definition 6.5. Suppose a function, $f(x)$, is defined on an interval $x \in [0, \ell]$ with $f(0) = f(\ell) = 0$. We define the *odd extension* of $f(x)$, $f_o(x)$ to be the unique function such that $f_o(-x) = -f(x)$ for $x \in [-\ell, \ell]$.

Exercise 6.1. Suppose $f(x)$ is defined on the interval $x \in [0, \ell]$ with $f(0) = f(\ell) = 0$. Show that

$$\mathbb{FSS}[f(x)] = \mathbb{FS}[f_o(x)]$$

Explain in words what this means.

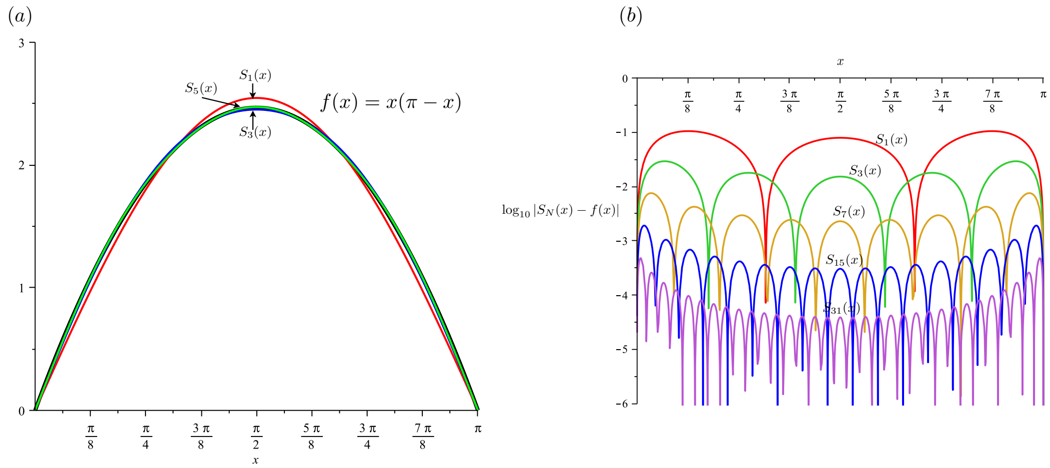


Figure 6.2: The partial sums, $S_N(x)$, for a Fourier Sine Series. (a) Note the partial sums $S_1(x)$,$S_3(x)$ and $S_5(x)$ converge rapidly to the function $f(x) = x(\pi - x)$(which is the black curve) in the range $0 < x < \pi$. (b) The log of the absolute value of the pointwise error, $\log_{10}|S_N(x) - f(x)|$ for $N = 1, 3, 7, 15$ and 31. Note that maximum error monotonically decreases as the number of terms in the approximation increases.

In addition, we can now define the *odd periodic extension* of $f(x)$, $\widetilde{f_o}(x)$ which is exactly what it sounds like – the periodic extension of the odd extension of $f(x)$. Note that $\widetilde{f_o}(x)$ is defined for all x

Exercise 6.2. Show that $\widetilde{f_o}(n\ell) = 0$ for any integer n.

Exercise 6.3. Show that $\widetilde{f_o}(2n\ell - x) = -\widetilde{f_o}(x)$ for any integer n.

We can now talk about the appropriate slogan for Fourier Sine Series:

Slogan 2. The Fourier Sine Series

$$\mathbb{FSS}[f(x)] \equiv \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) \qquad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx,$$

converges to $\widetilde{f_o}$, the odd periodic extension of $f(x)$ from $x \in [0, \ell]$.

Again, we will need to be cognizant of what *converges* means here.

6.3 The Fourier Cosine Series

Finally, for functions on the interval $x \in [0, \ell]$ whose derivatives vanish at the endpoint we showed the set of functions

$$\left\{ \frac{1}{2}, \ \cos\left(\frac{n\pi x}{\ell}\right), \right\}$$

are orthogonal in the L^2 inner-product. This allows us to propose the *Fourier Cosine Series* expansion for a function $f(x)$,

$$\mathbb{FCS}[f(x)] \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right). \tag{6.6}$$

as a natural expansion for functions that vanish at the endpoints of the interval $[0, \ell]$. We know that the choice of coefficients

$$\begin{aligned} a_0 &= \frac{\left\langle \frac{1}{2}, f(x) \right\rangle}{\left\| \frac{1}{2} \right\|^2} = \frac{2}{\ell} \int_0^\ell f(x)\, dx, \\ a_n &= \frac{\left\langle \cos\left(\frac{n\pi x}{\ell}\right), f(x) \right\rangle}{\left\| \cos\left(\frac{n\pi x}{\ell}\right) \right\|^2} = \frac{2}{\ell} \int_0^\ell f(x) \cos\left(\frac{n\pi x}{\ell}\right)\, dx, \end{aligned}$$

will minimize the L^2-error between the function and the Fourier Series approximation.

Example 6.3. Verify that the function

$$f(x) = 1 - 2x^2 + x^4$$

satisfies $f'(0) = f'(1) = 0$ Expand $f(x)$ in a Fourier Sine Series for the interval $x \in [0, 1]$. Graph some partial sums of the series and examine their convergence.
Solution: A simple calculation shows that

$$f'(x) = 4x^3 - 4x$$

so $f'(0) = f'(1) = 0$ and $f(x)$ is a natural candidate to expand as a Fourier Cosine Series. From the formulae above (with $\ell = 1$) we see that

$$\begin{aligned} a_0 &= 2\int_0^1 1 - 2x^2 + x^4\, dx = \frac{16}{15}, \\ a_n &= 2\int_0^1 \left(1 - 2x^2 + x^4\right) \cos(n\pi x)\, dx, = (-1)^{1+n} \frac{48}{\pi^4 n^4} \end{aligned}$$

so

$$\mathbb{FCS}[f(x)] \equiv \frac{8}{5} + \frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{1+n}}{n^4}\cos(n\pi x)\,. \tag{6.7}$$

■

For the Fourier Cosine Series (6.6) beyond the interval $x \in [0, \ell]$ we expect it to converge to an even 2ℓ-periodic function because the basis elements $\left\{\frac{1}{2}, \cos\left(\frac{n\pi x}{\ell}\right)\right\}$ are even and 2ℓ-periodic. Let's be a bit more specific about this.

Definition 6.8. Suppose a function, $f(x)$, is defined on an interval $x \in [0, \ell]$. We define the *even extension* of $f(x)$, $f_e(x)$ to be the unique function such that $f_e(-x) = f(x)$ for $x \in [-\ell, \ell]$.

Exercise 6.4. Suppose $f(x)$ is defined on the interval $x \in [0, \ell]$. Show that

$$\mathbb{FCS}[f(x)] = \mathbb{FS}[f_e(x)]$$

Explain in words what this means.

In addition, we can now define the *even periodic extension* of $f(x)$, $\widetilde{f_e}(x)$ – the periodic extension of the even extension of $f(x)$. Note that $\widetilde{f_e}(x)$ is defined for all x. The appropriate slogan for Fourier Cosine Series:

Slogan 3. The Fourier Cosine Series

$$\mathbb{FCS}[f(x)] \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) \qquad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx,$$

converges to $\widetilde{f_e}$, the even periodic extension of $f(x)$ from $x \in [0, \ell]$.

Again, we will need to be cognizant of what *converge* means here.

6.4 Bessel's Inequality, Parseval's Identity and Least Squares Convergence

In Chapter 4 we derive Bessel's Inequality which states that if F is an element of the inner product space $\mathcal{V}$ and $\{e_i\}$ are an orthogonal set in $\mathcal{V}$, then

$$\|F\|^2 \geq \sum_{n=1}^{N}(c_n)^2\|e_n\|^2 \tag{6.9}$$

where

$$c_n \equiv \frac{\langle e_n, F\rangle}{\|e_n\|^2}.$$

is the projection of F on e_n. Let us apply this idea to Fourier Series.

First, we define the *partial sums* of a Fourier series,

$$S_N \equiv \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

with coefficients

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx \qquad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx.$$

Bessel's Inequality now states that

$$\|f\|^2 \geq a_0^2\left\|\tfrac{1}{2}\right\|^2 + \sum_{n=1}^{N} a_n^2\left\|\cos\left(\frac{n\pi x}{\ell}\right)\right\|^2 + b_n^2\left\|\sin\left(\frac{n\pi x}{\ell}\right)\right\|^2$$

evaluating the norms yields

$$\left\|\tfrac{1}{2}\right\|^2 = \frac{\ell}{2} \qquad \left\|\cos\left(\frac{n\pi x}{\ell}\right)\right\|^2 = \left\|\sin\left(\frac{n\pi x}{\ell}\right)\right\|^2 = \ell$$

so Bessel's inequality now states that

$$\|f\|^2 \geq \frac{\ell}{2}a_0^2 + \sum_{n=1}^{N}\ell\left(a_n^2 + b_n^2\right)\,,$$

or if we divide by ℓ

$$\frac{1}{\ell}\|f\|^2 \geq \tfrac{1}{2}a_0^2 + \sum_{n=1}^{N} a_n^2 + b_n^2\,,$$

which should be true for the partial sums of any Fourier Series, independent of any notion of convergence of the series.

Example 6.4. Let's verify Bessel's inequality for the first example in the chapter; we found that for $g(x) = x^2$ on the interval $x \in [-1, 1]$ that

$$\mathbb{FS}[g(x)] = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx.$$

Note that

$$\frac{1}{\ell}\|g\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2)^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}x^4\,dx = \frac{2}{5}\pi^4$$

So that Bessel's inequality states that

$$\frac{2}{5}\pi^4 \geq \tfrac{1}{2}\left(\frac{2\pi^2}{3}\right)^2 + \sum_{n=1}^{N}\left(\frac{4(-1)^n}{n^2}\right)^2 = \frac{2\pi^4}{9} + \sum_{n=1}^{N}\frac{16}{n^4}$$

moving the a_0 term to the lefthand side and dividing by 16 yields

$$\frac{\pi^4}{90} \geq \sum_{n=1}^{N}\frac{1}{n^4}.$$

We invite the reader to verify this fact numerically. ■

Exercise 6.5. Write down Bessel's Inequality for the Fourier Sine Series and the Fourier Cosine Series.

6.4.1 Least Squares and Convergence in L^2.

In Chapter 4 we related the L^2 error of the N^{th} partial sum to Bessel's Inequality. For a Fourier Series this implies

$$(E_N)^2 \equiv \|f(x) - S_N\|^2 = \|f\|^2 - \ell\left[\tfrac{1}{2}a_0^2 + \sum_{n=1}^{N}a_n^2 + b_n^2\right]. \tag{6.10}$$

and Bessel's Inequality follows from the fact that the error is positive.

We would say the function *converges in* L^2 or in a mean-square sense if $\lim_{N\to\infty} E_N = 0$. An amazing result from analysis states that

Theorem 6.1. *If $f \in L^2[-\ell,\ell]$ (without smoothness assumptions), then for almost every $x \in [-\ell,\ell]$ we have that $S_N(x)$ converges to $f(x)$. That is $\lim_{N\to\infty} E_N = 0$ for all $f \in L^2[-\ell,\ell]$.*

Remark. This result is due to Lennart Carleson (in 1966) and has been called one of the hardest theorems in analysis. The notion of *almost every* x is precisely defined in terms of the Lebesgue integral.

This theorem means that one way the slogan above for convergence of the Fourier Series can be made rigorous by specifying that f is in $L^2[-\ell,\ell]$ and saying that convergence is in the mean-square sense. However, there are other notions of convergence that we will discuss in the next two chapters which apply for different classes of functions.

6.4.2 Parseval's Identity

A corollary to this theorem is:

Theorem 6.2. *(Parseval's Identity for Fourier Series) If $f \in L^2[-\ell, \ell]$ then*

$$\frac{1}{\ell}\|f\|^2 = \tfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

where a_n and b_n are the Fourier coefficients defined above.

The proof follows directly from the fact that $\lim_{N\to\infty} E_N = 0$ which turns Bessel's Inequality into an equality as $N \to \infty$.

So we can conclude from the example above that

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4},$$

an amazing result that is difficult to prove without the aid of Fourier Series.

Exercise 6.6. Write down Parseval's Theorem for the Fourier Sine Series and the Fourier Cosine Series.

Exercise 6.7. Apply Parseval's identity to the function $f(x) = x$ and conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Seven

Fourier Series II: Uniform Convergence, Pointwise Convergence and Gibb's Phenomena

In the previous Chapter we saw that Fourier Series are the best approximation to a function in a least square sense. However, this only tells us that the series converges to a function in some average sense. In fact we can say something much more concrete; Fourier series converge uniformly for sufficiently smooth functions.

For convenience in this Chapter we will use Fourier Series on the interval $x \in [-\pi, \pi]$; these results generalize quickly to an arbitrary interval. The big theorem we wish to state is:

Theorem 7.1. *(Uniform Convergence of Fourier Series) Let $f(x)$ be a continuous 2π-periodic function whose derivative is piecewise continuous. Then the Fourier Series, $\mathbb{FS}[f(x)]$, converges uniformly to f.*

Colloquially, this theorem says as the number of terms in the Fourier Series increases, the maximum difference between the partial sum and the function it approximates decays uniformly to zero.

To be more rigorous, we first need to define a piecewise continuous function,

Definition 7.1. A function $f(x)$ is *piecewise continuous* on $x \in [a, b]$ if:

(a) The function $f(x)$ is continuous on all but a finite number of points.

(b) For all $x_0 \in (a,b)$ the righthand limit,

$$f(x_0^-) \equiv \lim_{\epsilon \nearrow 0} f(x-\epsilon)$$

and the lefthand limit

$$f(x_0^+) \equiv \lim_{\epsilon \nearrow 0} f(x+\epsilon)$$

exist.

(c) The limits at the endpoints, $f(a^+)$ and $f(b^-)$ exist also.

Remember the partial sum of the Fourier Series (on the interval $x \in [-\pi,\pi]$) is defined as

$$S_N \equiv \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\ dx,$$

and now, we can define uniform convergence

Definition 7.2. We say a Fourier Series *converges uniformly* If for every $\epsilon > 0$ there exists N_ϵ such that

$$|f(x) - S_N| < \epsilon \qquad \text{for} \quad N > N_\epsilon$$

for all $x \in [-\pi,\pi]$.

Colloquially, this says we retain a sufficient number of terms of the Fourier Series, we can reduce the error to be less than any positive constant ϵ at every point in the interval.

7.1 The Riemann-Lebesgue Lemma and the Decay of Fourier Coefficients

In the previous sections we have seen that as the wavenumber n increases the Fourier coefficients a_n and b_n decrease to zero, which is clearly a necessary condition for a Fourier Series to converge. We will now prove that this is the case when $f(x)$ is a piecewise differentiable function.

Theorem 7.2 (Riemann-Lebesgue Lemma for Piecewise Differentiable Functions)**.** *Define*

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(nx\right)\, dx \qquad B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(nx\right)\, dx,$$

where $f(x)$ is a piecewise differentiable function. Then

$$\lim_{n\to\infty} a_n = 0, \qquad \lim_{n\to\infty} b_n = 0.$$

Note that this theorem does *not* depend on n being an integer or $f(x)$ being periodic. Moreover, the restriction to piecewise differentiable functions can be considerably weakened.

Proof. We begin by using integration by parts and exploiting the differentiability of $f(x)$. Note that

$$\begin{aligned} B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ &= \frac{1}{n\pi}\left\{ f(x)\sin(nx)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)\sin nx\,dx\right\} \end{aligned}$$

Define

$$P = \max_{x\in[-\pi,\pi]} |f(x)|, \qquad Q = \sup_{x\in[-\pi,\pi]} |f'(x)|.$$

We know that since $f(x)$ is continuous and periodic that P exists. For the derivative, Q is the *supremum* or least upper bound of $|f'(x)|$; it is the maximum of the absolute value of the function and its left- and right-hand limits at the discontinuities.

Now,

$$
\begin{aligned}
|B_n| &\leq \frac{1}{n\pi}\left\| f(x)\sin(nx)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)\sin nx\,dx \right\| \\
&\leq \frac{2}{n\pi}P + \frac{1}{n\pi}\int_{-\pi}^{\pi} |f'(x)\sin nx|\,dx \\
&\leq \frac{2}{n\pi}P + \frac{1}{n\pi}\int_{-\pi}^{\pi} |f'(x)|\,dx \\
&\leq \frac{2}{n\pi}P + \frac{1}{n\pi}\int_{-\pi}^{\pi} Q\,dx \\
&\leq \frac{2}{n\pi}P + \frac{2}{n}Q
\end{aligned}
$$

and as n tends to infinity the righthand side vanishes, proving the result for B_n. A nearly identical calculation shows $\lim_{n\to\infty} |A_n| \to 0$. □

Exercise 7.1. Prove the following theorem:

Theorem 7.3. *Let f be a 2π-periodic function with $k-1$ continuous derivatives, and whose k-th derivative is piecewise continuous. The Fourier coefficients of f decay like C/n^k, that is*

$$|A_n| < \frac{C}{n^k} \qquad |B_n| < \frac{C}{n^k}$$

for a constant C which depends only on f.

Show that if $k = 2$ that this implies the Fourier Series converges uniformly (although this doesn't imply that it converges to $f(x)$!!).

Note: In the previous section we showed that when $f(x)$ was the periodic extension of x^2, which is piecewise differentiable , the Fourier coefficients decayed like C/n^2 which suggests that this theorem isn't sharp. In fact, one can improve the bound on the coefficients to C/n^{k+1}.

A discussion of Gibb's phenomena goes here.

Eight

Complex Fourier Series

In this Chapter we will discuss the natural extension of the idea of Fourier Series to the space of complex functions. It turns out that expanding a function as a sum of complex exponentials can yield a more elegant formulation of the idea of Fourier Series, useful in many contexts.

8.1 Complex Vector Spaces

Previously we defined a real vector space as being a set of functions closed under addition and scalar multiplication, where it was understood that the scalars were real valued. We can extend this idea to a vector space closed under multiplication my complex scalars.

Definition 8.1. A *complex vector space* $\mathcal{V}$ is a set of functions that is closed under addition and scalar multiplication, that is if u and v are elements $\mathcal{V}$, so is $\alpha u + \beta v$ where α and β are complex scalars.

Note that if we have a function of x, f that is an element of a complex vector space that f will naturally have real and imaginary parts

$$f = u + iv$$

where u and v are real. We will refer to $u = \Re\{f\}$ as the *real part* of f and $v = \Im\{f\}$ as the *imaginary part* of f. We can also define the *complex conjugate* of the function as

$$\overline{f} = u - iv.$$

and the magnitude, $|f|$, as the positive value satisfying

$$|f|^2 \equiv \overline{f}f = (u - iv)(u + iv) = u^2 + v^2.$$

For most of this chapter we will consider the case when f is a function of x defined on some interval $[a, b]$.

8.2 The Complex Inner-Product

In Chapter 4 we introduced the inner product of two real functions p and q on an interval $x \in [a, b]$,

$$\langle p, q\rangle \equiv \int_a^b \bar{u}v\, dx. \tag{8.2}$$

Now, let us define a new inner-product appropriate for two complex functions u and v defined on the inteval $x \in [a, b]$,

$$[u, v] \equiv \int_a^b \bar{u}v\, dx. \tag{8.3}$$

This *complex inner product* is a function from two elements of a complex vector space to the complex numbers that satisfies three properties

- **Hermitian Symmetry:** The inner product is *hermitian symmetric* in its arguments; that is
$$[u, v] = \overline{[v, u]}$$
for all elements u and v in the vector space.

- **Linearity:** The inner product is linear in the second argument; that is
$$[u, \alpha v + \beta w] = \alpha\, [u, v] + \beta\, [u, w]$$
for all complex scalars α and β and any u, v and w in the vector space. Note the hermitian symmetry and linearity in the second argument guarantees that
$$[\alpha v + \beta w, u] = \overline{\alpha}\, [v, u] + \overline{\beta}\, [w, u]$$

- **Positive Definiteness:** We say an inner-product is positive definite if
$$[u, u] \geq 0$$
with equality when $u = 0$ only[1]. This allows to define the *norm* of a complex vector,
$$\|u\| \equiv \sqrt{[u, u]}.$$

[1]This assumes the function is continuous; if one chooses to venture into the intricate machinery of measure theory and allow for discontinuous functions one would write "$u = 0$ *almost everywhere*".

Remember an *inner product space* is simply a vector space together with an inner product defined on that vector space. We can nowdefine an inner product space that is the set of *complex square-integrable functions*. For our inner product (8.3) above, the set of complex functions u whose norm is finite,

$$\|u\| \equiv \sqrt{\int_a^b u^2\, dx},$$

is the vector space of square integrable functions.

Exercise 8.1. Show that if u and v are real functions that

$$[u, v] = \langle u, v \rangle .$$

8.3 COMPLEX FOURIER SERIES: THE FOURIER SERIES REVISITED

For periodic functions on the interval $x \in [-\ell, \ell]$ we showed the set of functions

$$\left\{\frac{1}{2},\ \cos\left(\frac{n\pi x}{\ell}\right),\ \sin\left(\frac{n\pi x}{\ell}\right)\right\}$$

are orthogonal in the real L^2 inner-product. Let us now expand the space they describe by considering them as a basis of a complex vector space. This allowed us to propose that the Fourier Series expansion can be expanded to a complex function $f(x)$,

$$\mathbb{FS}[f(x)] \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right). \tag{8.4}$$

Where now the coefficients are complex numbers, but still given by the same formulae

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\, dx, \quad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)\, dx. \tag{8.5}$$

This choice of coefficients still minimize the L^2-error between the function and the Fourier Series approximation when $f(x)$ is a complex function.

Exercise 8.2. If $f(x)$ is a complex function, show that the real part of the Fourier Series, $\mathbb{FS}[f(x)]$, is the Fourier Series of the real part of $f(x)$ and that the imaginary part of the Fourier Series is the Fourier Series of the imaginary part of $f(x)$. Use this to explain why the coefficient formulae for a_n and b_n remain the same. What does this mean for the convergence of a Fourier Series for a complex function?

Remembering Euler's Formula,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

we can introduce a new set of basis functions

$$y_n(x) = e^{i\frac{n\pi x}{\ell}} \qquad \text{for} \quad n = 0, \pm 1, \pm 2, \cdots \tag{8.6}$$

that span the same space. To see this note that

$$\begin{aligned}
\tfrac{1}{2} &= \tfrac{1}{2}\cdot 1 = \tfrac{1}{2}y_0(x) \\
\cos\left(\frac{n\pi x}{\ell}\right) &= \frac{e^{i\frac{n\pi x}{\ell}} + e^{-i\frac{n\pi x}{\ell}}}{2} = \frac{y_n(x) + y_{-n}(x)}{2} \\
\sin\left(\frac{n\pi x}{\ell}\right) &= \frac{e^{i\frac{n\pi x}{\ell}} - e^{-i\frac{n\pi x}{\ell}}}{2i} = \frac{y_n(x) - y_{-n}(x)}{2i}
\end{aligned}$$

The orthogonality of these functions under the complex inner product is easy to check

$$[y_n(x), y_m(x)] \equiv \int_{-\ell}^{\ell} \overline{e^{i\frac{n\pi x}{\ell}}} e^{i\frac{m\pi x}{\ell}}\, dxdx = \int_{-\ell}^{\ell} e^{i\frac{(m-n)\pi x}{\ell}}\, dx = \begin{cases} 0 & n \neq m, \\ 2\ell & n = m. \end{cases}$$

This allows us to propose the *Complex Fourier Series* expansion for a function $f(x)$,

$$\mathbb{CFS}[f(x)] = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{\ell}}. \tag{8.7}$$

Note that as the Fourier Series is the sum of periodic functions with period 2ℓ, it also is 2ℓ-periodic. We know that the choice of coefficients

$$c_n = \frac{\left[e^{i\frac{n\pi x}{\ell}}, f(x)\right]}{\left\|e^{i\frac{n\pi x}{\ell}}\right\|^2} = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-i\frac{n\pi x}{\ell}}\, dx, \tag{8.8}$$

will minimize the L^2-error between the function and the Fourier Series approximation.

8.4 Relationship Between Real and Complex Fourier Series

In general, using the fact that

$$e^{\pm i\frac{n\pi x}{\ell}} = \cos\left(\frac{n\pi x}{\ell}\right) + i\sin\left(\frac{n\pi x}{\ell}\right)$$

we can rewrite the complex Fourier series in terms of a real Fourier Series, with complex coefficients

$$\begin{aligned} CFS[f(x)] &= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{\ell}}, \\ &= \sum_{n=-\infty}^{\infty} c_n \left[\cos\left(\frac{n\pi x}{\ell}\right) \pm i\sin\left(\frac{n\pi x}{\ell}\right)\right], \\ &= c_0 + \sum_{n=1}^{\infty} (c_n + c_{-n})\cos\left(\frac{n\pi x}{\ell}\right) + i\,(c_n - c_{-n})\sin\left(\frac{n\pi x}{\ell}\right), \end{aligned}$$

Note that for $n > 0$

$$\begin{aligned} c_n &= \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-i\frac{n\pi x}{\ell}}\,dx \\ &= \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right) - i\sin\left(\frac{n\pi x}{\ell}\right)\,dx \\ &= \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)\,dx - i\frac{1}{2\ell}\int_{-\ell}^{\ell}\sin\left(\frac{n\pi x}{\ell}\right)\,dx \\ &= a_n - ib_n \end{aligned}$$

where a_n and b_n are the coefficients of the real Fourier Series. A similar calculation shows that $c_{-n} = a_n + ib_n$ and $c_0 = a_0/2$, so in summary

$$c_0 = \frac{a_0}{2} \qquad c_n = \frac{a_n - ib_n}{2} \qquad c_{-n} = \frac{a_n + ib_n}{2}$$

Solving for a_n and b_n yields

$$a_0 = 2c_0 \qquad a_n = c_n + c_{-n} \qquad b_n = i\,(c_n - c_{-n})$$

and one can quickly convince yourself from the expression above that the real and the complex Fourier series are equivalent.

Exercise 8.3. Show that if $f(x)$ is real that $c_n = \overline{c_{-n}}$.

Example 8.1. Consider a complex periodic function, $h(x)$, with period 2π and

$$h(x) = e^{ix/2} \qquad -\pi < x < \pi.$$

(a) Find a complex Fourier series for $h(x)$.

(b) Compute Fourier series for the real periodic functions, $C(x)$ and $S(x)$, with period 2π,

$$C(x) = \cos(x/2) \qquad -\pi < x < \pi$$
$$S(x) = \sin(x/2) \qquad -\pi < x < \pi$$

by taking the real and imaginary parts of the series you found in part (a).

(c) Show that $C'(x) = -S(x)/2$ but $S'(x) \neq C(x)/2$. Why is this?

Solution:

(a) The function $h(x)$ has the complex Fourier series representation

$$h(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)e^{-inx}dx$$

Evaluating, we see that

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(1/2-n)x}\,dx \\ &= \frac{1}{i\pi(1-2n)} e^{i(1/2-n)x}\Big|_{x=-\pi}^{x=\pi} \\ &= \frac{2\sin\left((1/2-n)\pi\right)}{\pi(1-2n)} \\ &= \frac{2(-1)^n}{\pi(1-2n)}. \end{aligned}$$

Or we can reconstitute the series to obtain

$$h(x) = e^{ix/2} = \frac{2}{\pi}\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1-2n} e^{inx}, \qquad -\pi < x < \pi.$$

(b) In general, using the fact that $e^{\pm inx} = \cos(nx) \pm i\sin(nx)$ we can rewrite the complex Fourier series in terms of a real Fourier Series, with complex coefficients

$$h(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, = c_0 + \sum_{n=1}^{\infty} \left(c_n + c_{-n}\right)\cos(nx) + i\left(c_n - c_{-n}\right)\sin(nx),$$

Noting from above that

$$c_0 = \frac{2}{\pi},$$

$$c_n + c_{-n} = \frac{2(-1)^n}{\pi(1-2n)} + \frac{2(-1)^{-n}}{\pi(1+2n)} = \frac{4(-1)^n}{\pi(1-4n^2)},$$

$$c_n - c_{-n} = \frac{2(-1)^n}{\pi(1-2n)} - \frac{2(-1)^{-n}}{\pi(1+2n)} = \frac{8n(-1)^n}{\pi(1-4n^2)},$$

we see that the series for $h(x)$ is

$$h(x) = e^{ix/2} = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi(1-4n^2)} \cos(nx) + i \frac{8n(-1)^n}{\pi(1-4n^2)} \sin(nx), \qquad -\pi < x < \pi.$$

Taking the real part of the series yields a Fourier cosine series for the even periodic function $C(x)$,

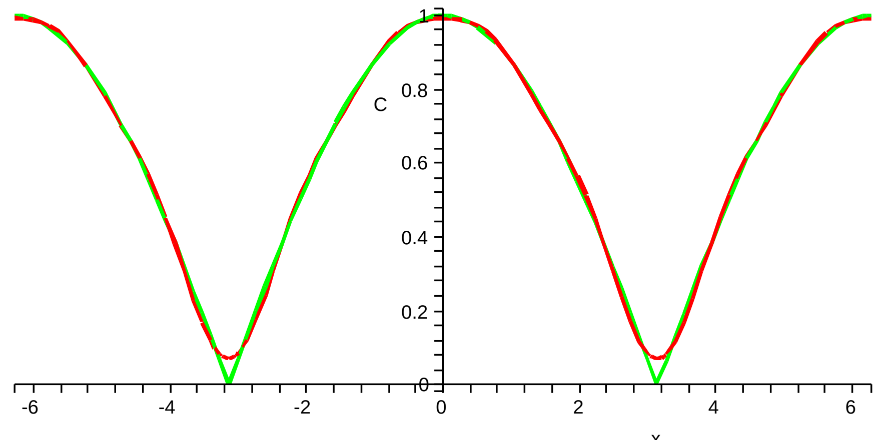
$$C(x) = \cos(x/2) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi(1-4n^2)} \cos(nx), \qquad -\pi < x < \pi,$$

whereas taking the imaginary part yields a Fourier sine series for the odd periodic function $S(x)$,

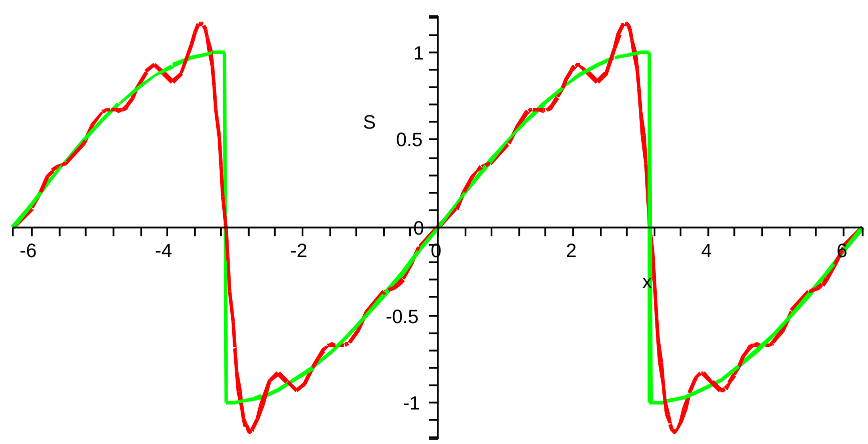
$$S(x) = \sin(x/2) = \sum_{n=1}^{\infty} \frac{8n(-1)^n}{\pi(1-4n^2)} \sin(nx), \qquad -\pi < x < \pi.$$

We graph a partial sum of five terms for $C(x)$ below.

Note that the sum is converging uniformly as the function is continuous and piecewise differentiable.



By contrast, consider the partial sum with ten terms for for $S(x)$.



The discontinuity at $x = \pm\pi$ leads to the persistent overshoot of Gibb's Phenomena.

(c) We can differentiate the fourier series for $C(x)$ term-by-term, using the fact that $(\cos(nx))' = -n\sin(nx)$ to obtain

$$C'(x) = -\sum_{n=1}^{\infty} \frac{4n(-1)^n}{\pi(1-4n^2)} \sin(nx), \qquad -\pi < x < \pi,$$

which is exactly $-S(x)/2$. However, differentiating $S(x)$ term-by-term,
using $(\sin(nx))' = n\cos(nx)$ yields

$$S'(x) = \sum_{n=1}^{\infty} \frac{8n^2(-1)^n}{\pi(1-4n^2)} \cos(nx), \qquad -\pi < x < \pi.$$

which not only is not $C'(x)$, but which is clearly divergent at $x = 0$ where

$$S'(0) = \sum_{n=1}^{\infty} \frac{8n^2(-1)^n}{\pi(1-4n^2)}$$

and the terms in the series approach $\pm 2/\pi$ as $n \to \infty$. The difference between these series is the fact that $C(x)$ is continuous and piecewise differentiable, so we expect it's derivative to have a convergent Fourier series. However, the same discontinuity in $S(x)$ that leads to Gibb's Phenomena is also associated with the derivative of $S(x)$ approaching infinity at $x = \pm\pi$, and the resulting Fourier Series diverging.

■

8.5 Galerkin Method for Solving the Heat Equation

A *Galerkin Method* is a method by which the solution a partial differential equation is expressed as an expansion in orthogonal functions in space with time-dependent coefficients. Let us consider an example.

Let $u(x,t)$ describe the temperature of a metal ring, where x parameterizes the angle, which we choose for convenience to span the interval $[-\pi, \pi]$. Suppose that the ring has some internal heating that is angle-dependent, so that $u(x,t)$ satisfies the inhomogeneous heat equation,

$$\begin{aligned} \mathbb{DE}: &\quad u_t = Du_{xx} + f(x) && -\pi \le x \le \pi, \quad t > 0 \\ \mathbb{IC}: &\quad u(x,0) = 0 && -\pi \le x \le \pi, \end{aligned}$$

where D is the thermal diffusivity and $f(x)$ describes the internal heating. Furthermore, we have assume that the temperature of the ring is initially zero.

Because the temperature $u(x,t)$ is parameterized by the angle x, the temperature must be a 2π-periodic function, $u(x,t) = u(x+2\pi,t)$, This suggests that we should write the temperature as a complex Fourier expansion with time-dependent coefficients,

$$u(x,t) = \sum_{n=-\infty}^{\infty} A_n(t) e^{inx}. \tag{8.9}$$

Let's substitute this expression into the heat equation, $u_t = Du_{xx} + f(x)$,

$$\sum_{n=-\infty}^{\infty} \frac{dA_n(t)}{dt} e^{inx} = -D \sum_{n=-\infty}^{\infty} A_n(t) n^2 e^{inx} + f(x)$$

If we further assume that $f(x)$ has the complex Fourier series representation

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{inx},$$

where

$$f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

then we obtain

$$\sum_{n=-\infty}^{\infty} \left[\frac{dA_n(t)}{dt} + Dn^2 A_n(t) \right] e^{inx} = \sum_{n=-\infty}^{\infty} f_n e^{inx}.$$

The only way for the two Fourier series on either side of this equation to match is if the coefficients of both series are identical. Therefore,

$$\frac{dA_n(t)}{dt} + Dn^2 A_n = f_n,$$

for all positive integers n. Effectively, we have used the assumption in (8.9) to turn the original PDE into an infinite system of ODEs. The initial condition for each of these ODEs is $A_n(0) = 0$, since the initial heat distribution is zero. The solution to each of these ODEs is

$$A_n(t) = \begin{cases} \dfrac{f_n}{Dn^2} \left[1 - e^{-Dn^2 t} \right] & \text{if } n \neq 0 \\ f_0 \, t & \text{if } n = 0, \end{cases}$$

and all that is left to do is to plug these into (8.9).

Example 8.2. Suppose $f(x) = \cos^2(x)$ for the problem above. Find an explicit solution for $u(x,t)$.
Solution: Suppose $f(x) = \cos^2(x)$. Since we already know the general solution, all that needs to be done is to figure out the complex Fourier series for $f(x)$. We'll do this using simple trigonometric identities:

$$\begin{aligned} f(x) = \cos^2(x) &= \frac{1}{2}\cos 2x + \frac{1}{2} \\ &= \frac{1}{4}\left[e^{2ix} + e^{-2ix}\right] + \frac{1}{2} \\ &= \frac{1}{4}e^{-2ix} + \frac{1}{2} + \frac{1}{4}e^{2ix}. \end{aligned}$$

So we see that the complex Fourier series for $f(x)$ only has three nonzero coefficients, which are $f_{-2} = f_2 = 1/4$, and $f_0 = 1/2$. Therefore, the solution in this particular case is

$$\begin{aligned} u(x,t) &= \frac{f_{-2}}{4D}\left[1 - e^{-4Dt}\right]e^{-2ix} + f_0\, t\, e^{0ix} + \frac{f_2}{4D}\left[1 - e^{-4Dt}\right]e^{2ix} \\ &= \frac{1}{16D}\left[1 - e^{-4Dt}\right]e^{-2ix} + \frac{t}{2} + \frac{1}{16D}\left[1 - e^{-4Dt}\right]e^{2ix} \\ &= \frac{t}{2} + \frac{1}{8D}\left[1 - e^{-4Dt}\right]\cos(2x). \end{aligned}$$

The temperature $u(x,t)$ grows linearly in time and without bound. One interpretation of this is that the internal heating is always positive and therefore the total heat must continually increase. ∎

Part IV

Separation of Variables and Well-Posed Problems

Nine

Separation of Variables and The Diffusion Equation

9.1 Outline of Lecture

- Separation of variables for the Dirichlet problem
- The separation constant and corresponding solutions
- Incorporating the homogeneous boundary conditions
- Solving the general initial condition problem

9.2 Solving the Diffusion Equation via Separation of Variables

In Lecture 5, we derived the homogeneous Dirichlet problem for the diffusion equation. This equation, also called the *Heat Equation*, governs the heat distribution in a finite metal bar of length π, where we keep the endpoints at a fixed temperature, in our case 0. The initial temperature at time $t = 0$ is given by $f(x)$. We derived the following conditions:

The Dirichlet Problem for the Diffusion Equation
(Homogeneous Boundary Conditions)

$$\begin{array}{rcll} & u_t = \kappa u_{xx} & 0 < x < \pi, t > 0 & \mathbb{DE}: \\ u(0,t) = 0 & u(\pi,t) = 0 & t > 0 & \mathbb{BC}: \\ & u(x,0) = f(x) & 0 < x < \pi, & \mathbb{IC}: \end{array}$$

We saw a few solutions to this system, but we didn't have a systematic way of solving the problem given a particular $f(x)$. In this lecture, we will discuss a method to solve the equation for (essentially) any initial $f(x)$.

9.2.1 A Solution to the Homogeneous Dirichlet Problem

In 1807 Jean Baptiste Joseph Fourier caused a big stir when he managed to solve a problem of heat dispersion using what are now called Fourier series. We will use the method he developed to solve our homogeneous Dirichlet problem.

When solving a differential equation, it is frequently advantageous to first look for special solutions that might be easier to find then the general case. Fourier's first step was to look for solutions in the special form

$$u(x,t) = X(x)T(t). \tag{9.1}$$

Plugging this form into the differential equation $u_t = \kappa u_{xx}$, we get

$$X(x)T'(t) = \kappa X''(x)T(t) \tag{9.2}$$

and dividing by $\kappa X(x)T(t)$ we find

$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)}. \tag{9.3}$$

Notice that the left hand side is a function of t alone, while the right is a function of x only. This implies that both sides must indeed be constant! Call this constant $-\lambda$. It is known as the *separation constant*. The reason for the negative sign in front of the λ will be apparent shortly.

Thus we have

$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} = -\lambda. \tag{9.4}$$

We can separate this equation into two equations, one involving only x, one involving only t:

$$\frac{T'(t)}{\kappa T(t)} = -\lambda,$$

and

$$\frac{X''(x)}{X(x)} = -\lambda.$$

Each of these equations is now an ordinary differential equation, and thus we can draw on the theory of ordinary differential equations to solve them. The first equation,

$$T'(t) = -\lambda\kappa T(t)$$

has the solution

$$T(t) = Ce^{-\lambda\kappa t}.$$

Note that we expect the temperature to remain finite as time goes to infinity, and thus the exponent to be negative. Thus λ should be non-negative. (Hence the choice of $-\lambda$ earlier.)

The ordinary differential equation in x,

$$X''(x) = -\lambda X(x), \;\; \lambda \geq 0,$$

has the solution

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

for $\lambda > 0$, and

$$X(x) = Ax + B$$

for $\lambda = 0$.

Putting these back together, we find that

$$\begin{array}{ll} u(x,t) = Ce^{-\lambda\kappa t}\left(A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)\right) & \lambda \geq 0 \\ u(x,t) = C(Ax+B) & \lambda = 0 \end{array} \tag{9.5}$$

solve the diffusion equation, though they do not in general satisfy the boundary or initial conditions.

9.3 Incorporating the homogeneous boundary conditions

We wish $u(x,t)$ to satisfy the homogeneous boundary conditions $u(0,t) = u(\pi,t) = 0$. In the case where $u(x,t) = C(Ax+B)$, this forces $u(x,t) = 0$. This is the trivial solution, and we will thus ignore it from now on.

In the case where

$$u(x,t) = Ce^{-\lambda\kappa t}\left(A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)\right),$$

$$u(0,t) = Ce^{-\lambda\kappa t}A.$$

Thus to have $u(0,t) = 0$ we must have $A = 0$. Thus

$$u(x,t) = Ce^{-\lambda\kappa t}B\sin(\sqrt{\lambda}x).$$

To satisfy $u(\pi,t)=0$, we must choose λ such that $\sin(\sqrt{\lambda}\pi)=0$. As $\sin(x)=0$ exactly when $x=n\pi,\ \ n=0,1,2,3,...$, this means that

$$\boxed{\lambda = n^2} \quad n = 0,1,2,3,... \tag{9.6}$$

To summarize, we now have a whole family of functions which satisfies both the differential equation, and the boundary values, namely

$$u_n(x,t) = e^{-n^2\kappa t}\sin(nx) \quad n = 1,2,3,...$$

and since the problem is homogeneous, any constant multiple of $u_n(x,t)$ is a solution also.

However, we now that the differential equation and boundary condition are homogeneous, so the most general solution is a linear combination of the u_n's.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\kappa t}\sin(nx)} \tag{9.7}$$

This solution functions has an initial value

$$u(x,0) = \sum_{n=1}^{\infty} a_n u_n(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) = f(x). \tag{9.8}$$

Thus we now know how to solve the Dirichlet problem for the homogeneous diffusion equation whenever the initial condition can be written as a sum of functions of the form $\sin(nx)$.

Strictly speaking, we need to formally prove that this series converges, and prove that this can represent any initial value $f(x)$. This is called a *Fourier Sine Series*. The numbers b_n are called the *Fourier Coefficients* of f. This proof is non-trivial, and we will not do it here.

9.4 The solution for general $f(x)$

If the initial condition $f(x)$ is the sum of a finite number of terms of the form $\sin(nx)$ the solution is straightforward.

Example 9.1. Find the solution to

$$\begin{array}{rcl} u_t = \kappa u_{xx} & 0 < x < \pi, t > 0 & \mathbb{DE}: \\ u(0,t) = 0 \quad u(\pi,t) = 0 & t > 0 & \mathbb{BC}: \\ u(x,0) = 5\sin 3x + 2.7\sin 100x & 0 < x < \pi, & \mathbb{IC}: \end{array}$$

Solution: The solution can be found by inspection; looking at the initial condition (9.12) associated with the general solution(9.7), we see that choosing $b_3 = 5$ and $b_{100} = 2.7$ and setting all the remaining terms to zero yields

$$\boxed{u(x,t) = 5e^{-3^2\kappa t}\sin(3x) + 2.7e^{-100^2\kappa t}\sin(100x).}$$

What about all the other possible initial conditions?

To solve the general problem we will make use of the following fact, the proof of which is left as an exercise:

The orthogonality condition

$$\boxed{\int_0^\pi \sin(nx)\sin(mx)dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}}$$

To calculate the Fourier coefficients, start with

$$f(x) = \sum_{n=1}^\infty b_n \sin(nx).$$

Multiply both sides by $\sin(mx)$ and integrate. We get

$$\begin{aligned} \int_0^\pi f(x)\sin(mx)dx &= \int_0^\pi \sum_{n=1}^\infty b_n \sin(nx)\sin(mx)dx \\ &= \sum_{n=1}^\infty b_n \int_0^\pi \sin(nx)\sin(mx)dx. \end{aligned}$$

Because of the orthogonality condition, all the terms in the sum are 0 except when $n = m$, in which case we get $\frac{\pi}{2}$. Thus

$$\sum_{n=1}^\infty b_n \int_0^\pi \sin(nx)\sin(mx)dx = b_m\frac{\pi}{2}\ !$$

Thus the formula to calculate the Fourier Coefficients is

$$\boxed{b_m = \frac{2}{\pi}\int_0^\pi f(x)\sin(mx)dx} \quad m = 1, 2, 3, \ldots\ .$$

Example 9.2. Find the solution to

$$\begin{array}{rcc} u_t = \kappa u_{xx} & 0 < x < \pi, t > 0 & \mathbb{DE}: \\ u(0,t) = 0 \quad u(\pi,t) = 0 & t > 0 & \mathbb{BC}: \\ u(x,0) = x(\pi - x) & 0 < x < \pi, & \mathbb{IC}: \end{array}$$

Solution: From the solution above, we see that

$$\begin{aligned} b_m &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)dx \\ &= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(mx)dx \\ &= \begin{cases} 0 & n \quad \text{e}ven \\ \dfrac{8}{\pi n^3} & n \quad \text{o}dd \end{cases} \end{aligned}$$

So the full solution is

$$\boxed{u(x,t) = \frac{8}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{e^{-n^2\kappa t}}{n^3}\sin(nx)} \tag{9.9}$$

9.5 The Neumann Problem

If instead of specifying the temperature at the endpoints, we specify the heat flux we obtain the Neumann problem:

The Neumann Problem for the Diffusion Equation
(Homogeneous Boundary Conditions)

$$\begin{array}{rlcl} & u_t = \kappa u_{xx} & 0 < x < \pi, t > 0 & \mathbb{DE}: \\ u_x(0,t) = 0 & u_x(\pi,t) = 0 & t > 0 & \mathbb{BC}: \\ & u(x,0) = f(x) & 0 < x < \pi, & \mathbb{IC}: \end{array}$$

The boundary conditions can be interpreted physically as saying the endpoints are insulated; basically there is no heat flux out of the ends or the bar.

We wish the solutions (9.5) we found for $u(x,t)$ to satisfy the homogeneous boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$. Note that when $\lambda = 0$ we found that

$$u(x,t) = C(Ax + B) \Rightarrow u_x(x,t) = CA,$$

and

$$u_x(0,t) = AC = 0, \qquad u_x(\pi,t) = AC = 0,$$

from which we deduce $AC = 0$ but we have a solution

$$u(x,t) = u_0(x,t) = 1,$$

or any multiple of this solution.

In the case where

$$u(x,t) = Ce^{-\lambda\kappa t}\left(A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)\right),$$

we find that

$$u_x(x,t) = Ce^{-\lambda\kappa t}\left(-A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)\right),$$

and

$$u_x(x,0) = CB\sqrt{\lambda} = 0,$$

From which we deduce $CB = 0$. At $x = \pi$ we now find

$$u_x(x, \pi) = -AC\sqrt{\lambda}\sin(\sqrt{\lambda}\pi),$$

To satisfy this condition we must choose λ such that $\sin(\sqrt{\lambda}\pi) = 0$. As $\sin(x) = 0$ exactly when $x = n\pi,\ n = 0, 1, 2, 3, ...$, this means that once again

$$\boxed{\lambda = n^2} \quad n = 0, 1, 2, 3, ... \tag{9.10}$$

which yields a whole family of functions which satisfies both the differential equation, and the boundary values, namely

$$u_n(x, t) = e^{-n^2\kappa t}\cos(nx) \quad n = 1, 2, 3, ...$$

and since the problem is homogeneous, any constant multiple of $u_n(x, t)$ is a solution also. However, we now that the differential equation and boundary condition are homogeneous, so the most general solution is a linear combination of the u_n's.

$$\boxed{u(x, t) = \sum_{n=0}^{\infty} a_n u_n(x, t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\kappa t}\cos(nx)} \tag{9.11}$$

This solution functions has an initial value

$$u(x, 0) = a_0 + \sum_{n=1}^{\infty} a_n u_n(x, 0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = f(x). \tag{9.12}$$

Thus we now know how to solve the Dirichlet problem for the homogeneous diffusion equation whenever the initial condition can be written as a sum of functions of the form $\cos(nx)$.

Strictly speaking, we need to formally prove that this series converges, and prove that this can represent any initial value $f(x)$. This is called a *Fourier Cosine Series*. The numbers b_n are called the *Fourier Coefficients* of f. This proof is non-trivial, and we will not do it here.

To solve the problem we again make use of an orthogonality condition,
The orthogonality condition

$$\boxed{\int_0^{\pi} \cos(nx)\cos(mx)dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m = 1, 2, 3, \dots \\ 1 & n = m = 0 \end{cases}}$$

To calculate the Fourier coefficients, start with

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx).$$

Multiply both sides by $\cos(mx)$ and integrate. We get

$$\begin{aligned} \int_0^\pi f(x)\cos(mx)dx &= \int_0^\pi \sum_{n=1}^{\infty} a_n \cos(nx)\cos(mx)dx \\ &= \sum_{n=1}^{\infty} a_n \int_0^\pi \cos(nx)\cos(mx)dx. \end{aligned}$$

Because of the orthogonality condition, all the terms in the sum are 0 except when $n = m$, from which we deduce the formula to calculate the Fourier Coefficients is

$$\boxed{a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx \qquad a_m = \frac{2}{\pi}\int_0^\pi f(x)\cos(mx)dx \;\; m = 1, 2, 3, ...}$$

Example 9.3. Find the solution to

$$\begin{array}{rccc} u_t = \kappa u_{xx} & 0 < x < \pi, t > 0 & \mathbb{DE}: \\ u_x(0,t) = 0 \quad u_x(\pi,t) = 0 & t > 0 & \mathbb{BC}: \\ u(x,0) = x & 0 < x < \pi, & \mathbb{IC}: \end{array}$$

Solution: From the solution above, we see that

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx = \frac{1}{\pi}\int_0^\pi x\,dx = \frac{1}{2}$$

and

$$\begin{aligned} a_m &= \frac{2}{\pi}\int_0^\pi f(x)\cos(mx)dx \\ &= \frac{2}{\pi}\int_0^\pi x\cos(mx)dx \\ &= \begin{cases} 0 & n \quad \text{e}ven \\ -\dfrac{4}{\pi n^2} & n \quad \text{o}dd \end{cases} \end{aligned}$$

So the full solution is

$$\boxed{u(x,t) = \frac{1}{2} - \frac{4}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{e^{-n^2\kappa t}}{n^2}\cos(nx)} \qquad (9.13)$$

9.5.1 Sturm-Liouville Eigenvalue Problems

In this section, we will consider the general eigenvalue problem as

$$\mathcal{L}y = \lambda y, \quad a < x < b \qquad y(a) = 0, \quad y(b) = 0,$$

where $\mathcal{L}y = -y''$. We introduce the L^2 inner-product

$$\langle u, v \rangle = \int_a^b uv \, dx.$$

We can show the differential operator $\mathcal{L}$ is self-adjoint via integration by parts,

$$\begin{aligned}
\langle u, \mathcal{L}v \rangle &= -\int_a^b uv'' \, dx \\
&= \int_a^b u'v' \, dx - uv'|_{x=a}^{x=b} \\
&= -\int_a^b u''v' \, dx + u'v - uv'|_{x=a}^{x=b}.
\end{aligned}$$

Now if u and v satisfy the DE's boundary conditions, that is $u(a) = v(a) = 0$ and $u(b) = v(b) = 0$, we can deduce the operator is self-adjoint,

$$\langle u, \mathcal{L}v \rangle = \langle \mathcal{L}u, v \rangle.$$

To show the eigenvalues are real, assume that y_n is an eigenvalue associated with λ_n. Then, since $\mathcal{L}$ is a real operator, we know that $\bar{y}_n$ and $\bar{\lambda}_n$ also satisfy the eigenvalue problem (here bars denote complex conjugates). That is

$$-(y_n)'' = \lambda_n y_n \qquad \Rightarrow \qquad -(\bar{y}_n)'' = \bar{\lambda}_n \bar{y}_n$$

Now from the self-adjointness of the operator

$$\lambda_n \langle \bar{y}_n, y_n \rangle = \langle \bar{y}_n, \mathcal{L}y_n \rangle = \langle \mathcal{L}\bar{y}_n, y_n \rangle = \bar{\lambda}_n \langle \bar{y}_n, y_n \rangle$$

or rearranging

$$(\lambda_n - \bar{\lambda}_n) \langle \bar{y}_n, y_n \rangle = (\lambda_n - \bar{\lambda}_n) \int_a^b \bar{y}_n y_n \, dx = (\lambda_n - \bar{\lambda}_n) \int_a^b |y_n|^2 \, dx = 0.$$

From which we deduce that either $\lambda_n = \bar{\lambda}_n$ or $y_n = 0$ identically for $a < x < b$ (assuming continuity). Consequently λ_n is real. Moreover, since the

DE has real coefficients, we can now conclude that there is a real-valued solution for y_n also.

To show that λ_n is positive, note that

$$\lambda_n \langle y_n, y_n \rangle = \langle y_n, \mathcal{L} y_n \rangle = -\int_a^b y_n (y_n)'' \, dx = \int_a^b |(y_n)'|^2 \, dx,$$

where we have used the fact that the boundary terms vanish when integrating by parts. More succinctly, we can write

$$\lambda_n = \frac{||(y_n)'||^2}{||(y_n)||^2} \, .$$

Clearly, the right-hand side is non-negative. Moreover, $\lambda_n = 0$ is only possible if $(y_n)' = 0$ for $a < x < b$, that is if y_n is constant. But $y_n(a) = 0$, so if y_n is constant it must vanish identically on the interval. Consequently, we conclude the eigenvalues are positive.

Finally, we wish to show that if we have two eigenfunctions y_n and y_m with distinct associated eigenvalues $\lambda_n \neq \lambda_m$ that the eigenfunctions are orthogonal, that is $\langle y_m, y_n \rangle = 0$. Note

$$\lambda_n \langle y_m, y_n \rangle = \langle y_m, \mathcal{L} y_n \rangle = \langle \mathcal{L} y_m, y_n \rangle = \lambda_m \langle y_m, y_n \rangle$$

or rearranging

$$(\lambda_n - \lambda_m) \langle y_m, y_n \rangle = 0.$$

As $\lambda_n \neq \lambda_m$ we conclude $\langle y_m, y_n \rangle = 0$, that is that the eigenfunctions are orthogonal.

Note that if we choose $a = 0$ and $b = \pi$ we find

$$\lambda_n = n^2 \qquad y_n = \sin(nx) \qquad for \quad n = 1, 2, 3, \cdots$$

The orthogonality conditions used to solve the Dirichlet problem now follow from the fact that the $\{y_n\}$ are eigenfunctions associated with different eigenvalues for this Sturm-Liouville problem.

9.6 Challenge Problems for Lecture 5

Problem 9.1. Use Maple to graph the solution to the homogeneous Dirichlet problem for the diffusion equation with initial condition $f(x) = (\sin(x))^2$ on $[0, \pi]$.

Problem 9.2. Rework the solution to the homogeneous Dirichlet problem for a bar of length L instead of length π. That is, solve

$$\begin{array}{rlcl} & U_t = \kappa U_{xx} & 0 < x < L, t > 0 & \mathbb{DE}: \\ U(0,t) = 0 & U(L,t) = 0 & t > 0 & \mathbb{BC}: \\ & U(x,0) = f(x) & 0 < x < L. & \mathbb{IC}: \end{array}$$

Problem 9.3. Solve the Neumann problem :

$$\begin{array}{rlcl} & u_t = \kappa u_{xx} & 0 < x < \pi, t > 0 & \mathbb{DE}: \\ u_x(0,t) = 0 & u_x(\pi,t) = 0 & t > 0 & \mathbb{BC}: \\ & u(x,0) = f(x) & 0 < x < \pi, & \mathbb{IC}: \end{array}$$

For $f(x) = 2 + 3\cos(2x)$ using separation of variables. Plot the solution in Maple. What happens as $t \to \infty$?

Problem 9.4. Solve the Dirichlet problem when the boundary conditions are not homogeneous. That is, solve it for

$$\begin{array}{rlcl} & U_t = \kappa U_{xx} & 0 < x < L, t > 0 & \mathbb{DE}: \\ U(0,t) = a & U(L,t) = b & t > 0 & \mathbb{BC}: \\ & U(x,0) = f(x) & 0 < x < L. & \mathbb{IC}: \end{array}$$

Hint: First find a simple solution $g(x)$ which satisfies the DE and the BC.

Ten

The Wave Equation

Parts of this chapter are based on notes by M. Vajiac & J. Tolosa

We are all familiar with the oscillation of a string under tension. A violin, a guitar and a piano all work in basically the same way. A string of length L oscillates with a small vertical displacement, which we call $u(x,t)$ where x which is between 0 and L represents the distance along the undisplaced string.

The oscillations can be modeled by a one-dimensional wave equation has the form:

$$\mathbb{DE}: u_{tt} = c^2 u_{xx} \qquad 0 < x < L, \quad t > 0 \tag{10.1}$$

where u_{tt} can be thought of as the (non-dimensional) acceleration and u_{xx} is the (non-dimensional) restoring force proportional to the curvature of the string. The speed $c = \sqrt{T/\rho}$ where T is the tension (with units of Force) and ρ is the (line) density of the string (with units of Mass/Length). Intuitively, if you increase the tension, the speed of the string and oscillation frequency increases, whereas if you increase the density, the oscillation frequency decreases.

We need to specify the position of the ends of the string. Most commonly one fixes both ends at zero displacement,

$$\mathbb{BC}: u(0,t) = u(L,t) = 0 \qquad t > 0,$$

but one can imagine more exotic boundary conditions where these positions are specified as a function of time. We also need to specify the strings initial velocity and acceleration,

$$\mathbb{IC}: u(x,0) = f(x), \quad u_t(x,0) = g(x) \qquad 0 < x < L.$$

Intuitively, since the wave equation gives the acceleration at each point we need to specify the initial position and velocity. Another way to think of this is by analogy with ordinary differential equations; this equation is second-order in time, so we need to specify the initial position and its derivative.

We can also consider the case where the string is *pushed* with an external force $h(x,t)$, which correspond to plucking or strumming.

$$u_{tt} = c^2 u_{xx} + h(x,t) \qquad 0 < x < L, \quad t > 0,$$

which yields the inhomogeneous *forced* wave equation.

Another important case is the *damped* wave equation; a plucked guitar string does not oscillate forever. this is due to air resistance and a simple model of this system is to include a linear damping term,

$$u_{tt} + k u_t = c^2 u_{xx} \qquad 0 < x < L, \quad t > 0,$$

where k is the damping constant.

10.1 Dirichlet Problem and Separation of Variables

The simplest model of the plucked string is the *Dirichlet Problem* for the wave equation:

The Dirichlet Problem for the Wave Equation

$$\begin{array}{rcl}
\mathbb{DE}: & u_{tt} = c^2 u_{xx}, & 0 < x < L, t > 0 \\
\mathbb{BC}: & u(0,t) = 0, \quad u(L,t) = 0, & t > 0 \\
\mathbb{IC}: & u(x,0) = f(x), \quad u_t(x,0) = g(x) & 0 < x < L
\end{array}$$

As you have seen in Chapter ** for the diffusion equation, the method of separation of variables yields a set of solutions for PDEs that (hopefully) form a basis for an arbitrary initial condition.

First, let us present the solution:

Theorem 10.1. *The Dirichlet problem for the wave equation:*

$$\begin{array}{rcl}
\mathbb{DE}: & u_{tt} = c^2 u_{xx}, & 0 < x < L, t > 0 \\
\mathbb{BC}: & u(0,t) = 0, \quad u(L,t) = 0, & t > 0 \\
\mathbb{IC}: & u(x,0) = f(x), \quad u_t(x,0) = g(x) & 0 < x < L
\end{array}$$

has a solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(\frac{n\pi ct}{L}) + B_n \frac{L}{n\pi c} \sin(\frac{n\pi ct}{L}) \right] \sin(\frac{n\pi x}{L}),$$

where:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L}),$$

and

$$g(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L}).$$

where the coefficients A_n and B_n are the Fourier coefficients of the initial velocity and displacement respectively:

$$A_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L})\, dx \qquad B_n = \frac{2}{L} \int_0^L g(x) \sin(\frac{n\pi x}{L})\, dx.$$

If $f(x)$ and $g(x)$ are continuous, piecewise differentiable and vanish at $x = 0$ and $x = L$, this solution converges uniformly.

We will derive this solution via the method of separation of variables, we look for solutions $u(x,t)$ that are a product of two function, one that depends only on the variable t and a second function that depends only on the variable x.

Let $u(x,t) = X(x)T(t)$ and substitute in the equation $u_{tt} = c^2 u_{xx}$, to obtain:

$$X(x)T''(t) = c^2 X''(x)T(t),$$

or dividing by $c^2 X(x)T(t)$, we obtain:

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)},$$

thus the equality is one of functions of different variables, so both quotients have to be constant,

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

Here λ is the *separation constant*. If we also separate the boundary conditions,

$$u(0,t) = X(0)T(t) = 0 \qquad u(0,t) = X(L)T(t) = 0$$

we see that either $T(t) = 0$ which yields the trivial solution $u(x,t) = 0$ or $X(0) = X(L) = 0$.

This yields the boundary value problem of Sturm-Liouville type

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = X(L) = 0$$

which we have examined previously. This eigenvalue problem has a countable number of positive eigenfuctions,

$$X(x) = X_n(x) \equiv \sin(\frac{n\pi x}{L}), \qquad \lambda_n = \mu_n^2 \equiv \left(\frac{n\pi}{L}\right)^2$$

where X_n is arbitrary up to a constant multiple.

We can now solve the T equation for every eigenvalue $\lambda_n = \mu_n^2$,

$$T''(t) + c^2\mu_n^2 T(t) = 0,$$

which has solution

$$T(t) = T_n(t) = a_n \cos(\mu_n ct) + b_n \sin(\mu_n ct) = a_n \cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L}) \; .$$

we can define now a solution

$$u(x,t) = u_n(x,t) \equiv X_n(x)T_n(t) = \left[a_n \cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L})\right] \sin(\frac{n\pi x}{L})$$

Since the equation is homogeneous, the most general solution is a linear combination of these solutions, namely:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L})\right] c_n \sin(\frac{n\pi x}{L}).}$$

The only conditions left to check are the initial conditions. Note that for our solution

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(\frac{\pi n}{L} x).$$

and comparing this to

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L})$$

we conclude $\boxed{a_n = A_n}$. Similarly, we note that

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin(\frac{n\pi}{L}x).$$

and comparing this to the initial velocity

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L}),$$

we see that $\boxed{b_n = \frac{L}{n\pi c} B_n}$. This yields our solution

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(\frac{n\pi ct}{L}) + B_n \frac{L}{n\pi c} \sin(\frac{n\pi ct}{L}) \right] \sin(\frac{n\pi x}{L}).}$$

Note that if $f(x)$ and $g(x)$ are continuous and piecewise differentiable this guarantees that the Fourier sine series converges uniformly.

10.2 The Example of the Plucked String

The *plucked string* refers to the initial condition for the Dirichlet problem, where the initial displacement $f(x)$ is a piecewise linear function, the equilibrium position assumed when a string is plucked at a point and that the velocity initially is zero.

$$\begin{aligned} \mathbb{DE}: &\quad u_{tt} = c^2 u_{xx}, &\quad 0 < x < L, t > 0 \\ \mathbb{BC}: &\quad u(0,t) = 0, u(L,t) = 0, &\quad t > 0 \\ \mathbb{IC}: &\quad u(x,0) = f(x), u_t(x,0) = 0 &\quad 0 < x < L \end{aligned}$$

where

$$f(x) = \begin{cases} U\frac{x}{\bar{x}}, & 0 \le x \le \bar{x} \\ U\frac{x-L}{\bar{x}-L}, & \bar{x} \le x \le L \end{cases}$$

Let's find a formal solution to the "plucked string" equation. Clearly $A_n = 0$ and the B_n's are the Fourier sine coefficients of $f(x)$.

$$\begin{aligned} B_n &= \int_0^L f(x) \sin(\frac{n\pi x}{L}) dx \\ &= \frac{2L^2 U}{\pi^2 \bar{x}(L-\bar{x})} \frac{1}{n^2} \sin(\frac{n\pi \bar{x}}{L}). \end{aligned}$$

Now we can write the formal solution to the plucked string equation:

$$u(x,t) = \frac{2L^2U}{\pi^2\bar{x}(L-\bar{x})}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(\frac{n\pi\bar{x}}{L})\cos(\frac{n\pi ct}{L})\sin(\frac{n\pi x}{L}).$$

An amazing fact is that this solution is actually a piecewise linear function, a fact we will prove using the d'Alembert solution to the wave equation.

10.2.1 Musical instruments

Many instruments produce sound by making strings vibrate; such are the harp, the piano, the harpsichord, the guitar, the violin, and others. Strings are kept fixed at the endpoints, but they way the instruments are played create different initial conditions. In instruments like the guitar, the string is plucked; this produces an initial perturbation with no initial velocity. In the piano, on the other hand, the string is hit, which creates an initial velocity but no initial perturbation from the initial position.

The oscillations of the string are described by

$$u(x,t) = \sum_{n=1}^{\infty}[A_n\cos\omega_n t + B_n\sin\omega_n t]\sin\lambda_n x,$$

where

$$\lambda_n = n\frac{\pi}{L} \qquad \text{and} \qquad \omega_n = c\lambda_n = cn\frac{\pi}{L}.$$

The sound we hear is thus a combination of the main harmonic sounds (eigenfunctions)

$$u_n(x,t) = (A_n\cos\omega_n t + B_n\sin\omega_n t)\sin\lambda_n t.$$

The contribution of each particular harmonic is measured by its *energy*, which turns out to be equal to:

$$E_n = \frac{\omega_n^2 M}{4}(A_n^2 + B_n^2),$$

where $M = DL$ is the total mass of the string (recall that D was the density).

For the plucked string, the energy is given by:

$$E_n = \frac{MU^2L^2c^2}{n^2\pi^2\bar{x}^2(L-\bar{x})^2}\sin^2\frac{\pi n\bar{x}}{L}.$$

The energy decreases as n^{-2}, so only the main tone u_1 and a few other harmonics are audible.

On the other hand, if we hit the string with a flat hammer of length 2δ with center at $\bar{x}$ and producing an initial velocity v_0, the energy of the n^{th} harmonic is:

$$E_n = \frac{4MV_0^2}{n^2}\pi^2 \sin^2 \frac{\pi n \bar{x}}{L} \sin^2 \frac{\pi n \delta}{L},$$

and the energy again decreases as n^{-2}. However, if the hammer is sufficiently narrow, letting δ tend to zero (the blade of a knife), we get the model of a string getting an impulse concentrated at a point $\bar{x}$. The corresponding energy is:

$$E_n = \frac{v_0^2}{L} \sin^2 \frac{\pi n \bar{x}}{L}.$$

Thus, for a very narrow hammer the energies of all harmonics are of the same order and the generated sound will be saturated with harmonics. This can be checked experimentally, by hitting a string with the blade of a knife. The sound will have a metallic quality.

Not all harmonics are desirable. The first ones, u_2 up to u_6, sound well together with the main harmonic u_1. However, the 7^{th} and the first harmonics sounding together produce a sense of dissonance.

There are several ways to try to "kill" those harmonics by percussion (as in the piano).

- **The position of the hammer.** The presence of the factor $\sin \frac{\pi n \bar{x}}{L}$ shows that by choosing the center $\bar{x}$ of the hammer at the node of the undesired harmonic we may make it disappear (make the corresponding A_n and B_n be equal to zero). In modern pianos the position of the hammer is chosen near the nodes of the 7^{th} and the 7^{th} harmonics, to "kill" them.

- **The shape of the hammer**. In modern pianos the hammers are not flat, but rather round. One can model this situation by choosing the initial velocity to be, say, a parabola on the interval $[\bar{x} - \delta, \bar{x} + \delta]$, instead of a horizontal line. Older pianos, which had flatter and narrower hammers, produced a more piercing, shrilled sound.

- **The rigidity of the hammer**. If instead of being rigid the hammer is softer the motion is not described by its initial position and velocity but rather by a short-time acting force,

Exercise 10.1. Can you model the percussion strategies above with our solution to the wave equation?

10.3 Challenge Problems

Problem 10.1. A string of length π is held fixed at both endpoints. Its initial position is $f(x) = \sin(x)$ and its initial velocity is $g(x) = \sin 2x$. Assuming that $c = 1$, find the position of the string $u(x,t)$ for every $x \in [0, \pi]$ and for every $t > 0$. Animate the approximation and draw a $3D$ plot.

Problem 10.2. Solve the following problem for the wave equation:

$$u_{tt} = u_{xx},$$

with boundary conditions

$$u(0,t) = 0, \quad u_x(\pi, t) = 0 \quad \text{for every } t > 0;$$

and initial conditions

$$u(x,0) = \sin(x), \quad u_t(x,0) = 0, \quad \text{for every } x \in [0, \pi]\,.$$

Notice the change in the boundary conditions. This will lead to different eigenvalues and eigenfunctions. Use Maple to animate the solution you found, to draw a $3D$ plot, and to check that the solution satisfies the conditions of the problem.

Problem 10.3. A damped string of length 1 has equation

$$u_{tt} = c^2 u_{xx} - \gamma u_t,$$

where γ is a small damping coefficient. Find the solution $u(x,t)$ assuming that both endpoints are fixed, the initial condition is $x(1-x)$ and the initial velocity is zero. Plot and animate the solution for the case when $c = \frac{1}{4}$ and $\gamma = \frac{1}{5}$.

Problem 10.4. Solve the string equation $u_{tt} = c^2 u_{xx}$ for $L = 1$, with the boundary conditions $u(0,t) = 0$ and $u(1,t) = 1$, with zero initial velocity, assuming that the initial position is

(a) $u(x,0) = 0$,

(b) $u(x,0) = x^2$.

Hint: You cannot use the superposition principle, since the boundary condition at $x = 1$ is not homogeneous. Try a change of coordinates first, $v(x,t) = u(x,t) + h(x)$, where $h(x)$ is a suitable (easy) function that would guarantee that v also satisfies the string equation, now with homogeneous boundary conditions.

Problem 10.5. Solve the equation

$$u_{tt} = c^2 u_{xx} + \sin x \qquad 0 \le x \le \pi, \quad t > 0.$$

With the boundary conditions

$$u(0,t) = u_t(\pi,t) = 0,$$

and the initial conditions

$$u(x,0) = 0. \qquad u_t(x,0) = 0.$$

Hint: Make the change of coordinates $u(x,t) = y(x) + v(x,t)$, where $y(x)$ is a steady solution that satisfies

$$c^2 y'' + \sin x = 0,$$

with $y(0) = y(\pi) = 0$. Find y and then show $v(x,t)$ satisfies a homogeneous wave equation.

Problem 10.6. Solve the wave equation

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < 1, t > 0$$

with the boundary conditions

$$u(0,t) = 0, \qquad u_x(1,t) + u(1,t) = 0 \qquad t > 0$$

which corresponds a string with the left end fixed and the right end being attached to an elastic hinge. Use the initial conditions are

$$u(x,0) = x - \frac{2}{3}x^2, \qquad u_t(x,0) = x.$$

Note. This exercise is hard! The eigenvalues λ_n will be solutions of a transcendental equation.

Eleven

Laplace's Equation in a Disk

If we seek the steady-state distribution of temperature in a two-dimensional region, this leads from the heat equation $u_t = \Delta u$ to the Laplace equation $\Delta u = 0$, since $u_t = 0$ for the steady-state distribution. In Electrostatics, according to Maxwell's equations the electrostatic potential ϕ satisfies the equation $\Delta\phi = -4\pi\rho$, where ρ is the density of the charges. Thus, if there are no electric charges inside the region, the potential will satisfy Laplace's equation $\Delta\phi = 0$. Let's examine this problem in some more detail.

Suppose we have a region, Ω, in $\mathbb{R}^n$ and we specify the temperature $u(\vec{x})$ (where $\vec{x} \in \mathbb{R}^n$) on the boundary of the region, $\partial\Omega$. The equilibrium temperature satisfies Laplace's Equation

$$\mathbb{DE} : \Delta u = 0 \qquad \vec{x} \in \Omega,$$

$$\mathbb{BC} : u = f \qquad \vec{x} \in \partial\Omega.$$

Below we will solve this problem for a disk in two-dimensions using separation of variables. Functions $u(x, y)$ that satisfy Laplace's Equation are called *harmonic* and play a central role in the study of functions of a complex variable.

11.1 The Laplacian in Polar Coordinates

When a problem has rotational symmetry, it is often convenient to change from Cartesian to polar coordinates. It is then useful to know the expression for the Laplacian acting on $u(x, y)$,

$$\Delta u = u_{xx} + u_{yy}$$

in polar coordinates. Recall that

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Using the chain rule,

$$u_x = u_r r_x + u_\theta \theta_x.$$

Let us find r_x and θ_x. Implicit functions help simplify the computations a little bit. Differentiating both sides of

$$r^2 = x^2 + y^2$$

with respect to x, we get

$$2rr_x = 2x,$$

whence $r_x = \frac{x}{r}$. You can find $r_y = \frac{y}{r}$ in a similar way.

To compute θ_x, we can start from the relation

$$y = r\sin\theta.$$

Differentiating both sides with respect to x, we get

$$0 = r_x \sin\theta + r\cos\theta \cdot \theta_x,$$

whence

$$\theta_x = -\frac{r_x \sin\theta}{r\cos\theta} = -\frac{r_x}{r}\tan\theta = -\frac{r_x}{r}\cdot\frac{y}{x}.$$

Substituting here the value of r_x we have just found, we get

$$\theta_x = -\frac{y}{r^2}.$$

You can find $\theta_y = \frac{x}{r^2}$ in a similar way, starting from the equation $x = r\cos\theta$.

Differentiating once again, we can show that

$$r_{xx} = \frac{y^2}{r^3}, \qquad \theta_{xx} = \frac{2xy}{r^4},$$

and find similar values for r_{yy}, θ_{xx}, and θ_{yy}.

This will yield the following expressions for u_{xx} and u_{yy}:

$$u_{xx} = \frac{x^2}{r^2}u_{rr} - \frac{2xy}{r^3}u_{r\theta} + \frac{y^2}{r^4}u_{\theta\theta} + \frac{y^2}{r^3}u_r + \frac{2xy}{r^4}u_\theta;$$
$$u_{yy} = \frac{y^2}{r^2}u_{rr} + \frac{2xy}{r^3}u_{r\theta} + \frac{x^2}{r^4}u_{\theta\theta} + \frac{x^2}{r^3}u_r - \frac{2xy}{r^4}u_\theta.$$

Exercise 11.1. Compute r_{xx}, r_{yy}, θ_{xx}, θ_{yy}, and obtain the above expressions for u_{xx} and u_{yy} using the chain rule.

Adding up both expressions, doing a couple of cancellations and regrouping, we obtain

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

This is the desired expression of the Laplacian in polar coordinates. Sometimes it is convenient to write it in a slightly different way:

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}; \tag{11.1}$$

for the second expression we combined the first two terms, using the product rule.

11.2 Separation of Variables

We will solve the Dirichlet problem for the Laplace equation on a circle, that is, the problem of finding a function that is harmonic inside a circle and has a prescribed value on the boundary; let us call a the radius of the circle which we will assume is centered at the origin,

$$\begin{aligned} \mathbb{DE}: &\quad \Delta u = 0 &\qquad x^2 + y^2 \le a^2 \\ \mathbb{BC}: &\quad u = f &\qquad x^2 + y^2 = a^2, \end{aligned}$$

where f is a known function on the boundary of the region. In polar coordinates

$$\begin{aligned} \mathbb{DE}: &\quad \Delta u(r,\theta) = 0 &\qquad \text{for every } \theta \text{ and for } r < a; \\ \mathbb{BC}: &\quad u(a,\theta) = f(\theta) &\qquad \text{for every } \theta, \end{aligned}$$

where $f(\theta)$ is now a specified periodic function with period 2π, (Periodicity is required because θ represents the polar angle, so $\theta + 2\pi$ and θ are measures of the same angle.)

Using the expression(11.1) of the Laplacian in polar coordinates, we can rewrite the problem as

$$\begin{aligned} \mathbb{DE}: &\quad u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} = 0 &\qquad \text{for every } \theta \text{ and for } r < a; \\ \mathbb{BC}: &\quad u(a,\theta) = f(\theta) &\qquad \text{for every } \theta, \end{aligned}$$

Using the method of separation of variables, we will first forget for a moment all about the boundary condition and seek nontrivial solutions (eigenfunctions) of the Laplace equation on the circle as a product:

$$u(r,\theta) = R(r)\Theta(\theta).$$

Substituting into the Laplace equation, we get

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Multiplying both sides by r^2 we can rewrite this as

$$(r^2R'' + rR')\Theta = -\Theta''R.$$

Dividing both sides by $R\Theta$, which is assumed to be nonzero, this becomes

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta}.$$

Now we use an argument you have already heard many times: since the left-hand side depends only on r, and the right-hand side depends only on θ, both sides must be constant, call it λ:

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda. \tag{11.2}$$

The equation for Θ reads

$$\Theta'' + \lambda\Theta = 0,$$

with the additional conditions that

$$\Theta(2\pi) = \Theta(0), \quad \Theta'(2\pi) = \Theta'(0); \tag{11.3}$$

these conditions arise, again, from the fact that θ represents the polar angle.

This is the Fourier Eigenvalue Problem for periodic boundary conditions (5.13) with a period of 2π; the eigenvalues and eigenfunctions are given by (5.14)

$$\begin{array}{lll} \Theta_0(\theta) = \frac{1}{2} & \lambda_0 = 0 & \\ \Theta_n^c(\theta) = \cos(n\theta) & \lambda_n = n^2 & \text{for} \quad n = 1,2,3,\ldots. \\ \Theta_n^s(\theta) = \sin(n\theta) & \lambda_n = n^2 & \text{for} \quad n = 1,2,3,\ldots. \end{array} \tag{11.4}$$

We can now solve for $R(r)$. Substituting $\lambda = \lambda_n \equiv n^2$ into (11.2) produces

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = n^2; \quad n = 0, 1, 2, \ldots \tag{11.5}$$

The equation for R is now $r^2R'' + rR' = n^2R$, or

$$r^2R'' + rR' - n^2R = 0.$$

This is an ordinary differential equation which you probably have seen in your ODE course; it is called an *Euler equation*. The main feature of an Euler equation is that each term contains a power of r that coincides with the order of the derivative of R.

Euler equations always admit particular solutions of the form $R(r) = r^\alpha$, where α is a suitable (possibly fractional) power, which we will now find. Differentiating twice and substituting into the equation, we get

$$\alpha(\alpha - 1)r^\alpha + \alpha r^\alpha - n^2r^\alpha = 0,$$

or, cancelling the common (positive) factor r^α,

$$\alpha(\alpha - 1) + \alpha - n^2 = 0,$$

or just

$$\alpha^2 - n^2 = 0.$$

This is the *characteristic equation* for the Euler equation. In this case, the solutions are particularly easy to find: $\alpha = \pm n$.

Accordingly, we have two particular solutions for our equation: $R_1(r) = r^n$ and $R_2(r) = r^{-n}$. The general solution of the factor $R(r)$ is a linear combination of these:

$$R(r) = P_nr^n + Q_nr^{-n}, n = 0, 1, 2, \ldots,$$

where P_n and Q_n are arbitrary constants.

For $n = 0$ the two solutions coincide and are equal to $r^0 = 1 = const$. In this case a second solution of the corresponding equation $r^2R'' + rR' = 0$ is $R(r) = \ln r$.

Exercise 11.2. Find the general solution of the Euler equation $r^2R''+rR' = 0$, corresponding to the case $n = 0$. We already know one particular solution, $R(r) = 1 = const$. To find the second solution, do the change of variables $R' = z$ and solve a first-order separable equation for z.

We will ignore this second solution, $R(r) = \ln r$, because it is not bounded at the center of the circle, when $r = 0$.

If Q_n is nonzero for some positive n, then $R(r)$ will contain the term r^{-n}, which blows up at the center of the circle. We don't want this kind of behavior, so we ask that $Q_n = 0$ for all nonzero n's.

Exercise 11.3. Can you imagine a problem involving $\Delta u = 0$ somewhere and $u = f$ on the boundary of a circle, for which Q_n nonzero could be useful? (Think outside the box!)

We have thus obtained the radial factor:

$$R_n(r) = P_n r^n, \quad n = 0, 1, 2, \ldots$$

Multiplying the solutions for $R_n(r)$ and $\Theta_n(\theta)$ (and dropping the arbitrary constants) yields

$$u_0(\theta) = \frac{1}{2}, \quad u_n^c(\theta) = r^n \cos(n\theta), \quad u_n^s(\theta) = r^n \sin(n\theta) \tag{11.6}$$

All these functions satisfy the Laplace equation, that is, all these functions are *harmonic*. In particular, we get the following interesting result: the functions

$$r^n \cos n\theta \qquad \text{and} \qquad r^n \sin n\theta$$

are harmonic for every $n = 1, 2, 3 \ldots$

Now we can get back to the original problem (if we still remember what it was!). Each u_n satisfies the Laplace equation. Since this equation is linear and homogeneous, any linear combination of the u_n's will also satisfy the same equation. This is even true for an infinite linear combination, provided the series converges nicely enough. So,

$$u(r, \theta) = A_0 u_0 + \sum_{n=1}^{\infty} A_n u_n^c(r, \theta) + B_n u_n^s(r, \theta) \tag{11.7}$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \tag{11.8}$$

will be a solution of $\Delta u = 0$. Our hope is to be able to pick the coefficients A_n and B_n so as to also satisfy the boundary condition $u(a, \theta) = f(\theta)$.

We have

$$u(a, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta). \tag{11.9}$$

To compare this with $f(\theta)$ it helps to expand this function in Fourier series:

$$\mathbb{FSS}[f(\theta)] = \frac{a_0}{2} + \sum_{n=0}^{\infty}(a_n \cos n\theta + b_n \sin n\theta), \tag{11.10}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\cos n\phi \, d\phi, \quad n = 0, 1, 2 \ldots \tag{11.11}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\sin n\phi \, d\phi, \quad n = 1, 2, 3 \ldots \tag{11.12}$$

The Fourier Series (11.10) is valid under mild assumptions on $f(\phi)$; for example, it holds if f is continuous and piecewise differentiable. For the moment we will assume that everything converges.

We want $u(a, \theta)$ to coincide with $f(\theta)$. Comparing (11.9) and (11.10), we get

$$A_0 = a_0; \qquad a^n A_n = a_n; \qquad a^n B_n = b_n, \quad n = 1, 2, \ldots$$

In turn, using (11.11) and (11.12), this implies that

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\, d\phi;$$

$$A_n = \frac{1}{a^n \pi}\int_{-\pi}^{\pi} f(\phi)\cos n\phi \, d\phi, \quad n = 1, 2, \ldots;$$

$$B_n = \frac{1}{a^n \pi}\int_{-\pi}^{\pi} f(\phi)\sin n\phi \, d\phi, \quad n = 1, 2, \ldots.$$

By substituting all these expressions into (11.8) we obtain a formula for the solution of our problem:

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{r}{a}\right)^n \cos n\theta + b_n \left(\frac{r}{a}\right)^n \sin n\theta. \tag{11.13}$$

Let's use this solution to solve some problems.

Example 11.1. As a concrete example, let us consider the problem of finding the steady-state distribution of temperature of a circular membrane, if the temperature is kept fixed and equal to 1 on half the boundary, and -1 on the other half. Namely, we will consider the problem of finding $u(r,\theta)$ on the circle $r \le a$ such that

$$\begin{aligned} \mathbb{DE}: &\quad \Delta u(r,\theta) = 0 \qquad r < a \\ \mathbb{BC}: &\quad u(a,\theta) = \begin{cases} -1, & -\pi < \theta < 0, \\ 1, & 0 < \theta < \pi. \end{cases} \end{aligned}$$

Solution: we can read our solution off directly from (11.13) we can say that the solution will look like:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n \cos n\theta + b_n \sin n\theta),$$

where:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta \quad n = 0, 1, 2, \ldots \tag{11.14}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta, \quad n = 1, 2, \ldots \tag{11.15}$$

In our case $f(\theta)$ is odd; therefore, all the coefficients a_n will vanish.

As regards the b_n, using the symmetry of the integrand and the interval of integration, we can write

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta \, d\theta \\ &= -\frac{2}{n\pi}(\cos n\pi - \cos 0) = \frac{2}{n\pi}\left(1 - (-1)^n\right) \\ &= \begin{cases} 0, & \text{if } n \text{ is even}, \\ \frac{4}{n\pi}, & \text{if } n \text{ is odd}. \end{cases} \end{aligned}$$

Therefore, the solution is

$$\begin{aligned} u(r,\theta) &= \frac{4}{\pi}\left(\frac{r\sin\theta}{a} + \frac{r^3 \sin 3\theta}{3a^3} + \frac{r^5 \sin 5\theta}{5a^5} + \cdots\right) \\ &= \frac{4}{\pi}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^{2n+1} \frac{\sin(2n+1)\theta}{2n+1}. \end{aligned}$$

We leave it as an exercise for the reader to examine the convergence property of the solutions; not surprisingly there is Gibb's phenomena near the boundary at $\theta = 0, \pi$ where the boundary data is discontinuous. ■

Example 11.2. We seek a function harmonic inside the unit circle and equal to (the restriction of) the polynomial $x^3 - y^3$ on the boundary of the unit circle. Let us write the problem in polar coordinates. Since on the unit circle we have $x = \cos\theta$ and $y = \sin\theta$, we must solve the problem

$$\begin{aligned} \mathbb{DE}: &\quad \Delta u(r,\theta) = 0 \quad && r < 1, && (11.16)\\ \mathbb{BC}: &\quad u(1,\theta) = \cos^3\theta - \sin^3\theta \quad && 0 \le \theta \le 2\pi && (11.17) \end{aligned}$$

Solution: To expand the boundary condition, $\cos^3\theta - \sin^3\theta$, in Fourier series, it is perhaps faster to use trigonometric identities. We begin with the identities

$$\begin{aligned} \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta && (11.18)\\ \sin 3\theta &= -\sin 3\theta + 3\cos^2\theta\sin\theta. && (11.19) \end{aligned}$$

which in turn imply that

$$\begin{aligned} \cos^3\theta &= \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta), && (11.20)\\ \sin^3\theta &= \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta). && (11.21) \end{aligned}$$

Identities (11.20,11.21) yield the desired Fourier expansion of the boundary condition:

$$\cos^3\theta - \sin^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta) - \frac{3}{4}\sin\theta + \frac{1}{4}\sin(3\theta). \qquad (11.22)$$

By using (11.13) with $r = 1$ we see the solution will look like

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n\cos n\theta + b_n\sin n\theta).$$

At $r = 1$, the solution will be

$$u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta).$$

Comparing this with (11.22) we conclude that

$$a_1 = \frac{3}{4}, \qquad b_1 = -\frac{3}{4}, \qquad a_3 = \frac{1}{4}, \qquad b_3 = \frac{1}{4},$$

and all the other coefficients will be equal to zero. In conclusion, the solution is

$$u(r,\theta) = \frac{3}{4}r(\cos\theta - \sin\theta) + \frac{1}{4}(\cos 3\theta + \sin 3\theta).$$

This solution can be written back in Cartesian coordinates, recalling that $r\cos\theta = x$ and $r\sin\theta = y$. Thus, the first two terms are just $\frac{3}{4}(x-y)$. To get the other two terms in Cartesians, we use again the identities (11.24) and (11.25):

$$r^3\cos 3\theta = r^3(\cos^3\theta - 3\cos\theta\sin^2\theta) = x^3 - 3xy^3;$$

$$r^3\sin 3\theta = r^3(-\sin^3\theta + 3\cos^2\theta\sin\theta) = -y^3 + 3x^2y.$$

Therefore, in Cartesian coordinates the solution is

$$u(x,y) = \frac{3}{4}(x-y) + \frac{1}{4}(x^3 - 3xy^2 - y^3 + 3x^2y). \tag{11.23}$$

This is an example of a *harmonic polynomial*. ■

Exercise 11.4. Use de Moivre's formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

with $n = 3$ to prove the formulas

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \tag{11.24}$$

$$\sin 3\theta = -\sin 3\theta + 3\cos^2\theta\sin\theta. \tag{11.25}$$

Exercise 11.5. Check directly that (11.23) is harmonic inside the unit circle (in fact, it is harmonic *everywhere*), and that it is equal to $x^3 - y^3$ on the boundary of the circle.

The previous example is a special case of a theorem for *harmonic polynomials*.

Definition 11.26. A *harmonic polynomial*, $H(x,y)$ is a polynomial that satisfies Laplace's equation, $H_{xx} + H_{yy} = 0$.

Theorem 11.1. *Suppose $H(x,y)$ is harmonic in a disk, Ω, that is $H_{xx} + H_{yy} = 0$. Moreover, suppose that $H(x,y)$ satisfies the boundary condition $H(x,y) = P(x,y)$ on $\partial\Omega$ where $P(x,y)$ is a polynomial in x and y of degree n. Then the unique solution for $H(x,y)$ is a harmonic polynomial of degree n.*

Remark. In the example above $P(x,y) = x^3 - y^3$ and

$$H(x,y) = \frac{3}{4}(x-y) + \frac{1}{4}(x^3 - 3xy^2 - y^3 + 3x^2y)$$

which demonstrates that the two polynomials are not necessarily equal.

11.3 The Poisson Kernel

Formula (11.13) is usually as far as one can go, using the method of separation of variables. Sometimes, though, one gets lucky and one can obtain a more compact expression. That's what we will do now. By substituting all the coefficients into (11.13) we obtain a formula for the solution of our problem:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\, d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \int_{-\pi}^{\pi} (\cos n\theta \cos n\phi + \sin n\theta \sin n\phi) f(\phi)\, d\phi.$$

Using the addition formula for the cosine, we can simplify a little this expression:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\, d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \int_{-\pi}^{\pi} \cos n(\theta - \phi) f(\phi)\, d\phi. \tag{11.27}$$

Let us work a little more with (11.27). To begin with, let us assume that it is legal to interchange summation and integration. Again, you need some assumptions on $f(\phi)$ for this to hold, but we will again ignore this fact for the time being. Then we can rewrite (11.27) as

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi)\right] f(\phi)\, d\phi. \tag{11.28}$$

The surprising fact is that you can actually compute the sum of this series. We don't want to spoil you the fun of actually doing it yourself.

Exercise 11.6. Compute the sum:

$$\frac{1}{2} + \sum_{n=1}^{\infty} t^n \cos n\alpha, \qquad |t| < 1. \tag{11.29}$$

by using complex variables to relate this to a geometric series. The idea is that this series is the real part of the series

$$\frac{1}{2} + \sum_{n=1}^{\infty} t^n e^{in\alpha}, \tag{11.30}$$

which converges for every t less than 1 in absolute value. Find the sum of (11.30) using geometric series. Then the real part of your answer will be the sum of (11.29).

The answer you should get:

$$\frac{1}{2} + \sum_{n=1}^{\infty} t^n \cos n\alpha = \frac{1 - t^2}{2(1 - 2t\cos\alpha + t^2)}, \qquad |t| < 1. \tag{11.31}$$

Continuing with our computations, let us apply (11.31) to find the sum of the series in (11.28):

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - \left(\dfrac{r}{a}\right)^2}{1 - 2\left(\dfrac{r}{a}\right)\cos(\theta - \phi) + r^2} f(\phi)\,d\phi,$$

or, multiplying top and bottom by a^2,

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2} f(\phi)\,d\phi. \tag{11.32}$$

This formula makes sense for every θ and every $r < a$. When $r = a$, expression (11.32) no longer makes sense.

The function

$$K(r,\theta,a,\phi) = \frac{1}{2\pi}\frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2} \tag{11.33}$$

is called the *Poisson kernel*. Using it, we can write the solution to the problem of finding u such that $\Delta u = 0$ inside the circle, and $u = f$ on the boundary, in a very compact form:

$$u(r,\theta) = \int_{-\pi}^{\pi} K(r,\theta,a,\phi) f(\phi)\,d\phi. \tag{11.34}$$

11.4 Validity of the Poisson Kernel Solution

Formula (11.32) or, equivalently, (11.34), looks very nice. However, we obtained it under unspecified assumptions for f, lighthearted assumptions that the infinite sum of solutions is still a solution, and a careless swapping of integration.

What one can try to do now is look at (11.32), forget how we obtained this formula, and see if it satisfies the given problem. One immediate problem we have already noticed is that (11.32) no longer makes sense when $r = a$, so there is little hope that (11.32) will directly satisfy the boundary condition. However, one can prove the following surprising result.

Theorem 11.2. *If $f(\theta)$ is continuous and periodic with period 2π then the function $u(r,\theta)$ given by (11.32) or (11.34) satisfies $\Delta u = 0$ for $r < a$ and:*

$$\lim_{r\to a^-,\theta\to\theta_0} = f(\theta_0), \tag{11.35}$$

for every θ_0.

Remark. In other words, if we define a function $u(r,\theta)$ for $r \le a$ and every θ as:

$$u(r,\theta) = \begin{cases} \int_{-\pi}^{\pi} K(r,\theta,a,\phi) f(\phi)\, d\phi, & \text{if } r < a, \\ f(\theta), & \text{if } r = a, \end{cases} \tag{11.36}$$

then we obtain a function that is continuous on the closed circle $r \le a$ and harmonic inside it. One of the reasons why this result is surprising is the fact that we know that the Fourier expansion for f, which was one of our key assumptions along the way, in general is not valid when f is just continuous. And important addition to this result is that *the solution is unique*.

What is more, one can prove that, in fact, if f is only *piecewise continuous* and has only jump discontinuities, then (11.32) *is still a harmonic function* inside the circle and satisfies the limit condition (11.35) at every point θ_0 at which $f(\theta)$ is continuous.

The proof of the theorem is very beautiful and uses several results of analysis and harmonic functions:

Proof. (a) If $f(\theta)$ is continuous or, even weaker, if $f(\theta)$ is bounded an integrable, the function $u(r,\theta)$ given by:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n \cos n\theta + b_n \sin n\theta), \tag{11.37}$$

where a_n and b_n are given by (11.11) and (11.12), is infinitely differentiable inside the circle. Indeed, the Fourier coefficients a_n and b_n are bounded, so any derivative of $u(r, \theta)$ is bounded above in absolute value by

$$\sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n (|a_n| + |b_n|),$$

which converges inside the circle. A theorem from analysis (the "Weierstrass M-series theorem") shows that $u(r, \theta)$ has as many derivatives as we want, and in fact satisfies $\Delta u = 0$ inside the circle, that is, it is harmonic inside the circle. The same applies to all subsequent formulas we got for $u(r, \theta)$; in particular, this proves that (11.32) is harmonic inside the circle, under these very weak assumptions on f.

(b) If $f(\theta)$ has many continuous derivatives, let us say three, to be on the safe side, then equality (11.37) is true, with a_n and b_n given by (11.11) and (11.12). Moreover, the coefficients a_n and b_n can be bounded by terms of the form $\frac{M}{n^3}$. This, again by the "Weierstrass M-series" result will imply that the series (11.37) converges uniformly on the *closed* circle $r \leq a$, and on the boundary is equal to $f(\theta)$. In other words, if f is sufficiently smooth, then (11.37) indeed produces a solution $u(r, \theta)$ to our problem: a function u harmonic inside the circle and equal to f on the boundary. This in turn implies that the same can be said about the function $u(r, \theta)$ defined by (11.32).

(c) If $f(\theta)$ is (only) continuous, then it can be uniformly approximated by a sequence of functions $f_k(\theta)$ that have as many derivatives as desired. This is a powerful result from analysis, the Weierstrass approximation theorem (this guy Weierstrass keeps popping up awfully often, don't you think?). In fact, one can choose f_k to be even a finite sum of trigonometric functions $\sin(m\theta)$ and $\cos(m\theta)$ (in general, it will *not* be a truncation of the Fourier series, though).

(d) For every $f_k(\theta)$, which is now as smooth as we want, the corresponding function $u_k(r, \theta)$ given by (11.32) or by (11.34) for f_k instead of f, will be a function harmonic inside the circle, and equal to f_k on the boundary of the circle. This means that $u_k - u_m$ will also be harmonic on the circle and equal to $f_k - f_m$ on the boundary of the circle, for every k and m.

(e) Remember now the Maximum Principle for harmonic functions: the maximum and the minimum of $u_k - u_m$ on the closed circle can only be attained on the boundary. This will mean that if $|f_k - f_m|$ can be made less than ϵ, then also $|u_k(r,\theta) - u_m(r,\theta)|$ will be less than ϵ for all $r \leq a$ and all θ. Since f_k converges uniformly (to f) on the boundary of the circle, this will imply that u_k converges uniformly to some function u on the whole closed circle.

(f) Finally, since f_k converges uniformly to f, this means that when taking the limit as $k \to \infty$ in

$$u_k(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2} f_k(\phi)\, d\phi$$

we can switch "$\lim_{k\to\infty}$" with integration at the right-hand side. This proves that the limit function u of the sequence u_k is precisely given by (11.32), and concludes the proof.

□

The proof of the more general assertion when f is piecewise continuous is omitted. We will only observe that one important ingredient in this proof if the fact that the Poisson kernel $K(r,\theta,a,\phi)$ is *positive* for $r < a$, a fact that we invite you to prove.

Exercise 11.7. Prove that the Poisson kernel $K(r,\theta,a,\phi)$ is always positive for $r < a$.

11.5 Interpretation of the Poisson Kernel

Every time you get a solution of a linear problem in the form (11.34), where $f(\theta)$ may be either a non-homogeneous boundary conditions (as in this case) or a non-homogeneous right-hand side (as in the case of a force acting on the system), the kernel K has an important mathematical and physical interpretation.

To save a little in the notation, since the radius of the circle is fixed and the function K depends on the *difference* $\theta - \phi$ rather than on the variables θ and ϕ independently, let us call:

$$k(r,s) = \frac{1}{2\pi}\frac{a^2 - r^2}{a^2 - 2ar\cos s + r^2},$$

so that $K(r,\phi,a,\theta) = k(r,\phi - \theta)$. Then (11.34) can be rewritten as

$$u(r,\theta) = \int_{-\pi}^{\pi} k(r,\theta-\phi) f(\phi)\, d\phi. \tag{11.38}$$

Imagine now a constant boundary condition f of height $\frac{1}{\Delta}$, concentrated on a small interval of length Δ with center at a point ϕ_0.

In other words, f is zero everywhere, except near ϕ_0, at which it is constant and equal to $\frac{1}{\Delta}$ on an interval of length Δ. The height of f has been chosen so as to have $\int_{-\pi}^{\pi} f(\phi)\, d\phi = 1$. Call δ_{ϕ_0} this function. (It also depends on Δ, but let us not complicate the notation.) According to (11.38), the solution of the Dirichlet problem on the circle for this f is given by

$$u(r,\theta) = \int_{-\pi}^{\pi} k(r,\theta-\phi)\delta_{\phi_0}(\phi)\, d\phi.$$

Applying the intermediate value theorem for integrals, we can rewrite this as

$$u(r,\theta) = k(r,\theta-\phi^*) \int_{-\pi}^{\pi} \delta_{\phi_0}(\phi)\, d\phi,$$

where ϕ^* is a number on the interval with length Δ and center ϕ_0. Since the integral of δ_{ϕ_0} is equal to 1, this becomes

$$u(r,\theta) = k(r,\theta-\phi^*).$$

Taking the limit as $\Delta \to 0$, we get

$$u(r,\theta) = k(r,\theta-\phi_0).$$

As $\Delta \to 0$, the function δ_{ϕ_0} itself tends to infinity in such a way that its integral is kept equal to 1 all the time. This is the so-called *the delta function* with center ϕ_0, and is denoted as $\delta(\theta-\phi_0)$. It is not a function but rather a generalized function, what mathematicians call a *distribution*. It is one of those extremely useful beasts that was born in the deranged mind of physicists and mathematicians had to work very hard to make any sense out of them.

Lemma 11.39. *We conclude that* the Poisson kernel

$$K(r,\theta,a,\phi) = k(r,\theta-\phi) = \frac{1}{2\pi}\frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}$$

is a harmonic function inside the circle $r < a$ that is equal to zero everywhere on the boundary, except at the point ϕ, at which it is equal to ∞. *Not only that, but $k(r,\theta-\phi)$ is a harmonic function that on the boundary coincides with the delta function with center at ϕ.*

Knowing the meaning of the Poisson kernel, one can now use an heuristic argument to obtain again formula (11.38).

Assume we are given a function $f(\theta)$ on the boundary of the circle. We divide the interval $[0, 2\pi]$ into N intervals of length Δ and approximate $f(\theta)$ by a piecewise constant function, writing it down as the sum of functions $f(\phi_i)\delta_{\phi_i}\Delta$:

$$f(\phi) \approx \sum_{i=1}^{N} f(\phi_i)\delta_{\phi_i}\Delta.$$

The factor Δ is needed to get the correct height $f(\theta_i)$; recall that δ_{ϕ_i} had height $\frac{1}{\Delta}$.

Since the Dirichlet problem is linear, the solution to the boundary condition $f(\phi_i)\delta_{\phi_i}\Delta$ will be $f(\phi_i)\Delta$ times the solution to the boundary condition δ_{θ_i}. Therefore, the solution will be

$$k(r, \theta - \phi_i)f(\phi_i)\Delta.$$

Again, since the Dirichlet problem is linear, the solution of a sum of boundary conditions $f(\phi_i)\delta_{\phi_i}\Delta$ will be the sum of the solutions for each boundary condition separately, that is,

$$\sum_{i=1}^{N} k(r, \theta - \phi_i)f(\phi_i)\Delta.$$

We recognize this as a Riemann sum, so if we take the limit as $\Delta \to 0$, we get the solution to the problem with boundary condition $f(\theta)$ as

$$\lim_{\Delta \to 0} \sum_{i=1}^{N} k(r, \theta - \phi_i)f(\phi_i)\Delta = \int_{-\pi}^{\pi} k(r, \theta - \phi)f(\phi)\, d\phi,$$

Which completes the proof.

11.6 Challenge Problems for Lecture 10

Problem 11.1. *Integrating the Poisson Kernel.* Prove that

$$\int_{-\pi}^{\pi} K(r, \theta, a, \phi)\, d\phi$$

is a harmonic function $u(r, \theta)$ inside the circle $r < a$, and tends to 1 for every θ, as $r \to a^-$. *Hint:* solve the Dirichlet problem $\Delta u = 0$ inside the circle and $u = 1$ on the boundary, and use uniqueness of the solution.

Problem 11.2. Solve the problem

$$\Delta u(r, \theta) = 0 \quad \text{if } r < a, \quad \textit{PDE}$$

$$u(a, \theta) = \begin{cases} 0, & \text{if} \quad -\pi < \theta < 0, \\ 1, & \text{if} \quad 0 < \theta < \pi. \quad \textit{BC} \end{cases}$$

Problem 11.3. Using the result you found in Problem 2, plus uniqueness of the solution of the Dirichlet problem for the Laplace equation, write down the integral

$$\int_{0}^{\pi} K(r, \theta, a, \phi)\, d\phi = \frac{1}{2\pi} \int_{0}^{\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2}\, d\phi,$$

as an infinite series, assuming $r < a$.

Problem 11.4. One can use polar coordinates, and separation of variables, for "pizza slices", that is, for sectors of a circle. As an example, find the steady-state temperature distribution of a thin plate over the sector

$$\Omega = \{(r,\theta)\,|\,0 < r < 1,\ 0 < \theta < \frac{\pi}{3}\}$$

given that

$$u(r,0) = 0, \qquad u\left(r, \frac{\pi}{3}\right) = 0;$$
$$u(1,\theta) = \theta\left(\frac{\pi}{3} - \theta\right).$$

We assume that the temperature at zero is bounded.

Repeat the process of separation of variables. What will be the boundary conditions for $\Theta(\theta)$? Notice that the eigenvalues will *not* be the same as for the case of the full circle.

Problem 11.5. Find the steady-state temperature distribution of a thin plate over the sector

$$\Omega = \{(r,\theta)\,|\,0 < r < 1,\ 0 < \theta < \frac{\pi}{3}\}$$

given that

$$u(r,0) = 0, \qquad u_\theta\left(r, \frac{\pi}{3}\right) = 0;$$
$$u(1,\theta) = \theta\left(1 - \frac{3\theta}{2\pi}\right).$$

Assume that the temperature at zero is bounded.

Twelve

Well-posed Problems: Existence, Uniqueness and Stability

When solving partial differential equations, mathematicians put a high premium on problems that are *well-posed*, that is problems which have a unique solution and which are stable in the sense that small perturbations to the initial and boundary conditions only yield small perturbations to the solution.

The heat equation, the wave equation and Laplace's Equation with appropriate boundary conditions are well-posed and this increases there usefulness when modeling physical problems. Below we will study well-posedness, concentrating primarily on the heat equation.

12.1 What is a well-posed problem?

A well-posed problem has three characteristics:

1) *Existence*: A solution exists to the problem. It satisfies the governing PDE and the associated boundary and initial conditions.

2) *Uniqueness*: The solution is the only solution satisfying the governing PDE and the associated boundary and initial conditions. That is the solution is unique.

3) *Stability*: If a small perturbation is made to the initial condition or boundary conditions, the solution changes by only a small amount.

These characteristics are physically desirable; for example, if we are modeling a physical problem, such as heat flow, we want there to be a unique solution of the governing equations and if perchance we make a small mistake in measuring the initial state we don't want the solution to change radically.

We also need to also talk about the *regularity* of the solutions. Solutions "live" in a function space. For example, for the heat equation, it is natural to talk about $u(x,t) \in C_x^2[0,L]$ - that is $u(x,t)$, $u_x(x,t)$, and the second derivative, $u_{xx}(x,t)$, are continuous, and $u(x,t) \in C_t^1[0,\infty)$ - that is $u(x,t)$ and $u_t(x,t)$ are continuous. One makes assumptions about the regularity of the solution implicitly when one solves the heat equation. After all, what would it mean for a function to solve the equation if these derivatives were undefined?

12.2 Existence

Existence can most easily be demonstrated by constructing an explicit solution. For the homogeneous Dirichlet problem,

$$\begin{array}{rccl} \mathbb{DE}: & & U_t = DU_{xx} & 0 < x < L, t > 0 \\ \mathbb{BC}: & U(0,t) = 0, & U(L,t) = 0, & t > 0 \\ \mathbb{IC}: & & U(x,0) = f(x) & 0 < x < L. \end{array}$$

we have already found a solution via separation of variables,

$$U(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt}\,. \tag{12.2}$$

Choosing

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

will satisfy the initial condition. If in addition, we specify that $f(x) \in C_x^2[0,L]$ with $f(0) = f(\pi) = 0$, then the Fourier sine series converges uniformly to the initial condition and the solution $u(x,t)$ is in $C_x^2[0,L]$ and $C_t^1[0,\infty)$. In this case, we have shown the solution exists and has the specified regularity.

Exercise 12.1. Show that for the homogeneous Neumann problem,

$$\begin{array}{rccl} \mathbb{DE}: & & U_t = DU_{xx} & 0 < x < L, t > 0 \\ \mathbb{BC}: & U_x(0,t) = 0, & U_x(L,t) = 0, & t > 0 \\ \mathbb{IC}: & & U(x,0) = f(x) & 0 < x < L. \end{array}$$

with $f(x) \in C_x^2[0, L]$ with $f_x(0) = f_x(L) = 0$ that a solution exists with $U(x, t)$ in $C_x^2[0, L]$ and $C_t^1[0, \infty)$. Why is the condition $f_x(0) = f_x(L) = 0$ necessary?

12.3 Energy Dissipation and Uniqueness

By looking at what is normally known as energy for the diffusion equation, we can show that the solution for the Dirichlet problem is unique. Note this energy is a mathematical construct, not to be confused with the thermal energy discussed in the derivation of the diffusion equation.

Let's define, the energy,

$$W \equiv \frac{1}{2} \int_0^L U^2 \, dx, \tag{12.4}$$

which is a function of t dependent on the particular solution $U(x, t)$ (technically it is a function of t and a *functional* of $U(x, t)$). Note that $W \geq 0$ and, assuming that U is continuous in x, $W = 0$ only for the trivial solution $U(x, t) = 0$.

If we differentiate the energy with respect to time, we find

$$\begin{aligned} \frac{dW}{dt} &= \int_0^L U U_t \, dx, \\ &= D \int_0^L U U_{xx} \, dx, \\ &= -\int_0^L (U_x)^2 \, dx + U U_x \left.\right|_{x=0}^{x=L}, \end{aligned}$$

where we have substituted the DE and used integration by parts. Now, applying the BC's, we fine that the boundary terms from the integration by parts vanish, so

$$\frac{dW}{dt} = -\int_0^L (U_x)^2 \, dx \leq 0.$$

Now, we can conclude that W is decreasing (that is energy is dissipated) *unless* $U_x = 0$, that is to say that U is constant. As the only constant solution satisfying the boundary conditions is $U = 0$, we might be tempted to conclude that the solution always decays to this trivial state. This turns out to be true, although one must invest some analysis to show it rigorously.

A second conclusion one can reach is that if $f(x) = 0$, that $U(x,t) = 0$ for all $t > 0$. This follows quickly because $W = 0$ at $t = 0$, it is non-increasing and non-negative. While this seems like a trivial result, it has a very powerful consequence.

Suppose we had two solutions to the non-homogeneous Dirichlet problem, call them U_1 and U_2. You should be able to convince yourself that there difference

$$V = U_1 - U_2$$

satisfies the homogeneous Dirichlet problem with $f(x) = 0$. Consequently, we know that $V(x,t) = 0$ for all $t > 0$, which implies $U_1 = U_2$. From this we conclude that: *The solution to the non-homogeneous Dirichlet problem is unique.* This is a powerful result indeed.

Exercise 12.2. Convince yourself the energy argument for uniqueness of solutions in the previous paragraph is correct. Show that a similar argument can be made for the Neumann problem.

12.4 The Maximum Principle

Looking at solutions to the heat equation, we note that they tend to average out maximums and minimums. We can develop some intuition for this by considering what the equation says. Basically, $U_t = DU_{xx}$ means: *The temperature is decreasing when the profile is convex down and the temperature is increasing when the profile is convex up*.

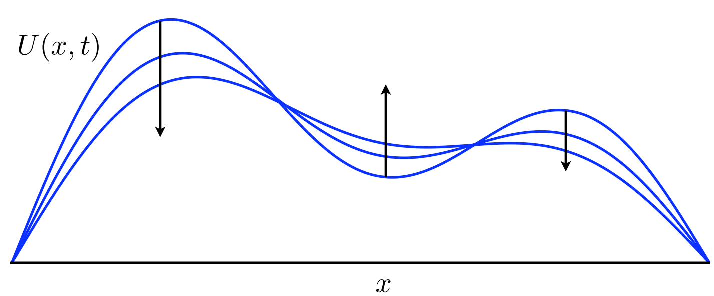


Figure 12.1: The heat equation interpreted graphically. The temperature is decreasing when the profile is convex down and the temperature is increasing when the profile is convex up.

From which we conclude that interior maximums in temperature are decreasing and interior minimums of temperature are increasing. For the Dirichlet problem,

$$\begin{array}{rlr} \mathbb{DE}: & U_t = DU_{xx} & 0 < x < L, t > 0 \\ \mathbb{BC}: & U(0,t) = a(t), \quad U(L,t) = b(t), & t > 0 \\ \mathbb{IC}: & U(x,0) = f(x) & 0 < x < L. \end{array}$$

We can give a rigorous statement of these ideas which is called the *Maximum Principle*:

Theorem 12.1 (Maximum Principle for the Diffusion Equation)**.** *Suppose $U(x,t)$ satisfies the Dirichlet problem for the diffusion equation in the rectangle $0 < x < L$, $0 < t < T$. Also, assume that $u(x,t)$ is in $C_x^2[0,L]$ and $C_t^1[0,T]$. Then it assumes its maximum value (as a function of x and t) either initially (when $t = 0$) or on the lateral boundaries (where $x = 0$ or $x = L$).*

For convenience we refer to the rectangle $0 \le x \le L$, $0 \le t \le T$ as R, and let

$$M = \max_{(x,t)\in R} U(x,t).$$

The maximum principle says that U obtains the value M either initially (when $t = 0$) or on the lateral boundaries of the rectangle R. Plausibly it could obtain the value of M at some points in the interior also.

First we prove a simple lemma about what a maximum would look like:

Lemma 12.5. *If $u(x,t)$ has a maximum $(\bar{x},\bar{t})$ in the interior of the rectangle R, then $U_t(\bar{x},\bar{t}) = U_x(\bar{x},\bar{t}) = 0$ and $U_{xx}(\bar{x},\bar{t}) \le 0$.*

Proof of Lemma. This follows from single variable calculus. First note that if we consider $U(\bar{x},t)$ as a function of t that it reaches a maximum at $\bar{t}$, and therefore U_t must vanish (where we have used the differentiability of U with respect to t). Similarly, we consider $U(x,\bar{t})$ as a twice continuously differentiable function of x. If it is a maximum, then $U_x(\bar{x},\bar{t}) = 0$. Moreover, if $U_{xx}(\bar{x},\bar{t}) > 0$, it is a strict minimum and therefore cannot be a maximum, from which we conclude $U_{xx}(\bar{x},\bar{t}) \le 0$. □

Now we proceed to the main event:

Proof of Maximum Principle. Suppose we found a maximum in the interior of the rectangle; we know that $U_t = 0$ and if $U_{xx} < 0$, we would have a contradiction because u satisfies the heat equation, $U_t = DU_{xx}$. The problem is that we could have a maximum where $U_{xx} = 0$. We deal with this by introducing the idea of a *subfunction*.

Define a new function,

$$V(x,t) = U(x,t) - \epsilon t x(L-x),$$

where ϵ is a positive constant; we call $V(x,t)$ a *subfunction* of $U(x,t)$ as it is slightly below it. Note that

(a) The function $V \leq U$ with equality only on the lateral and bottom boundary of R.

(b) The difference between U and V in R is

$$U - V = \epsilon t x(L-x) \leq \epsilon \frac{TL^2}{4},$$

which tends to zero uniformly as $\epsilon \to 0$.

(c) In the limit of decreasing ϵ,

$$\max_{x\in R} U(x,t) = \lim_{\epsilon\to 0}\left[\max_{x\in R} V(x,t)\right],$$

which follows from the uniform bound on the difference between U and V shown above.

Substituting into the $\mathbb{DE}$, we see that V satisfies

$$\begin{aligned}
\mathbb{DE}: &\quad V_t = DV_{xx} - \epsilon\left[2Dt + x(L-x)\right] && 0 < x < L, t > 0\\
\mathbb{BC}: &\quad V(0,t) = a(t), \quad V(L,t) = b(t), && t > 0\\
\mathbb{IC}: &\quad V(x,0) = f(x) && 0 < x < L.
\end{aligned}$$

Note that at a maximum $(\bar{x},\bar{t})$ of V in the interior of R that $V_t = 0$, which allows us to conclude that $V_{xx} = \epsilon[2t + x(L-x)/D] > 0$ from the $\mathbb{DE}$ which is a contradiction, so V does *not* have a maximum in the interior of the rectangle R.

If the maximum occurs at a point on the top boundary, $(\bar{x}, T)$, where $0 < \bar{x} < L$, we know that $V_x(\bar{x},T) = 0$ and $V_{xx}(\bar{x},T) \leq 0$, so

$$V_t = DV_{xx} - \epsilon\left[2Dt + x(L-x)\right] < 0$$

and consequently the function is decreasing in time which is a contradiction, so we know the maximum occurs on one of the other three boundaries.

So the maximum of $V(x,t)$ must occurs on the lateral boundaries or the bottom boundary where $U = V$. Consequently we have demonstrated the Maximum Principle for V, but as $\epsilon \to 0$, we know U converges uniformly to V (as does the maximum of U), so we conclude that the maximum of U must occur on the lateral or bottom boundary also, which completes the proof. □

There is a stronger version of the Maximum Principle that says if a maximum of U occurs in the interior of R then U is constant which leads us to the following exercise for the reader to contemplate.

Exercise 12.3. Show that while the Maximum Principles guarantees that the maximum of U occurs on the boundary of R, that it doesn't guarantee that it occurs *only* on the boundary of R. You may wish to do this by presenting an example.

Note the same result is also true of the minimum of $U(x,t)$, which is sometime called the *Minimum Principle*. This can be seen easily by considering $-U(x,t)$ which also satisfies the diffusion equation.

The Maximum Principle gives us another proof of uniqueness of the solution for the Dirichlet problem. Again, suppose we had two solutions to the non-homogeneous Dirichlet problem, call them U_1 and U_2 that satisfy the regularity conditions for the Maximum Principle ($U(x,t) \in C_x^2[0,L]$ and $U(x,t) \in C_t^1[0,\infty)$. The difference $V = U_1 - U_2$ satisfies the homogeneous Dirichlet problem with $f(x) = 0$. From the Maximum Principle (and the Minimum Principle) we know that $V(x,t)$ obtains its maximum (and minimum) either initially (at $t = 0$) or on the boundaries (where $x = 0$ or $x = L$) where $V = 0$. Consequently $V = 0$ for $0 < x < L$ and $t > 0$. Which means $U_1 = U_2$ and again we conclude the solution is unique.

Exercise 12.4. The Maximum Principle for the homogeneous Neumann problem,

$$\begin{aligned}
\mathbb{DE}: &\quad U_t = DU_{xx} &\quad 0 < x < L, t > 0\\
\mathbb{BC}: &\quad U_x(0,t) = 0, \quad U_x(L,t) = 0, &\quad t > 0\\
\mathbb{IC}: &\quad U(x,0) = f(x) &\quad 0 < x < L,
\end{aligned}$$

states that:

Theorem 12.2 (Maximum Principle for the Diffusion Equation)**.** *Suppose $U(x,t)$ satisfies the homogeneous Neumann problem for the diffusion equation in the rectangle $0 < x < L$, $0 < t < T$. Also, assume that $u(x,t)$ is in $C^2_x[0,L]$ and $C^1_t[0,T]$. Then $U(x,t)$ assumes its maximum value (as a function of x and t) initially (when $t = 0$).*

Prove this theorem using the subfunction $V(x,t) = U(x,t) - \epsilon t$.

Exercise 12.5. Show that the solution to the Neumann problem is unique using the results of Exercise 12.4.

12.5 Stability

Can a butterfly flapping its wings in Beijing alter the weather in San Francisco?

- Attributed to Ed Lorenz

Loosely speaking, we will say a system is *stable* if a small change in the initial condition induces only a small perturbation in the solution for $t > 0$

12.5.1 The backward heat equation - an example of instability

An example of instability is the backwards heat equation. Recall that in the heat equation we assume D is positive. Consider the Dirichlet problem that $D < 0$. In this case, the heat flows from cold to hot. This is the equivalent of running the heat equation backward in time.

Note that in our previous derivation of the solution, we did not make any use of the sign of D. Consequently, the solution to:

$$\begin{array}{rccl} \mathbb{DE}: & & U_t = -U_{xx} & 0 < x < L, t > 0 \\ \mathbb{BC}: & U(0,t) = 0 & U(L,t) = 0 & t > 0 \\ \mathbb{IC}: & & U(x,0) = \frac{1}{n}\sin(\frac{n\pi x}{L}) & 0 < x < L, \end{array}$$

where we have chosen $D = -1$ is

$$u(x,t) = \frac{1}{n}\sin\left(\frac{n\pi x}{l}\right)e^{\frac{n^2\pi^2}{l^2}t}.$$

Note that

$$\max_{0<x<l} u(x,0) = \frac{1}{n} \qquad 0 < x < L$$

but

$$\max_{0<x<l} u(x,t) = \frac{1}{n} e^{\frac{n^2\pi^2}{L^2}t}.$$

So given any $\epsilon > 0$, I can choose n such that $\frac{1}{n} < \epsilon$ and $|u(x,0)| < \epsilon$. But for any $T > 0$,

$$\max_{0<x<l} u(x,T) = \frac{1}{n} e^{\frac{n^2\pi^2}{l^2}T}$$

and as $n \to \infty$, this maximum of the solution tends towards infinity. Consequently, I can find solutions that, although arbitrarily small initially, that can be arbitrarily large at any fixed positive time T. In fact, for many initial conditions, the solution goes to infinity in a finite amount of time.

12.5.2 Stability for the forward heat equation

In fact, the forward heat equation is stable to perturbations of the initial condition. Suppose we consider two solutions to the homogeneous Dirichlet problem, $U_1(x,t)$ and $U_2(x,t)$ with initial conditions $U_1(x,0) = f_1(x)$ and $U_2(x,0) = f_2(x)$. If

$$\max_{0<x<l} |f_1(x,0) - f_2(x,0))| \leq \delta.$$

we can appeal to the dissipation of energy to get a bound on the difference between the solutions for all time.

Let

$$V(x,t) = U_1(x,t) - U_2(x,t) \ .$$

Note that $V(x,t)$ also satisfies a homogeneous Dirichlet problem, and we have shown previously that the energy associated with V is dissipated. Consequently,

$$W[V(x,t)] \leq W[V(x,0)]$$

or

$$\int_0^L \frac{V^2}{2}\, dx \leq \int_0^L \frac{\delta^2}{2}\, dx = \frac{\delta^2 L}{2}$$

Consequently,

$$||U_1(x,t) - U_2(x,t)||_{L^2} = ||V(x,t)||_{L^2} \leq \delta\sqrt{L}$$

so that the two solutions remain close in the L^2 or root-mean-square sense for all time.

One problem with this proof of stability is that just because solutions are close in L^2 does not mean that they remain close uniformly. We would refer to the result above as L^2 *stability to perturbations of the initial condition*.

A stronger result can be proven using the Maximum Principle. In fact, if

$$\max_{0<x<l} |f_1(x,0) - f_2(x,0))| \leq \delta$$

it follows immediately from the maximum principle that

$$\max_{0<x<l} |U_1(x,t) - U_2(x,t))| \leq \delta$$

which would be *uniform stability to perturbations of the initial condition*. The maximum principle can be used to prove stability with respect to perturbations to boundary conditions also.

Exercise 12.6. Show that the proof of L^2 stability for perturbation of the initial conditions of the Dirichlet problem using energy dissipation can be extended to the Neumann problem.

Exercise 12.7. Show that the proof of uniform stability can be extended to the Neumann problem using the results of Exercise 12.4.

Exercise 12.8. Show that the proof of uniform stability for the Dirichlet problem using the Maximum Principle can be extended to perturbations of the boundary condition also.

Exercise 12.9. Explain why uniform stability implies L^2 stability, but L^2 stability does *not* imply uniform stability.

Part V

The Cauchy Problem

Thirteen

The Cauchy Problem for the Wave Equation: d'Alembert's Solution

While the wave equation can be used to model the oscillations of a string, it also governs the propagation of electromagnetic waves over long distances. For these problems it makes more sense to consider the problem on the infinite line. This problem is called the *Cauchy Problem*, named in honor of Baron Augustin-Louis Cauchy (1789 – 1857) who considered a variety of problems in the absence of boundaries.

13.1 The Cauchy Problem for the Wave Equation

For the wave equation on an infinite domain we have the following:

The Cauchy Problem for the Wave Equation

$$\begin{array}{rccl} \mathbb{DE}: & & u_{tt} = c^2 u_{xx}, & -\infty < x < \infty, t > 0 \\ \mathbb{IC}: & u(x,0) = f(x), & u_t(x,0) = g(x) & -\infty < x < \infty \end{array}$$

Amazingly, we have an exact solution to this problem:

Theorem 13.1 (d'Alembert's Solution)**.** *The solution to the Cauchy problem for the wave equation is:*

$$u(x,t) = \tfrac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz,$$

when $f \in \mathcal{C}^2_x$ and $g \in \mathcal{C}^1_x$ on the real line, $-\infty < x < \infty$.

Exercise 13.1. Verify that the d'Alembert solution satisfies the Cauchy problem for the wave equation via direct substitution into the $\mathbb{DE}$ and the $\mathbb{IC}$s.

13.2 A Derivation of the d'Alembert Solution

This solution can easily be verified by direct substitution, but let us motivate the derivation. The wave equation can be written in operator notation,

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) u(x,t) = 0,$$

and we can factor the differential operator as a difference of two squares

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) &= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) \\ &= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\end{aligned}$$

Consequently we could write the wave equation as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)[u_t + cu_x] = 0$$

or

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)[u_t - cu_x] = 0$$

From which we decide that if $u(x,t)$ satisfies either first-order wave equation,

$$u_t - cu_x = 0 \qquad \text{or} \qquad u_t - cu_x = 0$$

then $u(x,t)$ also satisfies the wave equation. That is both

$$u(x,t) = A(x - ct) \qquad \text{and} \qquad u(x,t) = B(x + ct)$$

are solutions for any functions A and B. Since the equation is linear and homogeneous, we can deduce that

$$u(x,t) = A(x - ct) + B(x + ct)$$

is the most general solution.

We can derive this more rigorously via a change of variables; let

$$\xi = x - ct, \qquad \eta = x + ct \qquad u(x,t) \equiv U(\xi, \eta).$$

From the chain rule, we see that

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\end{aligned}$$

from which we can see that

$$\begin{aligned}\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) &= -2c\frac{\partial}{\partial \xi} \\ \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) &= 2c\frac{\partial}{\partial \eta}\end{aligned}$$

Thus, we can rewrite the $\mathbb{DE}$ as

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u(x,t) = 0 \quad \Rightarrow \quad -4c^2\frac{\partial^2}{\partial \eta \partial \xi}U(\xi,\eta) = 0$$

or (assuming $c \neq 0$) as just

$$\frac{\partial^2 U}{\partial \eta \partial \xi} = 0. \tag{13.1}$$

Integrating this equation with respect to η yields

$$\frac{\partial U}{\partial \xi} = C(\xi)$$

for some arbitrary function $C(\eta)$. Integrating with respect to ξ now yields

$$U(\xi, \eta) = A(\xi) + B(\eta)$$

where $A(z)$ is the indefinite integral of $C(z)$,

$$A(z) = \int C(z)\, dz,$$

which is again an arbitrary function. If we rewrite this in terms of the original variables, we obtain

$$\boxed{u(x,t) = A(x - ct) + B(x + ct)} \tag{13.2}$$

as previously advertised. Note that this solution can be interpreted as the superposition of a right-going wave and a left-going wave each propagating with a speed c. We will see this play out physically in some of the examples below.

Let us now apply the initial conditions; the initial position satisfies

$$u(x,0) = A(x) + B(x) = f(x). \tag{13.3}$$

The velocity can be computed by differentiating with respect to time

$$u_t(x,t) = -cA'(x-ct) + cB'(x+ct)$$

where A' denotes the derivate of A with respect to its argument and the factors of c appear due to the chain rule. Consequently, the initial velocity satisfies

$$u_t(x,0) = -cA'(x) + cB'(x) = g(x). \tag{13.4}$$

The initial velocity (13.4) can be integrated from 0 to x to yield

$$\int_0^x g(z)\,dz = \int_0^x [-cA'(x) + cB'(x)]\,dz = c[B(x) - A(x) - B(0) + A(0)]\,.$$

Remembering that we are trying to solve for A and B, we rewrite this as

$$-A(x) + B(x) = \frac{1}{c}\int_0^x g(z)\,dz + B(0) - A(0). \tag{13.5}$$

We can now solve (13.3) and (13.5) for $A(x)$ and $B(x)$ to yield

$$\begin{aligned} A(x) &= \tfrac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)\,dz - \tfrac{1}{2}[B(0) - A(0)] \\ B(x) &= \tfrac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(z)\,dz + \tfrac{1}{2}[B(0) - A(0)] \end{aligned}$$

substituting back into (13.2) yields

$$u(x,t) = A(x-ct)+B(x+ct) = \tfrac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\left[\int_0^{x+ct} g(z)\,dz - \int_0^{x-ct} g(z)\,dz\right]$$

and combining the two integrals finally yields the d'Alembert solution,

$$u(x,t) = \tfrac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz,$$

as described in the theorem above.

Exercise 13.2. Explain the appearance and subsequent disappearance of the constant $B(0) - A(0)$ in the above derivation; how is this related to A and B not being uniquely defined?

13.3 Some Examples of Solution to the Cauchy Problem

We can use the d'Alembert solution to investigate some solutions to the Cauchy problem for the wave equation.

Example 13.1. Find a solution to the wave equation

$$\begin{array}{rlll} \mathbb{DE}: & u_{tt} = c^2 u_{xx}, & & -\infty < x < \infty, t > 0 \\ \mathbb{IC}: & u(x,0) = e^{-x^2}, & u_t(x,0) = 0 & -\infty < x < \infty \end{array}$$

Solution: From the d'Alembert's solution we know that

$$U(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi,$$

where $U(x,0) = f(x)$ and $U_t(x,0) = g(x)$. We see that

$$U(x,t) = \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right].$$

The solution consists of two gaussians, one propagating to the left and one propagating to the right. ■

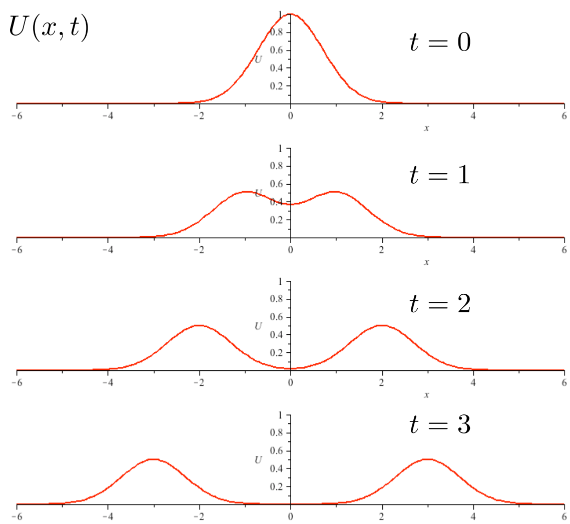


Figure 13.1: A graph of $U(x,t)$ for $t = 0, 1, 2, 3$ and $c = 1$. The initial gaussian splits into two, one propagating to the left at speed c and one propagating to the right at speed c, each of half the amplitude of the initial condition.

Example 13.2. Find a solution to the wave equation

$$\begin{array}{lllr} \mathbb{DE}: & & u_{tt} = c^2 u_{xx}, & -\infty < x < \infty, t > 0 \\ \mathbb{IC}: & u(x,0) = 0, & u_t(x,0) = xe^{-x^2} & -\infty < x < \infty \end{array}$$

Show that $U(0,t) = 0$ and graph the solution at a sample time.
Solution: From the d'Alembert's solution we know that

$$U(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \xi e^{-\xi^2}\, d\xi, = -\frac{e^{-\xi^2}}{4c}\bigg|_{x-ct}^{x+ct} = \frac{1}{4c}\left[e^{-(x-ct)^2} - e^{-(x+ct)^2}\right].$$

We can evaluate the solution at the origin to see that

$$U(0,t) = \frac{1}{4c}\left[e^{-(-ct)^2} - e^{-(ct)^2}\right] = 0,$$

which also follows from the odd symmetry of the solution at the origin, which is also evident in the graph below.

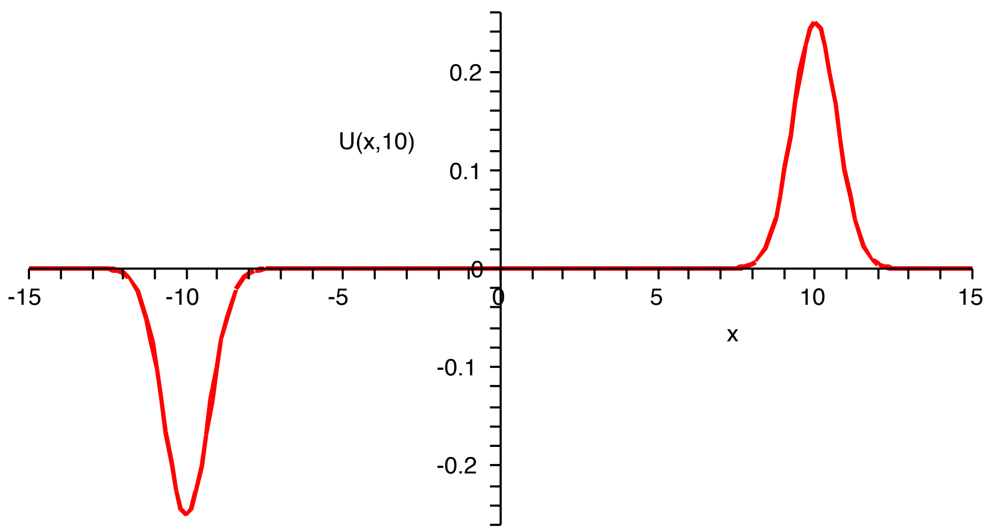


Figure 13.2: A graph of $U(x,10)$ for $c = 1$; we have a positive Gaussian propagating to the left centered at $x = 10$ and its negative image propagating to the right.

■

13.3.1 Periodic Initial Conditions

One can use the d'Alembert solution to find a solution to the wave equation on a finite interval by looking at equivalent periodic initial conditions on the infinite interval.

Example 13.3. In this problem we will use the d'Alembert solution to recover some solutions we found via separation of variables:

(a) Use the d'Alembert's solution to solve the problem

$$\begin{array}{rccl} \mathbb{DE}: & & U_{tt} = c^2 U_{xx}, & -\infty < x < \infty, t > 0 \\ \mathbb{IC}: & U(x,0) = \sin(nx), & U_t(x,0) = g(0) & -\infty < x < \infty \end{array}$$

where n is a positive integer. Show the solution can be written in the form

$$U(x,t) = \sin(nx)\cos(\omega_n t)$$

where ω_n is to be determined and relate the solution to the one we found via separation of variables on a finite interval.

(b) Use the same idea to solve the problem

$$U_{tt} = c^2 U_{xx} \qquad 0 < x < \pi, \quad 0 < t$$

$$U(x,0) = 0, \quad U_t(x,0) = \sin(nx). \qquad 0 < x < \pi$$

$$U(0,t) = U(\pi,t) = 0, \qquad 0 < t,$$

where, once again, n is a positive integer.

Solution:

(a) From d'Alembert's formula

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi,$$

where $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, we see that

$$u(x,t) = \frac{1}{2}\left[\sin(n(x-ct)) + \sin(n(x+ct))\right].$$

Using the sum formula $\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$ yields

$$U(x,t) = \frac{1}{2}\left[\sin(n(x-ct)) + \sin(n(x+ct))\right] = \sin(nx)\cos(nct).$$

That is

$$U(x,t) = \sin(nx)\cos(\omega_n t)$$

where $\omega_n = nc$.

Note that in addition to solving the $\mathbb{DE}$ the solution satisfies

$$U(x,0) = \sin(nx)\cos(0) = \sin(nx), \quad U_t(x,0) = -\omega_n \sin(nx)\sin(0) = 0,$$

and

$$U(0,t) = \sin(0)\cos(\omega_n t) = 0, \quad U(\pi,t) = \sin(n\pi)\cos(\omega_n t) = 0,$$

when n is a positive integer. We recognize the solution as that obtained via separation of variables for the Dirichlet Problem for the wave equation,

$$\begin{aligned}
\mathbb{DE}: &\quad U_{tt} = c^2 U_{xx}, && 0 < x < \pi, t > 0\\
\mathbb{IC}: &\quad U(x,0) = \sin(nx), \quad U_t(x,0) = g(0) && 0 < x < \pi\\
\mathbb{BC}: &\quad U(0,t) = U(\pi,t) = 0 && t > 0.
\end{aligned}$$

(b) We can look at the odd-periodic extension of the problem onto the real line,

$$\begin{aligned}
\mathbb{DE}: &\quad U_{tt} = c^2 U_{xx}, && -\infty < x < \infty, t > 0\\
\mathbb{IC}: &\quad U(x,0) = 0, \quad U_t(x,0) = \sin(nx) && -\infty < x < \infty,
\end{aligned}$$

for which we obtain the solution from the d'Alembert formula

$$U(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(n\xi)\,d\xi = \frac{1}{2nc}\left[\cos(n(x-ct)) - \cos(n(x+ct))\right].$$

Using the sum formula $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$ yields

$$U(x,t) = \frac{1}{nc}\sin(nx)\sin(nct)) = \frac{1}{\omega_n}\sin(nx)\sin(\omega_n t))$$

where $\omega_n = nc$. We leave it as an exercise for the reader to verify that the solution satisfies the wave equation, and the associated initial and boundary conditions.

■

In general, for a Dirichlet problem we can consider the odd periodic extension of the initial condition onto the real line. For the Neumann problem, we look at the even periodic extension onto the real line.

INSERT TRIANGLE WAVE EXAMPLE HERE

Part VI

The Fourier Transform

Fourteen

An Introduction to The Fourier Transform

Parts of this section evolved from earlier notes due to Jon Jacobsen.

The Fourier transform is the natural generalization of the idea of Fourier series to an infinite interval. It is extremely useful when consider problems on the real line, such as the Cauchy problem for the wave or heat equation.

14.1 The Fourier Transform

The Fourier Transform can be thought of as a the limit of a Complex Fourier Series on an infinite interval. Previously we defined the Complex Fourier Series for a function $f(x)$ defined on the interval $x \in [-\ell, \ell]$ as

$$\mathbb{CFS}[f(x)] = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{\ell}}.$$

where

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-i\frac{n\pi x}{\ell}}\, dx.$$

Suppose now that $f(x)$ is a complex function defined on the real line, $x \in \mathbb{R}$, and it is *absolutely integrable*, that is

$$\int_{-\infty}^{\infty} |f(x)|\, dx \equiv \lim_{R\to\infty}\int_{-R}^{R} |f(x)|\, dx = M < \infty.$$

we can define a wavenumber

$$k_n = \frac{n\pi x}{\ell}$$

and a scaled Fourier coefficient

$$\widehat{f}_\ell(k) = 2\ell c_n = \int_{-\ell}^{\ell} f(x) e^{-ik_n x}\, dx.$$

If $f(x)$ is absolutely integrable, we can let ℓ go to infinity while keeping the wavenumber k_n fixed to define

$$\begin{aligned} \widehat{f}(k_n) &\equiv \lim_{\ell\to\infty} \widehat{f}_\ell(k_n), \\ &= \lim_{\ell\to\infty} \int_{-\ell}^{\ell} f(x) e^{-ik_n x}\, dx, \\ &= \int_{-\infty}^{\infty} f(x) e^{-ik_n x}\, dx. \end{aligned}$$

This is important enough that we will write a separate definition:

Definition 14.1 (Fourier Transform)**.** Let $f : \mathbb{C} \to \mathbb{C}$. The Fourier transform of f at k is defined by

$$\mathcal{F}\{f\} = \widehat{f}(k) \equiv \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx, \tag{14.2}$$

provided this integral exists.

In general, $\widehat{f}(k)$ is a complex function of k, even if $f(x)$ is a real function. Note that

$$\begin{aligned} |\widehat{f}(k)| &= \left| \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx, \right| \\ &\le \int_{-\infty}^{\infty} |f(x) e^{-ikx}|\, dx, \\ &\le \int_{-\infty}^{\infty} |f(x)|\, dx. \end{aligned}$$

Thus if f is absolutely integrable, the Fourier transform $\widehat{f}(k)$ exists. In general we will also assume that $f \to 0$ as $x \to \pm\infty$.

14.2 Inverse Fourier Transform

A pleasing aspect of Fourier transforms is that there is a particularly nice form for the inverse transform:

Definition 14.3 (Inverse Fourier Transform)**.** Let $u : \mathbb{C} \to \mathbb{C}$. The inverse transform of u is defined by

$$\mathcal{F}^{-1}\{\hat{u}(k)\} = u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k)e^{ikx}\,dk. \tag{14.4}$$

It is not obvious that $\mathcal{F}^{-1}\{\mathcal{F}\{u\}\} = u$, or for that matter that the Fourier transform is unique. However, we will eventual show that both of these things are true. The Fourier inversion formula (14.4) represents u as a continuous superposition of e^{-ikx} with amplitudes $\widehat{u}(k)$. For instance, the transform of $\widehat{\delta_0}(k) = 1,$ implies the delta distribution contains equal strength of all modes.

The reader is warned that there are *six standard ways to define* $\widehat{f}$ and $\check{f}$. The essential choices to make are whether to put the $1/2\pi$ on $\widehat{f}$ or $\check{f}$ (or $1\sqrt{2\pi}$ on each) and whether to put the minus sign in $\widehat{f}$ or $\check{f}$. Other sources will put the 2π in the exponent. They are all equivalent, and the differences are not of any fundamental importance. On the other hand, it is important to remember this when consulting different sources.

14.3 Some Properties and Examples of Fourier Transforms

We can compute some examples of Fourier transforms.

Example 14.1. Find the Fourier transform of the top hat function

$$T(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

and relate it to the coefficients of the Complex Fourier Series.
Solution: We find the Fourier Transform by direct calculation

$$\widehat{T}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx = \int_{-1}^{1} e^{-ikx}\,dx = \frac{2\sin k}{k}.$$

Note that for $\ell > 1$ if we compute the coefficients of the Complex Fourier Series,

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-i\frac{n\pi x}{\ell}}\,dx = \frac{1}{2\ell}\int_{-1}^{1} e^{-i\frac{n\pi x}{\ell}}\,dx = \frac{1}{2\ell}\left[\frac{2\sin(\frac{n\pi}{\ell})}{n\pi/\ell}\right].$$

We see that

$$2\ell c_m = \frac{2\sin(k_m)}{k_m} \qquad k_m = \frac{m\pi x}{\ell}$$

which is exactly $\widehat{T}(k)$. ■

Another example that will prove useful is:

Example 14.2. Compute the Fourier Transform of

$$f(x) = H(x)e^{-ax}$$

for $a > 0$, where

$$H(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases},$$

is the Heaviside function, so

$$H(x)e^{-ax} = \begin{cases} e^{-ax} & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases}.$$

Solution: By definition,

$$\begin{aligned} \widehat{f}(k) &= \int_{-\infty}^{\infty} H(x)e^{-ax}e^{-ikx}\,dx \\ &= \int_{0}^{\infty} e^{-ax}e^{-ikx}\,dx \\ &= \int_{0}^{\infty} e^{-(a+ik)x}\,dx \\ &= \left.\frac{e^{-(a+ik)x}}{-(a+ik)}\right|_{x=0}^{\infty} \\ &= \frac{1}{a+ik}. \end{aligned}$$

where we have used the fact that $a > 0$ to assure the integral converges. ■

We can use some properties to increase the size of our list of Fourier transforms.

Theorem 14.1 (Linearity)**.** *The Fourier transform is a linear operator; that is*

$$\mathcal{F}\{af(x) + b(g(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{(g(x)\}$$

for complex scalars a and b, assuming the Fourier transforms of f and g exist.

The proof is left for the reader.

Theorem 14.2 (Reflection)**.** *If the Fourier transforms of f exists and $\mathcal{F}\{f(x)\} = \widehat{f}(k)$ then*

$$\mathcal{F}\{f(-x)\} = \widehat{f}(-k).$$

Proof. By direct calculation

$$\begin{aligned}\mathcal{F}\{f(-x)\} &= \int_{-\infty}^{\infty} f(-x)e^{-ikx}\,dx \qquad \boxed{\text{Let } y=-x,\, m=-k.}\\ &= \int_{-\infty}^{\infty} f(y)e^{-imy}\,dy\\ &= \widehat{f}(m),\end{aligned}$$

but $m = -k$ so $\widehat{f}(m) = \widehat{f}(-k)$. □

Example 14.3. Compute the Fourier transform of $g(x) = H(-x)e^{ax}$, $a > 0$.

Solution: This is the reflection of $f(x) = H(x)e^{-ax}$, so

$$\widehat{g}(k) = \widehat{f}(-k) = \frac{1}{a - ik}$$

■

Example 14.4. Compute the Fourier transform of $h(x) = e^{-a|x|}$, $a > 0$.

Solution: Since $h(x) = e^{-a|x|} = H(x)e^{-ax} + H(-x)e^{ax} = f(x) = g(x)$, we can use linearity to see that

$$\widehat{h}(k) = \widehat{f}(k) + \widehat{g}(k) = \frac{1}{a - ik} + \frac{1}{a - ik} = \frac{2a}{a^2 + k^2}$$

■

A dilation of a function, $f(ax)$ corresponds to a stretching of the x-axis,

Theorem 14.3 (Dilation)**.** *If the Fourier transforms of f exists and $\mathcal{F}\{f(x)\} = \widehat{f}(k)$ then*

$$\mathcal{F}\{f(ax)\} = \frac{1}{a}\,\widehat{f}\left(\frac{k}{a}\right).$$

Proof. By direct calculation

$$\begin{aligned}\mathcal{F}\{f(ax)\} &= \int_{-\infty}^{\infty} f(ax)e^{-ikx}\,dx \qquad \boxed{\text{Let } y = ax,\ m = k/a.}\\ &= \frac{1}{a}\int_{-\infty}^{\infty} f(y)e^{-imy}\,dy.\\ &= \frac{1}{a}\widehat{f}(m)\end{aligned}$$

But $m = k/a$, so

$$\mathcal{F}\{f(ax)\} = \frac{1}{a}\,\widehat{f}\left(\frac{k}{a}\right).$$

□

Read this and think about how varying a would change the profile of the function f and its transform. If you squish f by a factor of a, it's transform expands by the same amount. This principle is the essence behind the Heisenberg Uncertainty Principle, which basically states that *a function and its transform can not be highly localized*. In quantum mechanics, if f is the wave function, $|f|^2$ and $|\hat{f}|^2$ are probability distributions for position and momentum and the Heisenberg Uncertainty Principle says that localization in one of these variables implies dispersion in the other.

We can also use dilation to sneak up on the idea of a δ-function:

Example 14.5. For $\delta > 0$ consider

$$\phi_\delta(x) = \begin{cases} \frac{1}{2\delta} & -\delta < x < \delta \\ 0 & \text{otherwise.} \end{cases}$$

Find the Fourier transform of ϕ_δ.

Solution: Note that for $\phi_\delta(x) = \frac{1}{2\delta}T(x/\delta)$ where $T(x)$ is the top hat function with Fourier transform

$$\widehat{T}(k) = \frac{2\sin k}{k}.$$

Using the dilation theorem (with $a = 1/\delta$), we see that

$$\widehat{\phi}_\delta(k) = \frac{1}{2\delta}\left[\delta \widehat{T}(k\delta)\right] = \frac{1}{2}\widehat{T}(k\delta) = \frac{\sin(\delta k/2)}{\delta k/2}.$$

■

In particular, as $\delta \to 0$, the transform $\widehat{\phi}_\delta(k)$ approaches 1 for every fixed value of k.[1] Note that $\phi_\delta(x)$ has an area that is fixed at unity, but as δ tends to zero its width gets narrower and its height gets taller. This is an example of a δ-sequence that is used to define a Dirac δ-function, sometimes call a unit impulse.

Formally, the Fourier transform of the delta function,

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x) e^{-ikx}\, dx = 1,$$

is equal to 1 in agreement with the limit process above.

14.4 Transforms of Derivatives

Suppose $\widehat{y}(k) = \mathcal{F}\{y(x)\}$, what can we say about $\mathcal{F}\{y'(x)\}$?

$$\widehat{y}(k) = \int_{-\infty}^{\infty} y(x) e^{-ikx}\, dx$$

and

$$\mathcal{F}\{y'(x)\} = \int_{-\infty}^{\infty} y'(x) e^{-ikx}\, dx.$$

We integrate by parts;

$$\begin{aligned} y(x) &= u & y'(x)dx &= du \\ e^{-ikx} &= v & -ike^{-ikx} &= dv. \end{aligned}$$

So

$$\mathcal{F}\{y'(x)\} = y(x)e^{-ikx}\Big[_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} (-ikx) e^{-ikx} y(x)\, dx.$$

If $|y(x)| \to 0$ as $|x| \to \infty$ the first term vanishes,

$$\mathcal{F}\{y'(x)\} = ik \int_{-\infty}^{\infty} e^{-ikx} y(x)\, dx = ik\widehat{y}(k),$$

and the Fourier Transform turns differentiation into multiplication! Let's write this as a theorem

[1] Remember $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

Theorem 14.4 (Differentiation)**.** *Suppose the Fourier transforms of $y(x)$ and $y'(x)$ exists and $\mathcal{F}\{y(x)\} = \widehat{y}(k)$. Then*

$$\mathcal{F}\{y'(x)\} = ik\widehat{y}(k),$$

assuming that $y(x) \to 0$ as $|x| \to \infty$.

Example 14.6. Solve the differential equation for $y(x)$,

$$\begin{aligned} \mathbb{DE}: &\quad y' + y = H(x)e^{-2x} \qquad -\infty < x < \infty \\ \mathbb{BC}: &\quad |y(x)| \to 0 \qquad \text{as } x \to \pm\infty. \end{aligned}$$

Solution: Fourier Transform both sides

$$\begin{aligned} \mathcal{F}\{y' + y\} &= \mathcal{F}\{H(x)e^{-2x}\} \\ \mathcal{F}\{y'\} + \mathcal{F}\{y\} &= \frac{1}{2+ik}. \end{aligned}$$

But $\mathcal{F}\{y\} = \widehat{y}$ and $\mathcal{F}\{y'\} = ik\widehat{y}$, so

$$(1+ik)\widehat{y} = \frac{1}{2+ik}$$

and we can solve for $\widehat{y}$,

$$\widehat{y} = \frac{1}{(1+ik)}\frac{1}{(2+ik)}$$

But

$$\begin{aligned} y(x) &= \mathcal{F}^{-1}\{\widehat{y}(k)\} \\ &= \mathcal{F}^{-1}\{\frac{1}{(1+ik)}\frac{1}{(2+ik)}\} \\ &= \mathcal{F}^{-1}\{\frac{1}{1+ik} - \frac{1}{2+ik}\} \\ &= H(x)e^{-x} - H(x)e^{-2x}. \end{aligned}$$

So

$$y(x) = H(x)[e^{-x} - e^{-2x}].$$

■

Exercise 14.1. Suppose $y(x)$ and its first n derivatives are absolutely integrable. Show that

$$\mathcal{F}\left\{y^{(n)}(x)\right\} = (ik)^n\widehat{y}(k),$$

where $y^{(n)}(x)$ denotes the n^{th} derivative of $y(x)$.

14.5 Shifting Theorems

The shifting theorem allows us to compute the transform of a function that has been spatially translated or *shifted*. Consider:

$$\begin{aligned}
\mathcal{F}\{f(x+a)\} &= \int_{-\infty}^{\infty} f(x+a)e^{-ikx} && z = a+x \\
&= \int_{-\infty}^{\infty} f(z)e^{-ik(z-a)}\,dz && dz = dx \\
&= e^{ika}\int_{-\infty}^{\infty} f(z)e^{-ikz}\,dz \\
&= e^{ika}\widehat{f}(k)
\end{aligned}$$

Example 14.7. Use the Fourier Transform to find the solution to PDE:

$$\begin{aligned}
\mathbb{DE} &: u_t + cu_x = 0 && -\infty < x < \infty, t > 0 \\
\mathbb{IC} &: u(x,0) = F(x) && -\infty < x < \infty
\end{aligned}$$

Solution: If we Fourier Transform in x

$$\begin{aligned}
\mathcal{F}_x\{u(x,t)\} &= \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx \equiv \widehat{u}(k,t) \\
\mathcal{F}_x\{u_t(x,t)\} &= \int_{-\infty}^{\infty} u_t(x,t)e^{-ikx}\,dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx \\
&= \widehat{u}_t(k,t)
\end{aligned}$$

Also

$$\mathcal{F}\{u_x(x,t)\} = ik\widehat{u}(k,t).$$

the Fourier Transform of the $\mathbb{DE}$ yields

$$\mathcal{F}\{u_t + cu_x\} = \widehat{u}_t + ikc\widehat{u} = 0.$$

This is an ODE in t. We can solve this:

$$\widehat{u}(k,t) = A(k)e^{-ikct}.$$

To solve for $A(k)$, Fourier transform the $\mathbb{IC}$,

$$\mathcal{F}\{u(x,0)\} = \widehat{u}(k,0) = \mathcal{F}\{F(x)\} = \widehat{F}(k).$$

Now

$$\widehat{u}(k,0) = A(k) = \widehat{F}(k)$$

and

$$\widehat{u}(k,t) = \widehat{F}(k)e^{-ikct}.$$

Now by the shifting formula,

$$u(x,t) = F(x - ct).$$

■

14.6 The Convolution Theorem

Suppose we know the Fourier transform of two functions, $\mathcal{F}\{u\} = \widehat{u}(k)$ and $\mathcal{F}\{v\} = \widehat{v}(k)$. Can we compute the inverse transform of their product, $\mathcal{F}^{-1}\{\widehat{u}(k)\widehat{v}(k)\}$? Amazingly, the answer is yes!! The answer is called the *convolution* of u and v and corresponds to a sort of superposition of the two functions. Let us define the convolution.

Definition 14.5 (Convolution in $\mathbb{R}$)**.** Let $u, v : \mathbb{R} \to \mathbb{R}$. The convolution of u and v at x is defined by

$$u * v = \int_{-\infty}^{\infty} u(x-y)v(y)\,dy.$$

The fundamental theorem for convolutions is:

Theorem 14.5 (Convolution Theorem for Fourier Transform)**.** *The Fourier transform of the convolution of two functions is the product of the Fourier transforms of the functions, that is if*

$$\mathcal{F}\{u\} = \widehat{u}(k), \qquad \mathcal{F}\{v\} = \widehat{v}(k)$$

then

$$\mathcal{F}\{u * v\} = \widehat{u}(k)\widehat{v}(k).$$

First we will do an example.

Example 14.8. Consider the two functions

$$f(x) = H(x)e^{-ax}, \qquad g(x) = H(-x)e^{ax},$$

where $a > 0$.

(a) Compute the convolution $f * g$

(b) Verify the convolution theorem, that is if

$$\mathcal{F}\{f\} = \widehat{f}(k), \qquad \mathcal{F}\{g\} = \widehat{g}(k)$$

then

$$\mathcal{F}\{f * g\} = \widehat{f}(k)\widehat{g}(k).$$

Solution: We proceed by direct calculation.

(a) By definition,

$$\begin{aligned} f * g &= \int_{-\infty}^{\infty} f(x-y)g(y)\,dy, \\ &= \int_{-\infty}^{\infty} H(x-y)e^{-a(x-y)}H(-y)e^{ay}\,dy, \\ &= e^{-ax}\int_{-\infty}^{\infty} H(x-y)H(-y)e^{2ay}\,dy, \end{aligned}$$

Now note that

$$H(x-y)H(-y) = \begin{cases} H(-y) & x \geq 0 \\ H(x-y) & x \leq 0 \end{cases}$$

So

$$e^{-ax}\int_{-\infty}^{\infty} H(x-y)H(-y)e^{2ay}\,dy = e^{-ax}\int_{-\infty}^{0} e^{2ay}\,dy = \frac{e^{-ax}}{2a} \qquad x \geq 0$$

and

$$e^{-ax}\int_{-\infty}^{\infty} H(x-y)H(-y)e^{2ay}\,dy = e^{-ax}\int_{-\infty}^{x} e^{2ay}\,dy = e^{-ax}\frac{e^{2ax}}{2a} = \frac{e^{ax}}{2a} \qquad x \leq 0$$

from which we decide that

$$f * g = \frac{e^{-a|x|}}{2a}.$$

(b) We've shown previously that if $f(x) = H(x)e^{-ax}$ and $g(x) = H(-x)e^{ax}$ then

$$\widehat{f}(k) = \frac{1}{a+ik}, \qquad \widehat{g}(k) = \frac{1}{a-ik}.$$

Note that

$$\begin{aligned}
\mathcal{F}\{f * g\} &= \frac{1}{2a}\mathcal{F}\{e^{-a|x|}\} \\
&= \frac{1}{2a}\frac{2a}{a^2+k^2} \\
&= \frac{1}{a^2+k^2} \\
&= \frac{1}{a+ik}\frac{1}{a-ik} \\
&= \widehat{f}(k)\widehat{g}(k)
\end{aligned}$$

which verifies the convolution theorem.

■

Proof of Convolution Theorem. Proof goes here. □

14.7 δ-SEQUENCES AND THE TRANSFORM OF A δ-FUNCTION

A δ-sequence satisfies

$$\int_{-\infty}^{\infty} \delta_\epsilon(x)\, dx = 1 \quad \text{and} \lim_{\epsilon\to 0}\delta_\epsilon(x) = \begin{cases}\infty & x = 0 \\ 0 & x \neq 0\end{cases} .$$

Theorem 14.6. *Given any function $f(x)$ that is $f(x) > 0$, $f(x)$ is continuous,*

$$\int_{-\infty}^{\infty} f(x)\, dx = 1,$$

and

$$\lim_{x\to\pm\infty} f(x) = 0$$

then

$$f_\epsilon(x) = \frac{1}{\epsilon} f(\frac{x}{\epsilon})$$

is a δ-sequence.

Proof. Note that

$$\begin{aligned}\int_{-\infty}^{\infty} f_\epsilon(x)\, dx &= \int_{-\infty}^{\infty} f(\frac{x}{\epsilon}) \qquad \frac{x}{\epsilon} = z \qquad \frac{dx}{\epsilon} = dz \\ &= \int_{-\infty}^{\infty} f(z)\, dz \\ &= 1\end{aligned}$$

Moreover, if $f(x)$ is absolutely integrable, then

□

We can now compute the Fourier Transform of the δ function is the limit of the Fourier Transform of a δ-sequence. Let

$$\mathcal{F}\{f_\epsilon(x)\} = \widehat{f}_\epsilon(k).$$

First note

$$\widehat{f}_\epsilon(0) = \int_{-\infty}^{\infty} f_\epsilon(x) e^{-i0x}\, dx = \int_{-\infty}^{\infty} f_\epsilon(x)\, dx = 1.$$

Also,

$$\begin{aligned}\mathcal{F}\{f_\epsilon\} &= \mathcal{F}\{\frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right)\} \\ &= \frac{1}{\epsilon}\mathcal{F}\{f\left(\frac{x}{\epsilon}\right)\} \\ &= \frac{1}{\epsilon}\mathcal{F}\{f\left(\frac{x}{\epsilon}\right)\} \\ &= \frac{1}{\epsilon}[\epsilon \widehat{f}(k\epsilon)] \\ &= \widehat{f}(k\epsilon).\end{aligned}$$

Now

$$\lim_{\epsilon \to 0} \widehat{f}_\epsilon(k) = \lim_{\epsilon \to 0} \widehat{f}(k\epsilon)$$

and for any fixed k,[2]

$$\lim_{\epsilon \to 0} \widehat{f}_\epsilon(k) = \widehat{f}(0) = 1.$$

This is the Fourier Transform

$$\mathcal{F}\{\delta(x)\} = 1.$$

[2]Note that we assume the Fourier Transform is continuous, which can be proven.

14.8 Green's Functions

The Green's function is the response to a δ-function forcing:

$$\begin{aligned} g'' + 2g' + 5g &= \delta(x) \\ \mathcal{F}\{g'' + 2g' + 5g\} &= \mathcal{F}\{\delta(x)\} \\ (ik)^2 + 2ik + 9)\widehat{g} &= 1 \\ \widehat{g} &= \frac{1}{(ik)^2 + 2ik + 5} \end{aligned}$$

Note that

$$\begin{aligned} (ik)^2 + 2ik + 5 &= (ik+1)^2 + 4 \\ &= [(ik + 1 + 2i)(ik + 1 - 2i)] \end{aligned}$$

so

$$\begin{aligned} \widehat{g}(k) &= \frac{1}{(ik + 1 + 2i)(ik + 1 - 2i)} \\ &= \frac{A}{ik + 1 + 2i} + \frac{B}{ik + 1 - 2i} \\ &= \frac{-\frac{1}{4i}}{ik + 1 + 2i} + \frac{\frac{1}{4i}}{ik + 1 - 2i} \\ &= \frac{1}{4i}\left[\frac{1}{ik + 1 - 2i} - \frac{1}{ik + 1 + 2i}\right] \end{aligned}$$

We then need to figure out the inverse transform of this:

$$\begin{aligned} \mathcal{F}\{H(x)e^{-ax}\} &= \int_0^\infty e^{-ax - ikx} \\ &= -\frac{e^{-(a+ik)x}}{a + ik}\bigg|_{x=0}^{\infty} \\ &= \frac{1}{a + ik} \qquad \Re\{a\} > 0 \end{aligned}$$

Suppose $a = \alpha + i\beta$. Then

$$\mathcal{F}\{H(x)e^{-(\alpha + i\beta)x}\} = \frac{1}{\alpha + i\beta + ik} = e^{-\alpha x}.$$

so

$$\begin{aligned} g(x) &= \mathcal{F}^{-1}\left\{\frac{1}{4i}\left[\frac{1}{ik+1-2i}-\frac{1}{ik+1+2i}\right]\right\} \\ &= \frac{1}{4i}\left[H(x)e^{-x}e^{2ix}-H(x)e^{-x}e^{-2ix}\right] \\ &= \tfrac{1}{2}H(x)e^{-x}\sin(2x) \end{aligned}$$

14.9 The Cauchy Problem for the Heat Equation

We will solve the heat equation using the Fourier transform, but first we need to know the Fourier transform of a gaussian. We start out with a seemingly unrelated lemma which you may have seen in multivariable calculus.

Lemma 14.6. *Show*

$$I(a) = \int_{-\infty}^{\infty} e^{-ax^2/2}\,dx = \sqrt{\frac{2\pi}{a}}.$$

for $a > 0$

Solution: We use polar coordinates

$$\begin{aligned} [I(a)]^2 &= \int_{-\infty}^{\infty} e^{-ax^2/2}\,dx \int_{-\infty}^{\infty} e^{-ay^2/2}\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{a}{2}(x^2+y^2)}\,dx\,dy \\ &= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-a\frac{r^2}{2}}\,r\,dr\,d\theta \\ &= \frac{2\pi}{a}\int_{0}^{\infty} e^{\frac{-ar^2}{2}}\,a\,r\,dr \\ &= \frac{2\pi}{a}(-e^{-ar^2/2})\Big|_{r=0}^{\infty} \\ &= \frac{2\pi}{a} \end{aligned}$$

Now, as $[I(a)]^2 = \frac{2\pi}{a}$, we conclude that

$$I(a) = \sqrt{\frac{2\pi}{a}}$$

14.9.1 Fourier Transform of a Gaussian

Example 14.9. Show that

$$\mathcal{F}\{e^{-ax^2/2}\} = e^{-\frac{k^2}{2a}}\sqrt{\frac{2\pi}{a}}$$
$$\mathcal{F}\{\frac{1}{2\sqrt{\pi b}}e^{-x^2/4b}\} = e^{-bk^2}.$$

Solution: From the definition of the Fourier Transform, we know that

$$\mathcal{F}\{e^{-ax^2/2}\} = \int_{-\infty}^{\infty} e^{-ax^2/2+ikx}\,dx.$$

Let's complete the square

$$\begin{aligned}\frac{ax^2}{2} + ikx &= \frac{a}{2}\left[x^2 + \frac{2ik}{a}x\right]\\ &= \frac{a}{2}\left[\left(x+\frac{ik}{a}\right)^2 - (\frac{ik}{a})^2\right]\\ &= \frac{a}{2}\left(x+\frac{ik}{a}\right)^2 + \frac{k^2}{2a}\end{aligned}$$

So

$$\begin{aligned}\mathcal{F}\{e^{-ax^2/2}\} &= \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x+\frac{ik}{a}\right)^2 - \frac{k^2}{2a}}\,dx\\ &= e^{-\frac{k^2}{2a}}\int_{x=-\infty}^{\infty} e^{-\frac{a}{2}\left(x+\frac{ik}{a}\right)^2}\,dx\end{aligned}$$

Let

$$z = x + \frac{ik}{a} \Rightarrow dz = dx.$$

We need to be a little careful here, because we have shifted the integral into the complex plane an amount $\frac{ik}{a}$; complex variables tell us we can just drag the contour back down to the real axis to show that

$$\begin{aligned}\mathcal{F}\{e^{-ax^2/2}\} &= e^{-\frac{k^2}{2a}}\int_{-\infty}^{\infty} e^{-\frac{a}{2}z^2}\,dz\\ &= e^{-\frac{k^2}{2a}}\sqrt{\frac{2\pi}{a}}\end{aligned}$$

So

$$\mathcal{F}\{e^{ax^2/2}\} = \sqrt{\frac{2\pi}{a}}e^{-k^2/2a}.$$

Also if we set

$$b = \frac{1}{2a} \implies a = \frac{1}{2b}$$

it follows that

$$\mathcal{F}\left\{\frac{1}{2\sqrt{\pi b}}e^{-x^2/4b}\right\} = e^{-bk^2}.$$

■

Example 14.10. Use the Fourier transform to solve the Cauchy problem for the heat equation

$$\begin{aligned}
\mathbb{DE} &: u_t = Du_{xx} \qquad -\infty < x < \infty, t > 0 \\
\mathbb{IC} &: u(x,0) = f(x) \qquad -\infty < x < \infty \\
\mathbb{BC} &: \lim_{x\to\pm\infty} |u(x,t)| \text{ is bounded for all } t > 0
\end{aligned}$$

where

(a) $f(x) = \delta(x)$, the δ-function.

(b) $f(x) = f(x)$, an arbitrary functiom

(c) $f(x) = H(x)$ the Heaviside function.

Solution: We once again use the the Fourier Transform to turn a PDE into an ODE; define

$$\mathcal{F}\{u(x,t)\} = \widehat{u}(k,t)$$

then

$$\mathcal{F}\{u_t(x,t)\} = \widehat{u}_t(k,t) \qquad \mathcal{F}\{u_{xx}(x,t)\} = (ik)^2\widehat{u}(k,t) = -k^2\widehat{u}(k,t),$$

So the Fourier transform of the $\mathbb{DE}$ yields

$$u_t = Du_{xx} \implies \widehat{u}_t = -Dk^2\widehat{u},$$

which is easily solved to yield

$$\widehat{u}(k,t) = A(k)e^{-Dk^2t}$$

where the constant $A(k)$ depends on the transform variable. From the $\mathbb{IC}$, we see that

$$\mathcal{F}\{u(x,0)\} = \widehat{u}(k,0) = \mathcal{F}\{f(x)\} = \widehat{f}(k),$$

but from (14.10),

$$\widehat{u}(k,0) = A(k) = \widehat{f}(k) \implies \widehat{u}(k,t) = \widehat{f}(k)e^{-Dk^2t}.$$

We need to known $\mathcal{F}\{e^{(-(Dt)k^2}\}$; setting $(b = Dt)$ in the transform above, we see that

$$\mathcal{F}^{-1}\{e^{-Dk^2t}\} = G(x,t) \equiv \frac{1}{2\sqrt{\pi Dt}}e^{-x^2/4Dt}.$$

Here $G(x,t)$ is the *Green's function* (sometimes call the *kernel*) of the heat equation.

Let's now consider the three initial conditions given above,

(a) If $f(x) = \delta(x)$, $\mathcal{F}\{\delta(x)\} = 1$. Then

$$u(x,t) = G(x,t) = \frac{e^{-x^2/4Dt}}{2\sqrt{\pi Dt}}.$$

This is a spreading gaussian whose width scales like $\sqrt{Dt}$ and whose height decreases like $\frac{1}{2\sqrt{\pi Dt}}$. It's area $= 1$. The diffusion is *self-similar* - it spreads out, but maintains its characteristic shape.

(b) If $f(x)$ is a general function, we can still solve the problem using the convolution theorem. The solution

$$\widehat{u}(k,t) = \widehat{f}(k,t)e^{-Dtk^2}$$

implies $u(x,t)$ is the convolution of the initial condition and the green's function,

$$u(x,t) = f(x) * G(x,t) = \frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{\infty} f(x-y)e^{-y^2/4Dt}\,dy$$

which is known as the *Poisson Integral Formula* for the solution to the Cauchy problem of the heat equation.

(c) Finally, let us consider the specific case when the initial condition is the Heaviside function, $f(x) = H(x)$. Then from the convolution formula,

$$u(x,t) = \frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{\infty} H(x-y)e^{-y^2/4Dt}\,dy.$$

But

$$H(x-y) = \begin{cases} 1 & x > y \\ 0 & x < y \end{cases} .$$

So

$$u(x,t) = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{x} e^{-y^2/4Dt}\, dy.$$

We can rewrite the integral in terms of the error function. Let $z = \frac{y}{2\sqrt{Dt}}$ then $dz = \frac{dy}{2\sqrt{Dt}}$. So

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{Dt}}} e^{-z^2}\, dz.$$

Remember the error function

$$\operatorname{erf}(w) \equiv \frac{2}{\sqrt{\pi}} \int_0^w e^{-s^2}\, ds.$$

so

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-w^2}\, dw + \int_0^{\frac{x}{2\sqrt{Dt}}} e^{-w^2}\, dw\right] \\ &= \frac{1}{2}\left[-\operatorname{erf}(-\infty) + \operatorname{erf}(\frac{x}{2\sqrt{Dt}})\right] \\ &= \frac{1}{2}\left[1 + \operatorname{erf}(\frac{x}{2\sqrt{Dt}})\right] \end{aligned}$$

The reader is encouraged to graph this solution.

■

14.10 D'ALEMBERT SOLUTION TO THE WAVE EQUATION

In this section we will use the Fourier Transform to verify a portion of the d'Alembert solution. First, however we consider a warm-up problem

Exercise 14.2.

Show that if

$$I_a(x) = \begin{cases} 0 & |x| > a \\ 1 & |x| \le a \end{cases}$$

then the Fourier transform of $I_a(x)$ is given by

$$\mathcal{F}\{I(x)\} = \frac{2\sin(ka)}{k} \quad .$$

Solution: By direct calculation we see that

$$\begin{aligned}
\mathcal{F}\left\{I_a(x)\right\} &= \int_{-\infty}^{\infty} I(x)e^{-ikx}\,dx = \int_{-a}^{a} e^{-ikix}\,dx \\
&= -\frac{e^{-ikx}}{ik}\Big|_{x=-a}^{x=a} \\
&= \frac{e^{ika}-e^{-ika}}{ik} \\
&= \frac{2\sin(ka)}{k} \quad .
\end{aligned}$$

as advertised above. □

We now use the Fourier transform to solve the second-order wave equation for $W(x,t)$,

$$W_{tt} = c^2 W_{xx} \qquad -\infty < x < \infty,\ 0 < t$$

$$W(x,0) = f(x) \qquad W_t(x,0) = g(x)$$

Solution: Define $\widehat{W}(k,t)$ to be the Fourier transform of $W(x,t)$ with respect to x. The Fourier transform turns the partial differential equation problem into an ordinary differential equation problem:

$$\widehat{W}_{tt} + c^2k^2\widehat{W} = 0,$$

with

$$\widehat{W}(k,0) = \widehat{f}(k)\ , \qquad \widehat{W}_t(k,0) = \widehat{g}(k)\ ,$$

where $\widehat{f}(x)$ and $\widehat{g}(k)$ are the Fourier Transform of $f(x)$ and $g(x)$ respectively. The general solution to this ODE is

$$\widehat{W}(k,t) = A(k)\cos(kct) + B(k)\sin(kct)\ .$$

Applying the initial conditions yield

$$\widehat{W}(k,0) = A(k) = \widehat{f}(k)\ ,$$

and

$$\widehat{W}_t(k,t) = -kcA(k)\sin(kct) + kcB(k)\cos(kct) \Rightarrow \widehat{W}_t(k,0) = kcB(k) = \widehat{g}(k)$$

which implies $A(k) = \widehat{f}(k)$ and $B(k) = \widehat{g}(k)/(ck)$. Therefore

$$\widehat{W}(k,t) = \widehat{f}(k)\cos(kct) + \widehat{f}(k)\frac{\sin(kct)}{ck}$$

We can write

$$\widehat{W}(k,t) = \widehat{W}_1(k,t) + \widehat{W}_2(k,t)$$

where

$$\widehat{W}_1(k,t) == \widehat{f}(k)\cos(kct), \qquad \widehat{W}_2(k,t) = \widehat{g}(k)\frac{\sin(kct)}{ck}$$

$$\begin{aligned} W_2(x,t) &= \mathcal{F}^{-1}\{\widehat{W}_2(k,t)\} \\ &= \mathcal{F}^{-1}\{\widehat{f}(k)\cos(kct)\} \quad = \frac{1}{2}\mathcal{F}^{-1}\{\widehat{f}(k)\left[e^{ikct} + e^{-ikct}\right]\} = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] \end{aligned}$$

where we have used the first shifting theorem to compute the inverse transforms. Also, by the convolution theorem and the result in part (a) we see that

$$\begin{aligned} W_2(x,t) &= \mathcal{F}^{-1}\{\widehat{W}_2(k,t)\} \\ &== \frac{1}{2c}\mathcal{F}^{-1}\left\{\widehat{f}(k)\left[\frac{2\sin(kct)}{k}\right]\right\} \\ &= \frac{1}{2c}\, g(x) * I_{ct}(x) = \frac{1}{2c}\int_{-\infty}^{\infty} g(y) I_{ct}(x-y)\, dy. \end{aligned}$$

We see that $I_{ct}(x-y) = 1$ for $x - ct < y < x + ct$ and 0 otherwise, from which we deduce

$$W_2(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\, dy\ .$$

Now by linearity of the Fourier transform, $W(x,t) = W_1(x,t) + W_2(x,t)$ which yields

$$W(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

which is identical to the d'Alembert solution.

Part VII

Sturm-Liouville Theory

Fifteen

Heat Transfer in the Ball

This Section was mainly written by John Polking

In this Section we consider the diffusion of an axisymmetric distribution of heat in a ball. This leads naturally to a problem that can be solved via separation of variables and Sturm-Liouville theory

15.1 The Problem

By the ball of radius a we mean the set B in $\mathbb{R}^3$ defined by

$$B = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| x^2 + y^2 + z^2 \le a^2 \right\}.$$

We are interested in studying how the temperature $u(x, y, z, t)$ varies from point to point and with the time t. We already know that the temperature must satisfy the heat equation

$$u_t = k\Delta u = k \left[u_{xx} + u_{yy} + u_{zz} \right], \tag{15.1}$$

where $k > 0$ is a constant called the thermal diffusivity.

In general we are required to supply the initial temperature distribution,

$$u(x, y, z, 0) = u_0(x, y, z) \quad \text{for } (x, y, z) \in B, \tag{15.2}$$

and the temperature on the boundary,

$$u(x, y, z, t) = f(x, y, z) \quad \text{for } (x, y, z) \in \partial B \text{ and } t > 0. \tag{15.3}$$

Of course, the boundary of the ball is the sphere

$$S = \partial B = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| x^2 + y^2 + z^2 = a^2 \right\}.$$

One example of this phenomenon involves deciding how long it takes to cook a turkey (approximately spherical) that has been defrosted, allowed to reach room temperature, and then is placed into an oven where a constant ambient temperature is maintained. A second problem of the same type arises when the turkey is taken out of the oven and allowed to sit with its surface exposed to room temperature while its interior temperature distribution is what has been reached in the oven. In this case the center of the turkey continues to heat up for some time while the outer portion is cooling.

The problem in equations (15.1), (15.2), and (15.3) is called the *Dirichlet problem* for the heat equation in the ball. The solution is typically achieved in two stages. First we find the *steady-state temperature* u_s. This is a temperature which is independent of time and has the same boundary conditions as u. Since u_s satisfies $\partial u_s/\partial t = 0$ and also satisfies the heat equation (15.1)), we must have $\Delta u_s = 0$. Therefore u_s solves the problem

$$\begin{aligned} \Delta u_s(x,y,z) &= 0 \quad \text{for } (x,y,z) \in B, \text{ and} \\ u_s(x,y,z) &= f(x,y,z) \quad \text{for } (x,y,z) \in \partial B. \end{aligned} \tag{15.4}$$

The problem in (15.4) is called the Dirichlet problem for the Laplacian, Δ.

The second stage in the solution is to find the difference $v(x,y,z,t) = u(x,y,z,t) - u_s(x,y,z)$, which might be referred to as the *transient* temperature. Putting together the information in (15.1), (15.2), (15.3), and (15.4), we see that v must solve the problem

$$\begin{aligned} &v_t(x,y,z,t) = k\Delta v(x,y,z,t) \quad \text{for } (x,y,z) \in B \text{ and } t > 0, \\ &v(x,y,z,0) = u_0(x,y,z) - u_s(x,y,z) \quad \text{for } (x,y,z) \in B, \text{ and} \\ &v(x,y,z,t) = 0 \quad \text{for } (x,y,z) \in \partial B \text{ and } t > 0, \end{aligned} \tag{15.5}$$

Comparing (15.5) to the problem described in (15.1), (15.2), and (15.3), we see that v is a solution to the Dirichlet problem for the heat equation with homogeneous boundary conditions. This will enable us to use separation of variables in our solution in Section 15.3.

Having solved the problems in (15.4) and (15.5), the solution of the Dirichlet problem for the heat equation is $u(x,y,z,t) = u_s(x,y,z) + v(x,y,z,t)$.

15.2 The Problem for Radial Temperatures

The problems in (15.4) and (15.5) are rather daunting in the generality we have presented. We will limit ourselves to the case when the initial temperature u_0 and the boundary temperature f are both constants.

Let's turn first to the problem in (15.4) of finding the steady-state temperature, but now with a constant temperature f given on the boundary. Using either physical intuition or mathematical insight we are led to the suggestion that the steady-state temperature will also be a constant. We can verify directly that the constant function $u_s(x, y, z) = f$ is a solution to (15.4).

Next, in (15.5) we are looking for a temperature v which is equal to 0 everywhere on the boundary, and is initially equal to the constant $v_0 = u_0 - f$. Let's consider this solution along a radius of the ball. Since the initial and boundary conditions do not distinguish one radius from another, we are led to expect that the temperature distribution v will be the same along any radius. Therefore, in the spatial coordinates, v will depend only on the distance from the center of the ball, which is $r = \sqrt{x^2 + y^2 + z^2}$. Let's look for a solution to (15.5) of the from $v(r, t)$.

We need to compute Δv for a function of this form. If g is any function that depends only on r, so that $g = g(r)$, then by the chain rule

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial r} \cdot \frac{\partial r}{\partial x} = g_r r_x. \tag{15.6}$$

We need to compute r_x. That is most easily done by differentiating both sides of

$$r^2 = x^2 + y^2 + z^2 \tag{15.7}$$

to get

$$2rr_x = 2x \quad \text{or} \quad r_x = \frac{x}{r}.$$

Similarly,

$$r_y = \frac{y}{r} \quad \text{and} \quad r_z = \frac{z}{r}.$$

Therefore from (15.6), if g depends only on r,

$$\frac{\partial g}{\partial x} = \frac{x}{r}\frac{\partial g}{\partial r}. \tag{15.8}$$

In particular, $v_x = (x/r)v_r$. Using the product formula and (15.8) , we compute that the second derivative is

$$\begin{aligned} v_{xx} &= \frac{\partial v_x}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{r}\cdot v_r\right) \\ &= \frac{\partial}{\partial x}\left(\frac{x}{r}\right)\cdot v_r + \frac{x}{r}\cdot\frac{\partial v_r}{\partial x} \\ &= \frac{r - x(x/r)}{r^2}\cdot v_r + \frac{x}{r}\cdot\frac{x}{r}\frac{\partial v_r}{\partial r} \\ &= \frac{r^2 - x^2}{r^3}v_r + \frac{x^2}{r^2}v_{rr}. \end{aligned}$$

The second derivatives with respect to y and z have similar formulas. Adding them together using (15.7) we get

$$\Delta v = v_{xx} + v_{yy} + v_{zz} = \frac{2}{r}v_r + v_{rr}. \tag{15.9}$$

Thus, to solve (15.5) we are looking for a function $v(r,t)$ which satisfies

$$\begin{aligned} &v_t(r,t) = k\left[v_{rr}(r,t) + \frac{2}{r}v_r(r,t)\right] \quad \text{for } 0 \le r < a \text{ and } t > 0, \\ &v(r,0) = v_0 \quad \text{for } 0 \le r < a, \\ &v(a,t) = 0 \quad \text{for } t > 0. \end{aligned} \tag{15.10}$$

15.3 Solution by Separation of Variables

In view of past experience, it is natural to look for product functions $v(r,t) = R(r)T(t)$ which satisfy the differential equation and the boundary condition in (15.10), or

$$\begin{aligned} &v_t(r,t) = k\left[v_{rr}(r,t) + \frac{2}{r}v_r(r,t)\right] \quad \text{for } 0 \le r < a \text{ and } t > 0, \\ &v(a,t) = 0 \quad \text{for } t > 0. \end{aligned} \tag{15.11}$$

Separating variables in the usual way, we find that T must satisfy

$$T' = -\lambda k T, \tag{15.12}$$

while R must satisfy

$$-\left[R'' + \frac{2}{r}R'\right] = \lambda R \quad \text{and } R(a) = 0, \tag{15.13}$$

where λ is a constant.[1] The solution to (15.12) is

$$T(t) = e^{-\lambda kt}. \tag{15.14}$$

To find the solution to (15.13) we must work a little harder, but the solution is surprisingly easy.

First of all we must make the differential equation in (15.13) look like a Sturm-Liouville equation. This means we want to find a function $p(r)$ so that

$$p\left[R'' + \frac{2}{r}R'\right] = [pR']' = pR'' + p'R'.$$

This requires $p' = 2p/r$, so $p(r) = r^2$ will work. Multiplying the differential equation in (4.3) by r^2, it becomes

$$-\left[r^2R'' + 2rR'\right] = -\left[r^2R'\right]' = \lambda r^2 R. \tag{15.15}$$

Equation (15.15) now has the form of a Sturm-Liouville equation with weight function r^2. Notice that the coefficient $p(r) = r^2$ vanishes at $r = 0$, so this is a singular Sturm-Liouville equation.

Equation (15.13)) gives only the one boundary condition, $R(a) = 0$. However, there is a hidden condition that we see when we realize that the function $v(r,t) = T(t)R(r)$ is a temperature and has a finite value at $r = 0$. This means that $R(0)$ is also finite. Hence the complete Sturm-Liouville problem for R is

$$\begin{aligned} -\left[r^2R'' + 2rR'\right] = -\left[r^2R'\right]' &= \lambda r^2 R, \\ R(0) &\text{ is finite,} \\ R(a) &= 0. \end{aligned} \tag{15.16}$$

We can simplify the differential equation in (15.16) considerably by making the substitution[2]

$$S = rR. \tag{15.17}$$

---

[1]The functions T and R are each functions of one variable although the variable is different for each. We use the prime notation to indicate the derivative with respect to the one variable. Thus $T' = dT/dt$ and $R' = dR/dr$.

[2]While this substitution comes out of the blue here, mathematicians have discovered that if we have a Sturm-Liouville equation $(pu')' + qu = \lambda su$ on an interval of the form $(0, b)$, which is singular at the initial end point $x = 0$, and where the coefficient p can be factored as $p(x) = x^{2\alpha}P(x)$, where $P(0) \neq 0$, the substitution $w(x) = x^{\alpha}u(x)$ will sometimes change the problem to a Sturm-Liouville problem that is regular at $x = 0$.

Then $S' = rR' + R$ and $S'' = rR'' + 2R'$. Making this substitution into the differential equation, we get $-rS'' = \lambda rS$, or $-S'' = -\lambda S$. From (15.16) and (15.17), the boundary conditions satisfied by S are $S(0) = 0 \cdot R(0) = 0$ and $S(a) = a \cdot R(a) = 0$. To sum up, the function $S(r) = rR(r)$ must satisfy

$$-S'' = \lambda S, \quad \text{with} \quad S(0) = S(a) = 0.$$

We have seen this Sturm-Liouville problem several times. The solutions are

$$\lambda_n = \frac{n^2\pi^2}{a^2} \quad \text{and} \quad S_n(r) = \sin\frac{n\pi r}{a}, \quad \text{for } n = 1, 2, 3, \ldots.$$

Since $R = S/r$, the solutions to the Sturm-Liouville problem in (15.16) are

$$\lambda_n = \frac{n^2\pi^2}{a^2} \quad \text{and} \quad R_n(r) = \frac{1}{r}\sin\frac{n\pi r}{a}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{15.18}$$

Notice that the apparent singularity at $r = 0$ is not really there. We will set

$$R_n(0) = \lim_{r\to 0} R_n(r) = \lim_{r\to 0}\frac{\sin(n\pi r/a)}{r} = \frac{n\pi}{a}. \tag{15.19}$$

Let's say a few words about orthogonality that are not directly related to our train of thought. We have proved directly that

$$\int_0^a S_i(r)S_j(r)\,dr = \int_0^a \sin\frac{i\pi r}{a}\,\sin\frac{j\pi r}{a}\,dr = 0 \quad \text{if } i \neq j.$$

From Sturm-Liouville theory, we know that the eigenfunctions R_j of the problem in (15.16) are orthogonal with respect to the weight r^2. Thus

$$\int_0^a R_i(r)R_j(r)\,r^2\,dr = 0 \quad \text{if } i \neq j.$$

Since $rR_j = S_j$, this is in exact agreement with the orthogonality relation for S_j. Finally, Let's look at the orthogonality of the functions $R_j(r)$ on the ball B, where $r = \sqrt{x^2+y^2+z^2}$. Using polar coordinates to do the integration and integrating out the angles ϕ and θ, we have

$$\begin{aligned}\iiint_B R_i(r)R_j(r)\,dxdydz &= \int_0^a\int_0^{2\pi}\int_0^{\pi} R_i(r)R_j(r)\,r^2\sin\phi\,d\phi d\theta dr \\ &= 4\pi\int_0^a R_i(r)R_j(r)\,r^2\,dr \\ &= 0 \quad \text{if } i \neq j.\end{aligned}$$

The last line follows from the orthogonality relationship for the R_j with respect to the weight r^2. It is the orthogonality on the ball B which is really important here. Unfortunately, we do not have time to explore this.

Let's return to the solution of the heat equation. From (15.14) and (15.18) we see that the product solutions of (15.11) are

$$v_n(r,t) = e^{-kn^2\pi^2 t/a^2}\frac{\sin(n\pi r/a)}{r}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{15.20}$$

By superposition a convergent infinite series of the form

$$\begin{aligned} v(r,t) &= \sum_{n=1}^{\infty} A_n v_n(r,t) \\ &= \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/a^2}\frac{\sin(n\pi r/a)}{r} \end{aligned} \tag{15.21}$$

is also a solution to (15.11).

To complete the solution to (15.8), we must choose the coefficients A_n so that $v(r,0) = v_0$. Multiplying (4.11) by r and evaluating at $t = 0$, this becomes

$$v_0 r = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi r}{a} \quad \text{for } 0 < r < a.$$

This is the Fourier sine series for the function $v_0 r$. The coefficients are $A_n = (-1)^{n+1}\cdot 2av_0/\pi n$, so

$$v_0 = 2v_0\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(n\pi r/a)}{n\pi r/a}.$$

Notice for later reference that with $v_0 = 1/2$ this becomes

$$\frac{1}{2} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(n\pi r/a)}{n\pi r/a} \quad \text{for } 0 < r < a \quad . \tag{15.22}$$

Inserting the coefficients A_n into (15.21), we see that the solution to (15.5) is

$$v(r,t) = 2v_0\sum_{n=1}^{\infty}(-1)^{n+1}e^{-kn^2\pi^2 t/a^2}\frac{\sin(n\pi r/a)}{n\pi r/a}. \tag{15.23}$$

Recall that in our two stage approach, the solution to the original Dirichlet problem for the heat equation is the sum of the steady-state solution and the transient solution. Thus

$$\begin{aligned} u(r,t) &= u_s(r,t) + v(r,t) \\ &= f + 2(u_0 - f)\sum_{n=1}^{\infty}(-1)^{n+1}e^{-kn^2\pi^2 t/a^2}\frac{\sin(n\pi r/a)}{n\pi r/a}. \end{aligned} \tag{15.24}$$

Example. Let's look at a special case. Assume that we are cooking a turkey, roughly the shape of a sphere of radius 1 foot. The turkey has been defrosted and is uniformly at room temperature of $75°$. We put it into the oven at a temperature of $350°$. We want to cook it until the center has a temperature of $150°$. With time measured in hours, the thermal diffusivity in the proper units is $k = 0.02$. How long should we cook the turkey?

We have all of the information we need. The parameters are $f = 325$, $u_0 = 75$, $a = 1$, and $k = 0.02$. The solution is given in equation (15.24). Using Matlab, we sum the first 200 terms of the series in (15.24), and plot the results versus r at time intervals of 1 hour. The result is shown in Figure 1.

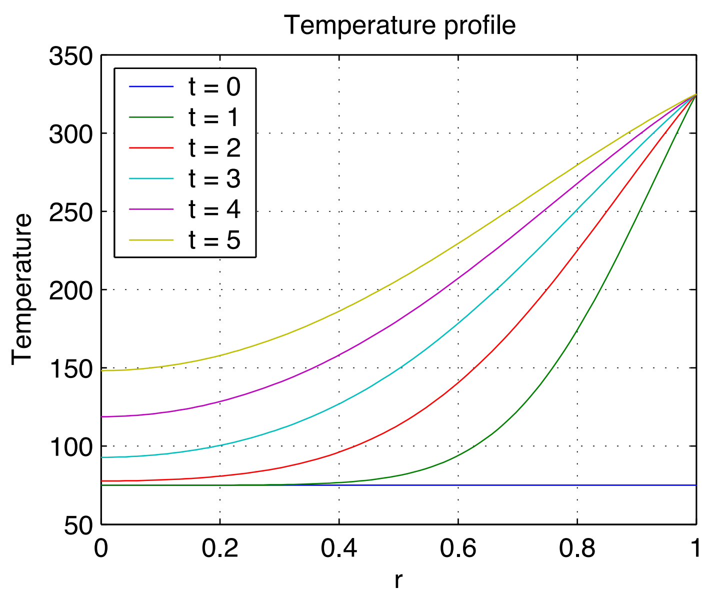


Figure 15.1: The temperature distribution in the turkey at 1 hour intervals.

A more efficient way to find the time when the center of the turkey is at $150°$ is to plot that temperature versus time. Setting $r = 0$ in (15.24) and using (15.19) we see that

$$\begin{aligned} u(0,t) &= \lim_{r\to 0} u(r,t) \\ &= f + 2(u_0 - f)\sum_{n=1}^{\infty}(-1)^{n+1}e^{-kn^2\pi^2 t/a^2}\lim_{r\to 0}\frac{\sin(n\pi r/a)}{n\pi r/a} \\ &= f + 2(u_0 - f)\sum_{n=1}^{\infty}(-1)^{n+1}e^{-kn^2\pi^2 t/a^2}, \end{aligned} \tag{15.25}$$

at least for $t > 0$. Notice that the series in (15.25) does not converge for $t = 0$. The result is plotted in Figure 2. Clearly we need to cook the turkey for a little more than 5 hours.

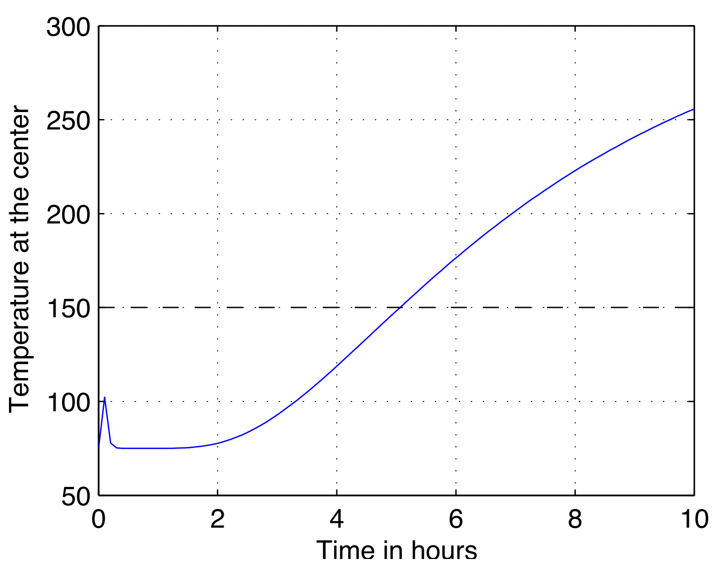


Figure 15.2: The temperature at the center of the turkey versus time.

15.4 Interchanging Infinite Sums and Limits

Let's return to the computation in (15.25). We skipped a step. The computation should read

$$\begin{aligned} u(0,t) &= \lim_{r\to 0} u(r,t) \\ &= f + 2(u_0 - f)\lim_{r\to 0}\sum_{n=1}^{\infty}(-1)^{n+1}e^{-kn^2\pi^2 t/a^2}\frac{\sin(n\pi r/a)}{n\pi r/a} \\ &= f + 2(u_0 - f)\sum_{n=1}^{\infty}(-1)^{n+1}e^{-kn^2\pi^2 t/a^2}\lim_{r\to 0}\frac{\sin(n\pi r/a)}{n\pi r/a} \\ &= f + 2(u_0 - f)\sum_{n=1}^{\infty}(-1)^{n+1}e^{-kn^2\pi^2 t/a^2}, \end{aligned} \tag{15.26}$$

Between the second and third lines in (15.26) we have interchanged a limit and an infinite sum. This should never be done without checking that it is legitimate. For $t > 0$ it is legitimate because the exponential terms decrease so rapidly, but for $t = 0$ it isn't.

Let's concentrate on the sum and limit terms and do the computation in both orders for $t = 0$. First, using (15.22) we have

$$\lim_{r\to 0}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(n\pi r/a)}{n\pi r/a} = \lim_{r\to 0}\frac{1}{2} = \frac{1}{2}. \tag{15.27}$$

If we interchange the sum and the limit, we get

$$\sum_{n=1}^{\infty}(-1)^{n+1}\lim_{r\to 0}\frac{\sin(n\pi r/a)}{n\pi r/a} = \sum_{n=1}^{\infty}(-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - \cdots, \tag{15.28}$$

which is a divergent series. Thus computing the limit of the sum we get the number $1/2$, while the sum of the limits leads to a divergent series. This is just one of the many strange things that can happen when we interchange a sum and a limit.

Although the series in (15.28) does not converge, the mathematician Leonhard Euler insisted that the sum is $1/2$. In his defense we should add that no rigorous definition of convergence existed at the time. However, setting the sum equal to $1/2$ makes the results in (15.27) and (15.28) equal. It also conveniently gives the correct answer $u(0,0) = u_0$ in the last line of (15.25)! It is amazing how often using $1/2$ as the sum of this strange, divergent series leads to a correct result.

Here is a theorem giving conditions under which a limit and an infinite sum can be interchanged.

Theorem. Suppose that $f_n(x)$ is a sequence of functions defined on an interval $I = [a, b]$, and that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I. Suppose in addition that $x_0 \in I$, and that $\lim_{x \to x_0} f_n(x)$ exists for each n. Then

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} f_n(x).$$

Sixteen

Sturm-Liouville Theory

In this lecture we will show that many eigenvalue problems can be put into Sturm-Liouville form,

$$(s(x)X'(x))' + (\lambda\rho(x) - q(x))\,X(x) = 0 \tag{16.1}$$

and that these problems generically have real eigenvalues and associated eigenvectors that are orthogonal with respect to a weighted inner-product. We begin with an example.

16.1 An example from population biology

Let's consider a simple model of fish populations:

Example 16.1. A population of trout, $u(x,t)$, is introduced into a river. The river flows at a speed c; a portion of the river of length L is isolated between a waterfall at $x = 0$ and a dam at $x = L$. The trout reproduce at a rate α and also diffuse up and downstream with a diffusion constant D. The population is modeled by the advection-growth-diffusion equation

$$\begin{aligned}
\mathbb{DE}:&\quad u_t = Du_{xx} + cu_x + \alpha u \qquad 0 < x < L, \quad t > 0\\
\mathbb{BC}:&\quad u(0,t) = u(L,t) = 0, \quad t > 0\\
\mathbb{IC}:&\quad u(x,0) = f(x) \quad 0 < x < L.
\end{aligned}$$

where the initial population satisfies $f(x) \geq 0$. Can we find a solution for $u(x,t)$? Does the population grow or decay? ■

16.1.1 Separation of Variables

We look for solutions of the form

$$u(x,t) = X(x)T(t). \tag{16.2}$$

Plugging this form into the differential equation we get

$$X(x)T'(t) = \left[DX''(x) + cX'(x) + \alpha X(x)\right]T(t)$$

and dividing by $X(x)T(t)$ we find

$$\frac{T'(t)}{T(t)} = \frac{DX''(x) + cX'(x) + \alpha X(x)}{X(x)}.$$

Notice that the left hand side is a function of t alone, while the right is a function of x only. This implies that both sides must indeed be constant. We will call this *separation constant*, $-\lambda$. Thus we have

$$\frac{T'(t)}{T(t)} = \frac{DX''(x) + cX'(x) + \alpha X(x)}{X(x)} = -\lambda. \tag{16.3}$$

We can separate this equation into two equations, one involving only t,

$$\frac{T'(t)}{T(t)} = -\lambda,$$

and one involving only x,

$$\frac{DX''(x) + cX'(x) + \alpha X(x)}{X(x)} = -\lambda.$$

We will now assume that λ is real. This assumption could plausibly cause us to lose some solutions, but eventually we will show that it is the only case that yields non-trivial solutions.

Each of these equations is now an ordinary differential equation, and thus we can draw on the theory of ordinary differential equations to solve them. The first equation,

$$T'(t) = -\lambda T(t)$$

has the solution

$$T(t) = Ae^{-\lambda t}.$$

Before we solve the second-order ordinary differential equation in x, we will derive some boundary conditions for this equation by apply the separation of variable ansatz to the boundary conditions on the PDE. Note that

$$u(0,t) = X(0)T(t) = 0$$

which implies either $X(0) = 0$ or $T(t) = 0$. If we choose $T(t) = 0$ then $u(x,t) = X(x)T(t) = 0$, which, while true, is just the trivial solution. Therefore we conclude that

$$u(0,t) = X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0$$

for us to find non-trivial solutions. Similarly

$$u(L,t) = X(L)T(t) = 0 \quad \Rightarrow \quad X(L) = 0$$

by analogous reasoning. Which yields the boundary value problem,

$$\mathbb{DE}: \quad DX''(x) + cX'(x) + (\alpha + \lambda)X(x) = 0 \qquad 0 < x < L \tag{16.4}$$

$$\mathbb{BC}: \quad X(0) = 0, \quad X(L) = 0. \tag{16.5}$$

This is an example of a *Sturm-Liouville Eigenvalue Problem*; we will show that this problem only has non-trivial solutions for certain values of λ, called the eigenvalues.

To solve (16.4), we look for solutions of the form $X(x) = e^{rx}$ which yields the characteristic equation

$$Dr^2 + cr + (\alpha + \lambda) = 0$$

which implies

$$r = \frac{-c \pm \sqrt{c^2 - 4(\alpha+\lambda)D}}{2D}.$$

As before, we will assume that λ is real, a fact that we will prove later. There are now three cases:

- **Two real roots:** When

  $$c^2 - 4(\alpha+\lambda)D > 0 \quad \Rightarrow \quad \lambda < \frac{c^2}{4D} - \alpha$$

  there are two real roots

  $$r_{\pm} = \frac{-c \pm \sqrt{c^2 - 4(\alpha+\lambda)D}}{2D}.$$

In this case

$$X(x) = Be^{r_+ x} + Ce^{r_- x},$$

and to satisfy the two boundary conditions

$$X(0) = B + C = 0 \qquad X(L) = Be^{r_+ L} + Ce^{r_- L} = 0$$

which written in matrix form yields

$$\begin{bmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which would only have a non-trivial solution if the determinant of the coefficient matrix is zero, that is if

$$e^{r_- L} = e^{r_+ L}$$

which implies $r_+ = r_-$ which is a contradiction.

- **Two equal roots:** When

$$c^2 - 4(\alpha + \lambda)D = 0 \quad \Rightarrow \quad \lambda = \frac{c^2}{4D} - \alpha$$

then the two roots are real, that is

$$r = -\frac{c}{2D}.$$

In this case

$$X(x) = Be^{-cx/2D} + Cxe^{-cx/2D},$$

and to satisfy the two boundary conditions

$$X(0) = B = 0 \qquad X(L) = Be^{-cL/2D} + CLe^{-cL/2D} = 0$$

which implies $B = C = 0$ and again there is no non-trivial solution.

- **A complex conjugate pair of roots:** When

$$c^2 - 4(\alpha + \lambda)D < 0 \quad \Rightarrow \quad \lambda > \frac{c^2}{4D} - \alpha$$

there is a complex conjugate pair of roots,

$$r_\pm = -\frac{c}{2D} \pm i\Omega \qquad \Omega = \frac{\sqrt{4(\alpha + \lambda)D - c^2}}{2D}.$$

In this case

$$X(x) = Be^{-cx/2D}\cos(\Omega x) + Ce^{-cx/2D}\sin(\Omega x),$$

the first boundary condition implies

$$X(0) = B = 0$$

while the second implies that

$$X(L) = Ce^{-cx/2D}\sin(\Omega L) = 0$$

or that

$$\Omega = \Omega_n \equiv \frac{n\pi}{L} \qquad n = 1, 2, 3\cdots.$$

Solving

$$\Omega_n = \frac{n\pi}{L} = \frac{\sqrt{4(\alpha+\lambda)D - c^2}}{2D}$$

yields

$$\lambda = \lambda_n \equiv -\alpha + \frac{c^2}{4D} + \frac{Dn^2\pi^2}{L^2} \qquad n = 1, 2, 3\cdots.$$

Summarizing, we have found a countable set of eigenfunctions and eigenvalues,

$$\boxed{X_n(x) = e^{-cx/2D}\sin\left(\frac{n\pi x}{L}\right) \qquad \lambda_n = -\alpha + \frac{c^2}{4D} + \frac{Dn^2\pi^2}{L^2}} \tag{16.6}$$

for $n = 1, 2, 3, \cdots$. We can associate with each eigenvalue a solution to the ODE for $T(t)$,

$$T_n(t) \equiv e^{-\lambda_n t},$$

where we note that again we have chosen the multiplicative constant $A = 1$ and the index n recognizes that we restricting ourselves to the case when $\lambda = \lambda_n$.

We now have a countable set of solutions which satisfy both the differential equation, and the boundary values, namely

$$u_n(x,t) \equiv X_n(x)T_n(t) = e^{-\lambda_n t}\cdot e^{-cx/2D}\sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, ...$$

However, we know that as the differential equation and boundary condition are homogeneous the solutions form a vector space!! So the most general solution is a linear combination of the u_n's.

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n u_n(x,t) \\ &= \sum_{n=1}^{\infty} a_n X_n(x) T_n(t) \\ &= \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \cdot e^{-cx/2D} \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

where the a_n are arbitrary constants. So the general solution is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \cdot e^{-cx/2D} \sin\left(\frac{n\pi x}{L}\right)} \tag{16.7}$$

16.1.2 Solving the initial value problem

To summarize, so far we have found a general solution (16.7) that satisfies both the differential equation and the associated boundary conditions for the homogeneous Dirichlet problem for the diffusion equation. We still need to satisfy the initial condition, $u(x,0) = f(x)$, which we will argue determines the arbitrary constants b_n.

Apply the initial condition to the general solution yields

$$u(x,0) = \sum_{n=1}^{\infty} a_n X_n(x) = \sum_{n=1}^{\infty} a_n e^{-cx/2D} \sin\left(\frac{n\pi x}{L}\right) = f(x). \tag{16.8}$$

At first this seems daunting; we have the solution as the sum of a set of functions we have never seen before. However, note what happens if we multiply through by $e^{cx/2D}$,

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = e^{cx/2D} f(x), \qquad 0 < x < L.$$

This is just a Fourier Sine Series for the function $e^{cx/2D} f(x)$. Consequently, we know that

$$\boxed{a_n = \frac{2}{L} \int_0^{\pi} e^{cx/2D} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx \qquad n = 1, 2, 3, \ldots.}$$

16.1.3 Analyzing the solution: Does the population grow or decay?

Looking at the solution,

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \cdot e^{-cx/2D} \sin\left(\frac{n\pi x}{L}\right),$$

we see the n^{th} term in the sum will grow only if

$$\lambda_n < 0 \quad \Rightarrow \frac{c^2}{4D} + \frac{Dn^2\pi^2}{L^2} < \alpha$$

The smallest eigenvalue is λ_1, so when the growth rate α satisfies

$$\alpha > \alpha_c \equiv \frac{c^2}{4D} + \frac{Dn^2\pi^2}{L^2}$$

we will have growth. Note that the threshold growth rate increases as c increases (think about fish being swept downstream) and decreases as the domain gets longer. We also note that

$$a_0 = \frac{2}{L}\int_0^{\pi} e^{cx/2D} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

is positive for any non-zero initial population, $f(x) \geq 0$ (remember populations are non-negative!!).

16.2 Solution to a Population Problem: What Just Happened?

To summarize, we have suggested that the solution to

$$\begin{aligned}
\mathbb{DE}: &\quad u_t = Du_{xx} + cu_x + \alpha u \qquad 0 < x < L, \quad t > 0\\
\mathbb{BC}: &\quad u(0,t) = u(L,t) = 0, \quad t > 0\\
\mathbb{IC}: &\quad u(x,0) = f(x) \quad 0 < x < L.
\end{aligned}$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \cdot e^{-cx/D} \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{2}{L}\int_0^{\pi} e^{cx/D} f(x) \sin(nx)\, dx$$

However, the derivation has left us with a number of questions:

- Are the eigenvalues always real? Are they always positive?
- Have we found all the eigenvalues and eigenfunctions?
- Is there a systematic way to find the coefficients a_n? In particular, can we found an orthogonality condition for the set of eigenfunctions, $\{X_n(x)\}$?

We will attempt to answer these questions (or at least indicate the answers) below using Sturm-Liouville Theory.

16.3 Sturm-Liouville Eigenvalue Problems

We begin by defining the Sturm-Liouville Eigenvalue Problem:

Definition 16.9 (Sturm-Liouville Eigenvalue Problem)**.** Let $y(x)$ be a twice continuously differentiable function on the interval $a \leq x \leq b$ (i.e. $y(x) \in C^2[a,b]$) . Let $\mathcal{L}$ be the Sturm-Liouville differential operator defined by

$$\mathcal{L}y \equiv -\left(s(x)y'\right)' + q(x)y$$

where the functions $s(x), q(x), \rho(x)$ are continuous functions on $[a,b]$ with $s(x)$ and $\rho(x)$ both positive-valued on $[a,b]$. The regular *Sturm-Liouville Eigenvalue Problem* is defined by the differential equation

$$\mathbb{DE}: \qquad \mathcal{L}y = \lambda\rho(x)y, \quad a < x < b, \tag{16.10}$$

together with the boundary conditions

$$\mathbb{BC}: \qquad \alpha y(a) + \beta y'(a) = 0, \quad \gamma y(b) + \delta y'(b) = 0 \tag{16.11}$$

where α and β are not both zero and γ and δ are not both zero. We call a constant λ and a non-zero function y that satisfy this problem an *eigenvalue/eigenfunction pair*.

Exercise 16.1. Show that the Sturm-Liouville form stated at the beginning of the chapter,

$$\left(s(x)X'(x)\right)' + \left(\lambda\rho(x) - q(x)\right)X(x) = 0 \tag{16.12}$$

and the form (16.10) stated in the definition above

$$\mathcal{L}y = \lambda\rho(x)y, \tag{16.13}$$

are equivalent when $X(x)$ is replaced $y(x)$.

We will prove a set of theorems about the eigenfunctions and eigenvalues of this problem, but first we need to introduce an appropriate vector space of real functions and an inner product. Define

$$\mathcal{U} \equiv \{y(x) \in C^2[a,b],\ \alpha y(a) + \beta y'(a) = 0, \quad \gamma y(b) + \delta y'(b) = 0\},$$

The idea here is we only consider the set of functions that satisfy the boundary conditions.

Exercise 16.2. Show that $\mathcal{U}$ is a vector space.

In the same way that we generalized the eigenvalue problem, we also need to generalize the inner-product associated with the eigenfunctions.

Definition 16.14 (Weighted Inner-product)**.** If $\rho(x)$ is defined on $x \in [a,b]$ with $\rho(x) > 0$ for $x \in (a,b)$ we can define the weighted inner-product,

$$\langle u, v \rangle_\rho = \int_a^b uv\ \rho(x)\ dx.$$

Note that if $\rho(x) = 1$ this is the normal L^2 inner-product. The vector space $\mathcal{U}$ is a real inner product space with respect to this weighted inner-product. We will show the eigenfunctions for the Sturm-Liouville eigenvalue problem are orthogonal with respect to a weighted inner-product.

Definition 16.15 (Weighted norm)**.** We defined the weighted norm of a function, as

$$\|u\|_\rho \equiv \sqrt{\langle u, u \rangle_\rho} = \sqrt{\int_a^b u^2\ \rho(x)\ dx}.$$

16.3.1 Examples of Sturm-Liouville equations

(a) The Fourier Eigenvalue Problem for $X(x)$ (Dirichlet boundary conditions):

$$\begin{aligned} \mathbb{DE}: &\qquad X'' + \lambda X = 0 \\ \mathbb{BC}: &\quad X(a) = 0, \quad X(b) = 0 \end{aligned}$$

for $x \in [a,b]$ satisfies the formal definition (16.10) for $s(x) = 1,\ q(x) = 0$, and weight function $\rho(x) = 1$. Note that this is a regular Sturm-Liouville problem.

(b) Bessel's equation for $\phi(x)$:

$$\begin{aligned}\mathbb{DE}: &\quad (x\phi'(x))' + \lambda\,(x\phi(x)) = 0\\ \mathbb{BC}: &\quad \phi(1) = 0, \quad \phi(2) = 0\end{aligned}$$

is a Sturm-Liouville equation with $s(x) = x,\ q(x) = 0$, and weight function $\rho(x) = x$. This is a regular Sturm-Liouville problem.

Sturm-Liouville equations naturally result from separation of variables applied to many problems of physical and mathematical interest. Fortunately, these equations with appropriate boundary conditions provide a wealth of orthogonal functions. In fact, we will see that regular Sturm-Liouville problems have an infinite number of eigenvalues, and the corresponding eigenfunctions form a *complete*, orthogonal set.

16.3.2 The Self-adjoint Sturm Liouville operator

Remember the differential operator $\mathcal{P}$ acting on elements in an inner product space is *self-adjoint* if

$$\langle u, \mathcal{P}v\rangle = \langle \mathcal{P}u, v\rangle\,.$$

We now show the Sturm-Liouville operator

$$\mathcal{L}y \equiv -\left(s(x)y'\right)' + q(x)y$$

is self-adjoint in $\mathcal{U}$.

Theorem 16.1. *The differential operator $\mathcal{L}$ is self-adjoint in the inner-product space $\mathcal{U}$ with the standard L^2 inner-product.*

Proof. The proof follows by integration by parts; suppose u and v are elements of one of the inner product spaces. Then

$$\begin{aligned}\langle u, \mathcal{L}v\rangle &= -\int_a^b u[(sv')']\,dx + \int_a^b u[qv]\,dx\\ &= \int_a^b su'v'\,dx - suv'|_{x=a}^{x=b} + \int_a^b quv\,dx\\ &= -\int_a^b su''v\,dx + s(u'v - uv')|_{x=a}^{x=b} + \int_a^b quv\,dx\\ &= \langle \mathcal{L}u, v\rangle + s(u'v - uv')|_{x=a}^{x=b}\,.\end{aligned}$$

Let us consider the boundary at $x = a$. Remember that u and v are in $\mathcal{U}$ and therefore satisfy the boundary conditions

$$\alpha u(a) + \beta u'(a) = 0, \quad \alpha u(a) + \beta u'(a) = 0$$

with α and β not both zero. To show the boundary term vanishes at $x = a$ it suffice to show that $u'(a)v(a) - u(a)v'(a) = 0$, If $\alpha = 0$ (which means $\beta \neq 0$) then $u'(a) = v'(a) = 0$ and the boundary term vanishes. Otherwise

$$\begin{aligned}
\alpha(u'(a)v(a) - u(a)v'(a)) &= u'(a)[\alpha v(a)] - v'(a)[\alpha u(a)] \\
&= u'(a)[-\beta v'(a)] - v'(a)[-\beta u'(a)] \\
&= \beta[-u'(a)v'(a) + v'(a)u'(a)] \\
&= 0
\end{aligned}$$

and again the boundary term vanishes. A similar argument holds at $x = b$. Therefore

$$\langle u, \mathcal{L}v \rangle = \langle \mathcal{L}u, v \rangle ,$$

and we have shown the operator is self-adjoint. □

16.3.3 The Eigenvalues of the Sturm-Liouville Problem are Real

The proof that the eigenvalues of the Sturm-Liouville problem are real follows analogously to the proof for the Fourier Eigenvalue problem.

Theorem 16.2. *Suppose $\mathcal{L}$ is self-adjoint linear operator associated with the inner product space $\mathcal{U}$. Then the eigenvalue problem*

$$\mathcal{L}y = \lambda \rho y$$

where

$$y = p + iq \qquad p, q \in \mathcal{U}$$

has only real eigenvalues, λ.

Remark. Note that we have temporarily expanded the eigenvalue problem to allow y to be a complex function with real part p and imaginary part q (we know p and q are real functions because they are in the real inner product space $\mathcal{U}$). Also, for the Sturm-Liouville Eigenvalue Problem remember that the boundary conditions are hidden in the definition of the vector space.

Proof. First we need to be clear about how $\mathcal{L}$ acts on the complex function y; it is linear so

$$\mathcal{L}y = \mathcal{L}(p + iq) = \mathcal{L}p + i\mathcal{L}q.$$

next define the complex conjugate of y,

$$\bar{y} = p - iq.$$

We now use linearity to extend the definition of the real inner product to complex functions. Consider

$$\begin{aligned}
\langle \bar{y}, \mathcal{L}y \rangle &= \langle p - iq, \mathcal{L}p + i\mathcal{L}q \rangle \\
&= \langle p, \mathcal{L}p \rangle + \langle q, \mathcal{L}q \rangle + i(\langle p, \mathcal{L}q \rangle - \langle q, \mathcal{L}p \rangle) \\
&= \langle p, \mathcal{L}p \rangle + \langle q, \mathcal{L}q \rangle
\end{aligned}$$

where we have used the fact that the operator $\mathcal{L}$ is self-adjoint and the symmetry of the inner product to see that $\langle p, \mathcal{L}q \rangle = \langle q, \mathcal{L}p \rangle$. From the eigenvalue problem and linearity we also know

$$\begin{aligned}
\langle \bar{y}, \mathcal{L}y \rangle &= \langle \bar{y}, \lambda \rho y \rangle \\
&= \lambda \left\langle \bar{y}, y \right\rangle_\rho \\
&= \lambda \left\langle p - iq, p + iq) \right\rangle_\rho \\
&= \lambda \left[\left\langle p, p \right\rangle_\rho + \left\langle q, q \right\rangle_\rho + i(\left\langle p, q \right\rangle_\rho - \left\langle q, p \right\rangle_\rho) \right] \\
&= \lambda(\|p|_\rho^2 + \|q\|_\rho^2)
\end{aligned}$$

Now, solving for λ from these expressions yields

$$\lambda = \frac{\langle p, \mathcal{L}p \rangle + \langle q, \mathcal{L}q \rangle}{\|p\|_\rho^2 + \|q\|_\rho^2}.$$

The quotient on the righthand side is real, therefore λ is real. As a side note, if we now rewrite the eigenvalue problem,

$$\mathcal{L}y = \lambda \rho y \Rightarrow \mathcal{L}(p + iq) = \lambda(p + iq),$$

and equate real and imaginary parts,

$$\mathcal{L}p = \lambda \rho p, \qquad \mathcal{L}q = \lambda \rho q,$$

we see that both the real and imaginary parts of y are eigenfunctions; that is, the real eigenvalue λ can be associated with a real eigenfunction (either p and/or q which can't both be zero). □

16.3.4 Orthogonality of Eigenfunctions

Eigenfunctions associated with self-adjoint operators inherit a natural orthogonality from the inner product space.

Theorem 16.3. *Suppose $\mathcal{L}$ is self-adjoint linear operator associated with the inner product space $\mathcal{U}$. If y_n and y_m are eigenfunctions with distinct associated eigenvalues $\lambda_n \neq \lambda_m$ for the eigenvalue problem*

$$\mathcal{L}y = \lambda \rho y$$

then the eigenfunctions are orthogonal, that is

$$\langle y_m, y_n \rangle_\rho = 0.$$

Proof. From the self-adjointness of $\mathcal{L}$ we see that

$$\langle y_m, \mathcal{L}y_n \rangle = \langle \mathcal{L}y_m, y_n \rangle$$

and from the eigenvalue problem and linearity this implies

$$\lambda_n \langle y_m, y_n \rangle_\rho = \lambda_m \langle y_m, y_n \rangle_\rho .$$

Rearranging yields

$$(\lambda_n - \lambda_m) \langle y_m, y_n \rangle_\rho = 0.$$

As $\lambda_n \neq \lambda_m$ we conclude $\langle y_m, y_n \rangle_\rho = 0$, that is the eigenfunctions are orthogonal. □

The fact that the eigenvalues are real and the eigenfunctions are orthogonal depended solely on the operator $\mathcal{L}$ being self-adjoint. Like for the Fourier Eigenvalue Problem we can sometimes deduce some results about the sign of the eigenvalues.

Theorem 16.4. *Suppose y and λ are an eigenvalue/eigenfunction pair for the Sturm-Liouville Eigenvalue Problem with $q(x) \geq 0$, $\alpha = 0$ or $\beta = 0$, and $\gamma = 0$ or $\delta = 0$. Then $\lambda \geq 0$. Moreover, $\lambda = 0$ is an eigenvalue if and only if $q = 0$ and the associated eigenfunction is constant.*

Remark. The condition that $\alpha = 0$ or $\beta = 0$, and $\gamma = 0$ or $\delta = 0$ implies that the eigenfunction satisfies Dirichlet or Neumann boundary conditions at each end point. It is possible to show positivity of the eigenvalue for some other boundary conditions also.

Proof. We know that

$$\langle y, \mathcal{L}y \rangle = \lambda \langle y, \rho y \rangle = \lambda \|y\|_\rho^2$$

however, using integration by parts, we also know that

$$\begin{aligned}
\langle y, \mathcal{L}y \rangle &= -\int_a^b y[(sy')'] \, dx + \int_a^b y[qy] \, dx \\
&= \int_a^b sy'y' \, dx - syy'|_{x=a}^{x=b} + \|y\|_q^2 \\
&= \|y'\|_s^2 + \|y\|_q^2 - syy'|_{x=a}^{x=b} \\
&= \|y'\|_s^2 + \|y\|_q^2
\end{aligned}$$

where we have used the fact that the boundary term yy' vanishes at each end point for Dirichlet or Neumann boundary conditions. Now, solving for λ from these expressions yields

$$\lambda = \frac{\|y'\|_s^2 + \|y\|_q^2}{\|y\|_\rho^2} \, .$$

Clearly, the right-hand side is non-negative. Moreover, if $\lambda = 0$ then the numerator must be zero which implies $\|y\|_q^2 = 0$ which means $q = 0$ and also that $\|y'\|_s^2 = 0$ for $a < x < b$, that is y is constant. So we conclude that $\lambda \geq 0$ and if $\lambda = 0$, then $q = 0$ and y is constant. □

16.3.5 Putting equations in Sturm-Liouville form

The general second-order differential equation for $y(x)$,

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + [b(x) + \lambda c(x)] \, y = 0,$$

where $a(x), b(x)$ and $c(x)$ are arbitrary functions, can be put into Sturm-Liouville form,

$$\frac{d}{dx}\left(s(x)\frac{dy}{dx}\right) + [\lambda\rho(x) - q(x)] \, y(x) = 0.$$

By choosing

$$s(x) = \exp[\int a(x')dx'], \qquad q(x) = -b(x)s(x), \qquad \rho(x) = c(x)s(x)$$

which transforms the Sturm-Liouville problem into the general second-order DE. We leave this as an exercise for the reader

Exercise 16.3. Consider the general second-order differential equation above.

(a) Show that choosing $s(x) = \exp[\int a(x')dx']$, $q(x) = -b(x)s(x)$, and $\rho(x) = c(x)s(x)$ transforms the Sturm-Liouville problem into the general second-order DE.

(b) Write the following equations in Sturm-Liouville form:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left[\lambda - \frac{n^2}{x^2}\right] y = 0 \quad x > 0 \qquad \text{(Bessel's Equation)}$$

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{\lambda}{1-x^2} y = 0 \quad 1 > x > -1 \qquad \text{(Legendre's Equation)}$$

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0 \quad \infty > x > -\infty \qquad \text{(Hermite's Equation)}$$

Returning to our population example, consider the boundary value problem,

$$\mathbb{DE}: \quad DX''(x) + cX'(x) + (\alpha + \lambda)X(x) = 0 \qquad 0 < x < L \quad (16.16)$$

$$\mathbb{BC}: \quad X(0) = 0, \quad X(L) = 0. \quad (16.17)$$

If we multiply (16.16) by $\exp(cx/D)$ we can rewrite it as

$$\frac{d}{dx}\left(e^{cx/D}\frac{dX}{dx}\right) + \left[\lambda e^{cx/D} + \alpha e^{cx/D}\right] X(x) = 0.$$

which is now in standard Sturm-Liouville form. Consequently we know the eigenvalues are real (as previously claimed). Note also that $\rho(x) = e^{cx/D}$. Consequently, we have the orthogonality condition (with a weighted inner-product)

$$\langle X_m, X_n \rangle_\rho = 0$$

when $m \neq n$.

Now, to solve for a_n when

$$f(x) = \sum_{n=1}^{\infty} a_n X_n(x) = \sum_{n=1}^{\infty} a_n e^{-cx/2D} \sin\left(\frac{n\pi x}{L}\right)$$

we use orthogonality to yield

$$\begin{aligned}
a_n &= \frac{\langle X_n(x), f(x)\rangle_\rho}{\|X_n(x)\|_\rho^2} \\
&= \frac{\int_0^L e^{-cx/2D}\sin\left(\frac{n\pi x}{L}\right)\cdot f(x)\cdot e^{cx/D}\,dx}{\int_0^L \left[e^{-cx/2D}\sin\left(\frac{n\pi x}{L}\right)\right]^2\cdot e^{cx/D}\,dx} \\
&= \frac{\int_0^L e^{cx/2D}\sin\left(\frac{n\pi x}{L}\right)\cdot f(x)\,dx}{\int_0^L \left[\sin\left(\frac{n\pi x}{L}\right)\right]^2\,dx} \\
&= \frac{2}{L}\int_0^L e^{cx/2D}\sin\left(\frac{n\pi x}{L}\right)\cdot f(x)\,dx
\end{aligned}$$

which is exactly the result we found above. Consequently, the population example is a nice example of a Sturm-Liouville Eigenvalue Problem and we have a justification for several of the ad hoc assumptions made above.

Seventeen

Playing the Bongos: Bessel Functions and Oscillations of a Circular Membrane

Parts of this section evolved from earlier notes due to Darryl Yong.

17.1 An Introduction to Bessel's Equation and Bessel Functions

Bessel's Equation,

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0, \tag{17.1}$$

arises in many problems in mathematical physics such as the vibration of a circular membrane or the oscillations of a swinging chain. The constant n here is called the *order* of the equation, and the case where n is a positive integer arises most frequently. The equation has a regular singular point at $x = 0$, which suggest that we can find solutions using the method of Frobenius. For this discussion we will assume that $x > 0$.

17.1.1 Frobenius Series Solutions

We will now look for a Frobenius series solutions to Bessel's equation at $x = 0$. Let's multiply by x^2 to obtain

$$x^2y'' + xy' + \left(x^2 - n^2\right) y = 0, \tag{17.2}$$

We substitute the usual ansatz,

$$y = x^p \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+p}$$

which upon differentiation yields

$$y' = \sum_{k=0}^{\infty} (k+p) a_k x^{k+p-1}, \qquad xy' = \sum_{k=0}^{\infty} (k+p) a_k x^{k+p},$$

and a second differentiation yields

$$y'' = \sum_{k=0}^{\infty} (k+p)(k+p-1) a_k x^{k+p-2}, \qquad x^2 y' = \sum_{k=0}^{\infty} (k+p)(k+p-1) a_k x^{k+p},$$

Finally, we need to shift the sum for $x^2 y$,

$$x^2 y = \sum_{k=0}^{\infty} a_k x^{k+p+2} \quad \overset{k'=k+2}{=} \quad \sum_{k'=2}^{\infty} a_{k'-2} x^{k'+p}$$

into (17.2) to get

$$\begin{aligned}
0 &= x^2 y'' + xy' + \left(x^2 - n^2\right) y \\
&= \sum_{k=0}^{\infty} (k+p)(k+p-1) a_k x^{k+p} + \sum_{k=0}^{\infty} (k+p) a_k x^{k+p} + \sum_{k=2}^{\infty} a_{k-2} x^{k+p} - n^2 \sum_{k=0}^{\infty} a_k x^{k+p}, \\
&= \sum_{k=0}^{\infty} a_k \left[(k+p)^2 - n^2\right] x^{k+p} + \sum_{k=2}^{\infty} a_{k-2} x^{k+p}.
\end{aligned}$$

Now we combine the sums and collect the leftover initial ($k = 0, 1$) terms,

$$a_0 (p^2 - n^2) x^p + a_1 \left[(p+1)^2 - n^2\right] x^{p+1} + \sum_{k=2}^{\infty} \left\{ a_k \left[(k+p)^2 - n^2\right] + a_{k-2} \right\} z^{k+p+2} = 0 \tag{17.3}$$

Finally, we set all coefficients to zero:

$$\begin{aligned}
a_0 (p^2 - n^2) &= 0, \\
a_1 \left[(p+1)^2 - n^2\right] &= 0, \\
a_m \left[(m+p)^2 - n^2\right] + a_{m-2} &= 0 \qquad \text{for } m = 2, 3, 4, \cdots .
\end{aligned}$$

The value p is supposed to represent the lowest exponent present in the series, so a_0 should not be zero (or else p would not be the lowest exponent present in the series). Since $a_0 \neq 0$, equation (17.4a) implies that $p^2 - n^2 = 0$, which is the *indicial equation*. The indicial equation has roots $p = \pm n$. For $n > 0$, the indicial equation indicates that one possible solution to the ODE has a power series that begins with z^n, and one that begins with z^{-n}. The former would be well-defined (and bounded) at $z = 0$, the other would blow up at $z = 0$. Let's try and find a solution for $p = n$.

Note that equation (17.4b) now implies that

$$a_1 \left[(n+1)^2 - n^2\right] = a_1 \left[n(2n+1)\right] = 0$$

Consequently, we choose[1] $a_1 = 0$, which will imply that the odd terms of the series vanish.

Equation (17.4c), can be rewritten as

$$a_k = -\frac{a_{k-2}}{(k+n)^2 - n^2} = -\frac{a_{k-2}}{(k)(k+2n)},$$

is a *recurrence relation* for $k = 2, 3, 4 \cdots$. Because it relates a_k to a_{k-2}, it effectively links all odd coefficients together, and all even coefficients together. As a result, $a_1 = 0$ causes $a_3 = a_5 = \cdots = 0$, so all odd coefficients are zero. To determine the even coefficients, we use the recurrence relation repeatedly,

$$a_2 = -\frac{a_0}{2(2n+2)} = -\frac{a_0}{2^2(n+1)},$$

and

$$a_4 = -\frac{a_2}{4(4+2n)} = -\frac{a_2}{2^2 \cdot 2(n+2)} = \frac{a_0}{2^4 \cdot 2 \cdot 1 \cdot (n+2) \cdot (n+1)},$$

until the general pattern can be inferred:

$$a_{2j} = \frac{(-1)^j a_0}{2^{2j} j!(n+j) \cdots (n+1)}.$$

So,

$$y = a_0 x^n \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{2^{2j} j!(n+j) \cdots (n+1)}.$$

We note that this solution is well defined as long as n is **not** a negative integer; if n is a negative integer we discover that the coefficient a_{-2n} is undefined as we end up dividing by zero!! We now examine a set of cases.

[1]When $n = -1/2$, we are not forced to choose $a_1 = 0$ but in this case a_1 would just multiply the second solution generated by the index $n = 1/2$.

17.1.2 The Bessel Function $J_n(x)$ where $n = 0, 1, 2, 3\cdots$.

Arguably, this case is the most important and commonly occurs in physical problems. If n is a positive integer, by tradition we choose $a_0 = 1/(2^n n!)$ to obtain the specific solution $y = J_n(x)$ where

$$J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+n)!} \left(\frac{x}{2}\right)^{2j+n},$$

which is known as the n^{th} *order Bessel function of the first kind.* Note that J_0 has a particularly simple form,

$$J_0(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j!)^2} \left(\frac{x}{2}\right)^{2j},$$

and that $J_0(0) = 1$ (which is clear below).

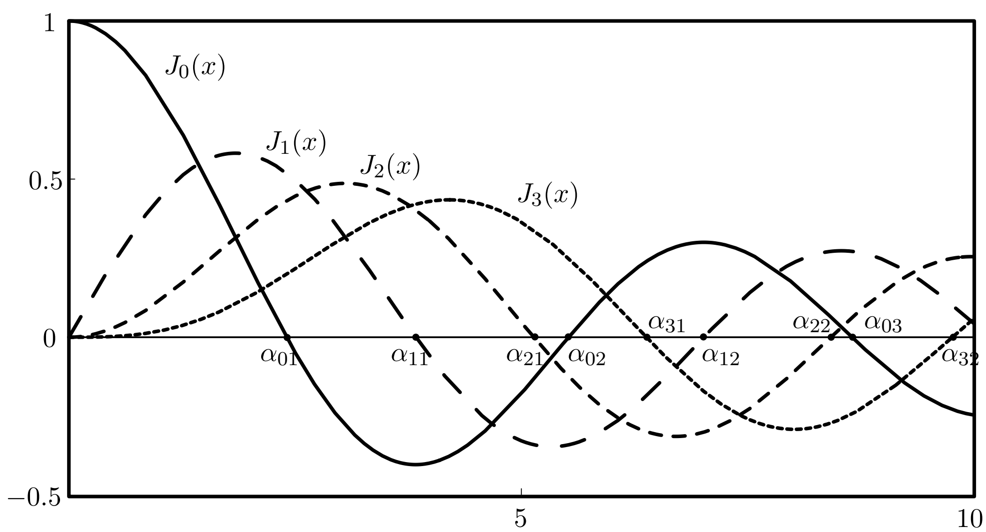


Figure 17.1: The first four Bessel functions: $J_n(x)$ for $n = 1, 2, 3$ and 4.

In general, this expansion above shows that

$$J_n(x) \sim \frac{1}{n!} \left(\frac{x}{2}\right)^n,$$

for small values of x. which is evident in Figure 17.1. Note that J_n has an infinite number of positive zeroes; the m^{th} positive zero is denoted α_{nm},

$$J_n(\alpha_{nm}) = 0 \qquad 0 < \alpha_{n1} < \alpha_{n2} < \alpha_{n3} \cdots .$$

As m tends to infinity, the distance between successive zeroes of J_n tends to π. The zeroes of the Bessel functions play a critical role in the eigenvalue problems in which they arise.

17.1.3 The Bessel Function $Y_n(x)$, $n = 0, 1, 2 \cdots$.

When $\alpha = -n$ a negative integer, the coefficients in the Frobenius series for J_α become infinite. In addition, when $n = 0$, the inidicial equation for the Frobenius series, $p^2 - n^2 = 0$, has a double root at zero. In these cases we must rely on other methods to generate the second independent solution to Bessel's equation.

One can use reduction of order[2] to obtain

$$Y_n(z) = \frac{2}{\pi}\left[\ln\left(\frac{z}{2}\right) + \gamma\right] J_n(z) - \frac{(z/2)^{-n}}{\pi}\sum_{j=0}^{n-1}\frac{(n-j-1)!(z/2)^{2j}}{j!}$$
$$- \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\left[\left(1 + \frac{1}{2} + \frac{1}{3}\cdots + \frac{1}{k}\right) + \left(1 + \frac{1}{2} + \frac{1}{3}\cdots + \frac{1}{k+n}\right)\right]\frac{(z/2)^{2j+n}}{k!(n+k)!}$$

The feature of $Y_n(z)$ that will generally be most relevant to us is that as z decreases to zero, $Y_n(z)$ tends to minus infinity because of the n^{th} order pole and logarithmic singularity at $z = 0$.

To summarize, the general solution to (17.2) is

$$y(x) = c_1 J_n(x) + c_2 Y_n(x),$$

where the functions J_n and Y_n are the nth order Bessel functions of the first and second kind, respectively. For more information, a good reference is *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by Abramowitz and Stegun.

[2] If $y_1(x)$ is a known solution to a linear n-order differential equation, the substitution $y(x) = u(x)y_1(x)$ will result in a linear differential equation of order $n-1$ for $u'(x)$.

17.2 Separation of Variables for the Axisymmetric Heat Equation

Suppose we wish to solve the axisymmetric heat equation for the temperature, $u(r,t)$, in a disc of radius d and initial temperature $f(r)$ where the outer edge of the disc is held at $u=0$. Then

$$u_t = u_{rr} + \frac{1}{r}u_r \qquad 0 < r < d$$

$$u(d,t) = 0 \qquad u(r,0) = f(r)$$

In addition, we know that the temperature at the origin, $u(0,t)$, is bounded.

As usual, we solve for $u(r,t)$ using separation of variables. We plug $u(r,t) = R(r)T(t)$ into the PDE, and obtain

$$\frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R}$$

For this equality to hold for all r and t, each side of the equation must be a constant. Therefore, we let

$$\frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda$$

where λ is a constant. We choose the separation constant in this particular way, because we will find that the radial eigenvalue problem has only non-negative eigenvalues. The solution to the $T(t)$ differential equation is $T(t) = \exp(-\lambda t)$.

The differential equation for $R(r)$,

$$r^2R'' + rR' + \lambda r^2 R = 0,$$

which we recognize as a form of Bessel's equation of order 0, which can be written in Sturm-Liouville form

$$(rR')' + \lambda r R = 0.$$

In addition,

$$u(d,t) = R(d)T(t) = 0 \quad \Rightarrow R(d) = 0$$

which means we have the singular Sturm-Liouville Eigenvalue Problem

$$DE: \qquad (rR')' + \lambda r R = 0 \quad 0 < r < d \tag{17.5}$$

$$BC: \qquad R(d) = 0 \quad \text{and} \quad R(0) \text{ is bounded.} \tag{17.6}$$

we can show that the eigenvalues to this problem are positive.

We can find the solutions to (17.5,17.6) by first making a change of variables. Let $z = \sqrt{\lambda} r$, then the equation becomes

$$(zR_z)_z + zR = 0.$$

This equation has two independent solutions, best defined via their Frobenius series

$$J_0(z) = 1 - \frac{z^2}{(1!)^2 2^2} + \frac{z^4}{(2!)^2 2^4} - \frac{z^6}{(3!)^2 2^6} \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{(n!)^2}$$

$$Y_0(z) = \frac{2}{\pi} \log(z) + \cdots \qquad z > 0$$

where J_0 is a Bessel function of the first kind and Y_0 is a Bessel function of the second kind. Undoing the change of variables, we find that the general solution for $R(r)$ is

$$R(r) = CJ_0(\sqrt{\lambda} r) + DY_0(\sqrt{\lambda} r)$$

but as Y_0 has a logarithmic singularity at $r = 0$, we must choose $D = 0$. So we let

$$R(r) = CJ_0(\sqrt{\lambda} r)$$

and to satisfy $R(d) = 0$ we see that we must choose λ such that

$$J_0(\sqrt{\lambda} d) = 0.$$

Fortunately, we know that J_0 has an infinite number of zeros; that is

$$J_0(\alpha_n) = 0 \qquad 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$$

from which we deduce that

$$\sqrt{\lambda} d = \alpha_n \quad \Rightarrow \quad \lambda = \lambda_n \equiv \frac{\alpha_n^2}{d^2} \qquad n = 1, 2, 3 \cdots$$

So this equation has eigenvalues and eigenfunctions,

$$R_n(r) = J_0\left(\frac{\alpha_n r}{d}\right) \qquad \lambda_n = \frac{\alpha_n^2}{d^2},$$

where $n = 1, 2, 3, \ldots$.

To satisfy the initial condition, we need to use a linear combination of these solutions:

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_n}{d} r\right) \exp\left(-\frac{\alpha_n^2}{d^2} t\right). \tag{17.7}$$

The initial condition requires that

$$u(r,0) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_n}{d} r\right) = f(r).$$

Because the differential equation for $R(r)$ is a Sturm-Liouville eigenvalue problem, we know that the weighted inner-product of any two distinct eigenfunctions will be zero. Define

$$\langle P(r), Q(r)\rangle_r \equiv \int_0^d P(r)Q(r)\, r\, dr \qquad \|P(r)\|_r^2 \equiv \langle P(r), P(r)\rangle_r$$

Then the orthogonality condition for the eigenfunctions is

$$\langle R_m(r), R_n(r)\rangle_r = \int_0^d J_0\left(\frac{\alpha_m}{d} r\right) J_0\left(\frac{\alpha_n}{d} r\right) r\, dr = \begin{cases} 0 & m \neq n \\ \|R_n(r)\|_r^2 = \frac{d^2}{2} J_1(\alpha_n)^2 & m = n. \end{cases}$$

where the $\|R_n(r)\|_r^2$ is evaluated above from integral identities for Bessel functions.

We use the orthogonality condition to determine that

$$A_n = \frac{\langle f(r), R_n(r)\rangle_r}{\|R_n(r)\|_r^2} = \frac{\int_0^d f(r) J_0(\alpha_n r/d)\, r\, dr}{\frac{d^2}{2} J_1(\alpha_n)^2}.$$

See the MAPLE worksheet for plots of this solution.

17.3 Playing the Bongos: A Vibrating Circular Membrane

Drums from small bongos to orchestral kettle drums consist of a membrane tautly stretched over a circular frame. As a first approximation, the vibrations of the membrane can be modeled by the wave equation,

$$u_{tt} = c^2 \nabla^2 u,$$

where u is the vertical displacement of the membrane and c is the speed of waves traveling on the membrane. As we are describing the vibrations of a circular membrane, it is convenient to use polar coordinates. Let the displacement of the membrane be $u = u(r, \theta, t)$, in which case the wave equation can be more explicitly written as

$$u_{tt} = c^2 \left[u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right], \qquad r < a. \tag{17.8}$$

Since the membrane is fixed a the boundary of the drumhead, where $r = a$, we specify

$$u(a, \theta, t) = 0.$$

We will look for solutions of a particular form in which the variables *separate*. One can look for solutions that are periodic in θ and oscillatory in time. Let

$$u(r, \theta, t) = R(r) \cos(n\theta) \cos(\omega t)$$

where $n = 0, 1, 2, 3 \cdots$ is an non-negative integer and ω is the oscillation frequency which is to be determined. Substituting into the wave equation (17.8) yields

$$-\omega^2 [R \cos(n\theta) \cos(\omega t)] = c^2 \left[R_{rr} + \frac{1}{r} R_r - \frac{n^2}{r^2} R \right] \cos(n\theta) \cos(\omega t)$$

which after dividing by c^2 and rearranging can be rewritten as

$$\left[R_{rr} + \frac{1}{r} R_r + \left(\lambda^2 - \frac{n^2}{r^2} \right) R \right] \cos(n\theta) \cos(\omega t) = 0$$

where we have let $\lambda = \omega / c$. Which means that $R(r)$ must satisfy

$$R_{rr} + \frac{1}{r} R_r + \left(\lambda^2 - \frac{n^2}{r^2} \right) R = 0, \qquad r < a. \tag{17.9}$$

In addition, from the boundary condition we see that we must have $R(a) = 0$.

At this point we make a change of variables; let $r = x/\lambda$ and let $R(r) = R(x/\lambda) \equiv y(x)$. This yields Bessel's Equation,

$$\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{n^2}{x^2} \right) y = 0, \tag{17.10}$$

for which the general solution is

$$y(x) = c_1 J_n(x) + c_2 Y_n(x),$$

where J_n and Y_n are Bessel's functions of the first and second kind, respectively. Changing variables back to $R(r)$, we see that the general solution to (17.9) is

$$R(r) = CJ_n(\lambda r) + DY_n(\lambda r),$$

Bessel's functions of the second kind, $Y_n(\lambda r)$ have singularities at $r = 0$, so for $R(0)$ to remain finite we must choose $D = 0$. The other boundary condition $R(a) = 0$ requires that

$$CJ_n(\lambda a) = 0.$$

which means that λa must be a zero of the Bessel function. This is an *eigenvalue problem*; remembering that we denote the m^{th} zero of the n^{th} Bessel function as α_{nm}, that is $J_n(\alpha_{nm}) = 0$, we see that we must choose

$$\lambda = \lambda_{nm} = \frac{\alpha_{nm}}{a}$$

which corresponds to an oscillation frequency

$$\omega_{nm} \equiv c\lambda_{nm} = \alpha_{nm}\frac{c}{a}.$$

Consequently, we conclude that

$$u(r, \theta, t) = J_n\left(\alpha_{nm}\frac{r}{a}\right)\cos(n\theta)\cos(\omega_{nm}t), \qquad \omega_{nm} = \alpha_{nm}\frac{c}{a}$$

is a solution to the wave equation for $n = 0, 1, 2, 3\cdots$ and $m = 1, 2, 3\cdots$.

Geometrically, n is an angular wavenumber indicating the number of oscillations in θ. The second wavenumber m indicates the number of radial maxima and minima. The figure on the next page illustrates some sample eigenmodes, each which oscillates with it's own characteristic frequency, $\omega_{nm} = \alpha_{nm}\frac{c}{a}$ determined by the wavenumbers.

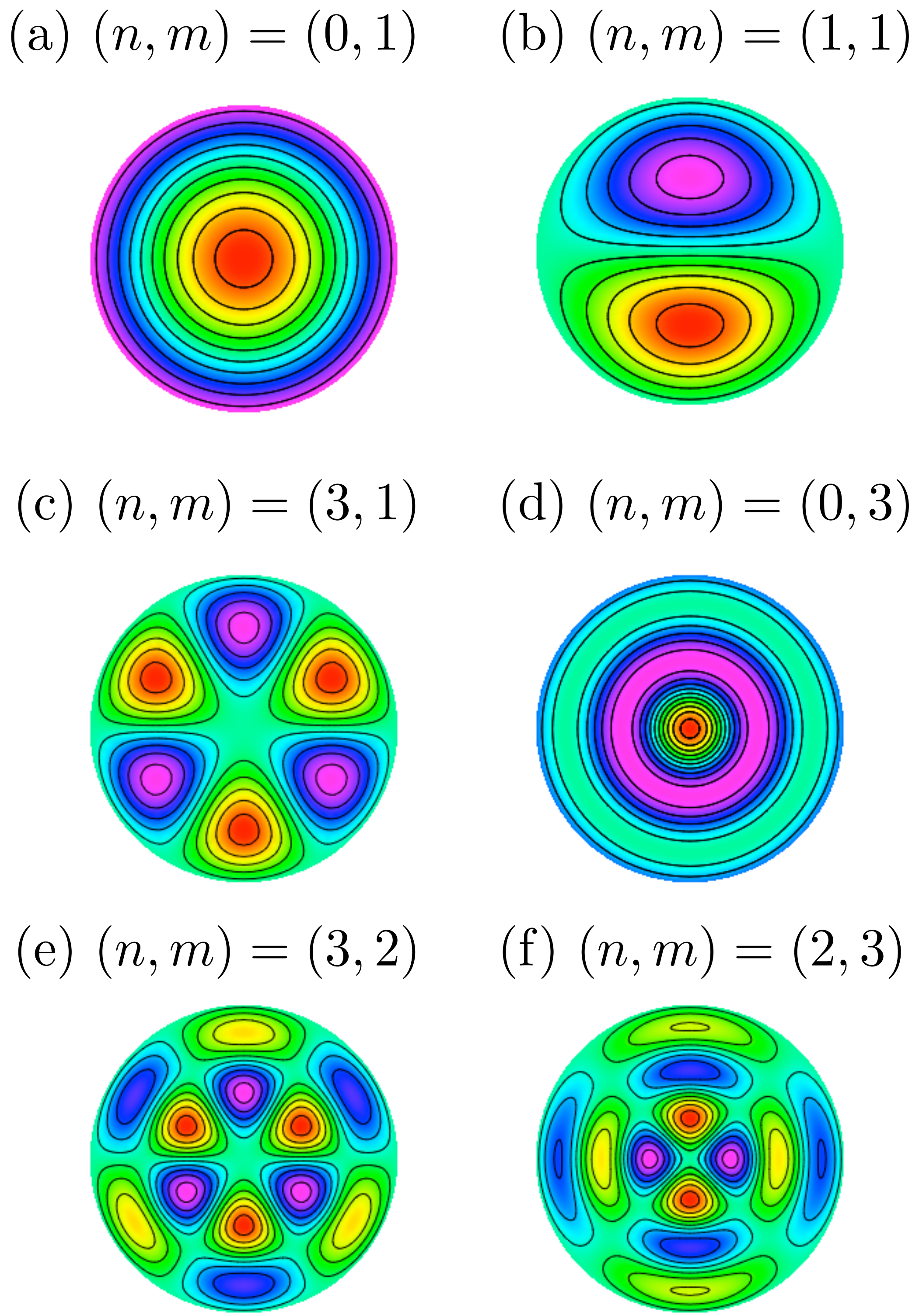


Figure 17.2: Six eigenmodes for the oscillation of a circular membrane. The displacement is proportional to $J_n\left(\alpha_{nm}\frac{r}{a}\right)\cos(n\theta)$ where the wavenumbers (n,m) are indicated on each figure.