One

Complex Arithmetic

CHAPTER OUTLINE

- Complex numbers, complex conjugates.
- Polar representation and Euler's formula.
- Regions of the complex plane.
- Roots of complex numbers.

1.1 Complex Numbers

You remember that the imaginary number, $i = \sqrt{-1}$, is a useful tool when evaluating expressions that contain negative numbers under a radical symbol. From previous classes, we know that

$$\begin{aligned} i &= \sqrt{-1} \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \\ &\vdots \end{aligned}$$

Also remember that $\mathbb{C}$ represents the set of all complex numbers and can be notated as

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\} \tag{1.1}$$

There is a *bijection* between a pair of real numbers (a, b) (which can be thought of as points in the plane $\mathbb{R}^2$) and complex numbers. We can recover the Cartesian representation of a complex numbers by identifying the real and imaginary part,

$$\text{Re}[a+bi] = a \qquad \text{Im}[a+bi] = b$$

which yields the point (a, b) in the plane $\mathbb{R}^2$.

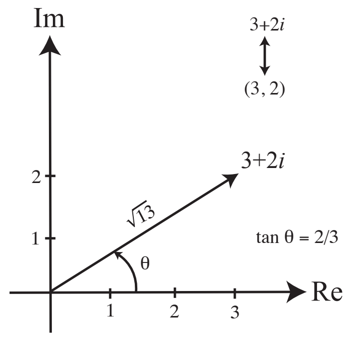

Figure 1.1: The Cartesian, complex and polar representation of $z = 3 + 2i$.

1.2 Complex Conjugate

Definition 1.2. The *complex conjugate* of a complex number $z = a + bi$ is defined as $\bar{z} = a - bi$.

Exercise 1.1. Show that graphically, the complex conjugate of a number z is its reflection in the real axis.

We can also use the complex conjugate to compute the distance of $z = x + yi$ from the origin. Note that

$$\begin{aligned} z\bar{z} &= (x+yi)(x-yi) \\ &= x^2 + xyi - xyi - y^2 i^2 \\ &= x^2 + y^2 \\ &\equiv |z|^2 \end{aligned}$$

This allows us to define the *magnitude* of a complex number.

Definition 1.3. The *magnitude* or *length* of $z = x + iy$ is defined as

$$|z| \equiv \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

Note that it is the distance of the point (x, y) from the origin in the plane.

1.3 Polar Representation

Complex numbers can also be represented using polar coordinates. If the point (x, y) is a distance r and an angle θ from the polar axis, then we know that

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

We write the complex number $z = x + iy$ as

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) \equiv re^{i\theta}.$$

That is we define

$$e^{i\theta} \equiv \cos\theta + i\sin\theta$$

which is commonly known as Euler's formula

Just as with polar coordinates, we can extract the magnitude r and the angle θ, sometimes called the *argument*, associated with a complex number $z = x + iy$.

$$r = |z| = |x + iy| = r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}\ .$$

Here we write $\theta = arg(z)$ and one must be careful to choose θ in the proper quadrant of the complex plane; θ is also arbitrary up to integer multiples of 2π, an issue we will address in detail in this and later lectures.

Example 1.1. Write $-3 + 3i$ in polar coordinates.
Solution: We see that $r = \sqrt{(-3)^2 + 3^2} = 3\sqrt{2}$ and $\tan\theta = 3/(-3) = -1$.

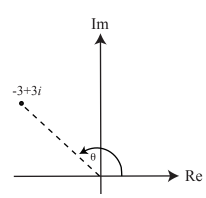

Figure 1.2: The point $z = -3 + 3i$.

Since z lies in the second quadrant, we choose that $\theta = 3\pi/4$ and

$$\boxed{-3 + 3i = 3\sqrt{2}e^{i\frac{3\pi}{4}}}$$

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So what happens geometrically when you multiply two complex numbers together? We compute

$$\begin{aligned} z_1 &= r_1 e^{i\theta_1} \\ z_2 &= r_2 e^{i\theta_2} \\ z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1+\theta_2)} \end{aligned}$$

where we leave it as an exercise to show from Euler's formula that

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

We see that the magnitude of the product is the product of the magnitudes and the argument of the product is the sum of the arguments.

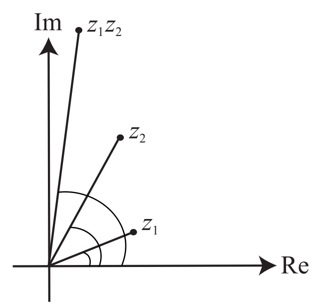

Figure 1.3: Multiplication of complex numbers

Example 1.2. Compute $(-3+3i)^{40}$.

Solution: We could compute this by multiplying out the product

$$(-3+3i)^{40} = (-3+3i)(-3+3i)\cdots(-3+3i)$$

which is a very tedious way to find the product. However, we can use the polar form; remember $-3+3i = 3\sqrt{2}e^{\frac{i3\pi}{4}}$, so

$$\begin{aligned} (-3+3i)^{40} &= \left(3\sqrt{2}e^{\frac{i3\pi}{4}}\right)^{40} \\ &= \left(3\sqrt{2}\right)^{40} \left(e^{\frac{i3\pi}{4}}\right)^{40} \\ &= 3^{40} 2^{20} e^{i30\pi} \\ &= 3^{40} 2^{20}. \end{aligned}$$

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1.4 Regions of the Complex Plane

There are many different regions of the complex plane. Three important ones are

- The *upper-half plane* (UHP), $Im[z] \geq 0$
- The *right-half plane* (RHP), $Re[z] \geq 0$
- The *unit disc*, $|z| \leq 1$ centered at the origin

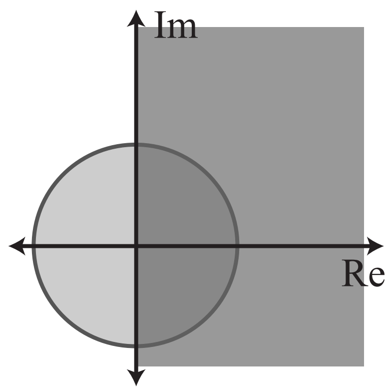

Figure 1.4: The unit disc and right-half plane regions.

We can also describe curves in the complex plane. Two examples:

Circle: Let's construct a circle centered at z_0 or radius a:

(a) A circle is the set of points z a fixed distance, a, from the point z_0 in the complex plane

$$|z - z_0| = a \quad (non - parametric)$$

(b) A circle is the set of points z that are displaced by a distance a at an angle θ in the complex plane

$$z = z_0 + ae^{i\theta} \quad (parametric)$$

where θ is a parameter that varies from 0 to 2π for one rotation around the circle.

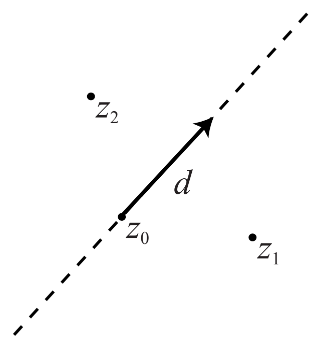

Figure 1.5: A line in the complex plane.

Line: We can construct a line in several ways.

(a) The set of points z equidistant from z_1 and z_2 - that is the perpendicular bisector.

$$|z - z_1| = |z - z_2| \quad (non - parametric)$$

(b) The set of points z displaced from z_0 be a constant multiple, r of the complex number d

$$z = z_0 + rd \quad (parametric)$$

where $r \in \mathbb{R}$ is the parameter.

1.5 ROOTS OF COMPLEX NUMBERS

A classic mathematical brain teaser asks if $i = \sqrt{-1}$, what is $\sqrt{i}$? The well-known science fiction writer, Isaac Asimov, claimed that this was a hypercomplex number. Fortunately, we know better! Computing roots of complex numbers is easy if we use the polar representation. Using the polar representation $i = e^{i\pi/2}$, we see

$$\begin{aligned}\sqrt{i} &= (e^{i\pi/2})^{1/2} \\ &= e^{i\pi/4} \\ &= \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\end{aligned}$$

This is one answer, but in fact there are two distinct solutions. To see this, one must first remember that we can write $1 = e^{2\pi in}$ for any integer n. So

$$\begin{aligned}i &= e^{i\pi/2+2\pi in} \\ \sqrt{i} &= e^{i\pi/4+\pi in}\end{aligned}$$

Remember that $e^{2\pi i} = 1$, we see this yields two distinct solutions for $n = 0, 1$, that is $\{e^{\pi/4},\ e^{5i\pi/4}\}$ where

$$\begin{aligned}e^{5i\pi/4} &= \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) \\ &= -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\end{aligned}$$

and the remaining values of n just alternate between these two values. Therefore, any nonzero complex number has two distinct square roots.

Example 1.3. Find all solutions to $z^3 + 8 = 0$.

Solution: We rewrite the problem in the form

$$z^3 = -8$$

and the first step is to write -8 as a complex number in polar coordinates. Remember $-1 = e^{i\pi}$ or, more generally, $-1 = e^{i(\pi+2\pi n)}$ for any integer n. So,

$$-8 = 8 \cdot e^{i(\pi+2\pi n)}$$

for any integer n. The equation can now be written as

$$z^3 = -8 = 8 \cdot e^{i(\pi+2\pi n)}$$

so

$$\begin{aligned} z = (-8)^{1/3} &= (8)^{1/3} e^{i(\pi/3+2\pi n/3)} \\ &= \quad (8)^{1/3} e^{i\pi/3}, \quad (8)^{1/3} e^{i3\pi/3}, \quad (8)^{1/3} e^{i5\pi/3} \cdots \\ &= 2 \cdot \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), \quad -2, \quad 2 \cdot \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \cdots \\ &= 1 + i\sqrt{3}, \quad -2, \quad 1 - i\sqrt{3} \cdots \end{aligned}$$

You might ask yourself the question, what happens for other values of n?

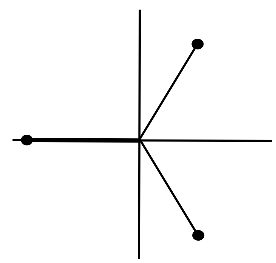

Figure 1.6: The cube roots of -8. Note that the roots lie on a circle of radius 2 at angles $\theta = \pi/3 + 2\pi n/3$.

Do you think we found all the solutions? ■

Example 1.4. What is the value of i^i ?
Solution: Remember that

$$\begin{aligned} i &= e^{i\pi/2} \quad \text{so} \\ i^i &= \left(e^{i\pi/2}\right)^i = e^{i^2\pi/2} = e^{-\pi/2} \\ &\approx 0.20788\ldots \end{aligned}$$

but also

$$\begin{aligned} i &= e^{i(\pi/2+2n\pi)} \qquad n \in \mathbb{Z} \\ i^i &= e^{-(\pi/2+2n\pi)} \end{aligned}$$

So there are infinitely many values (all real) of i^i. Weird, huh? ■