Ten

Convolutions and Delta Functions

CHAPTER OUTLINE

- Convolution Theorem
- Borel's Theorem
- Delta Functions
- Transfer Functions and Green's functions

10.1 CONVOLUTION THEOREM

Suppose we wish to solve a differential equation of the form

$$y' + y = f(t) \qquad y(0) = 0 \qquad (f(t) \text{ is given})$$

If we apply a Laplace transform to the equation, we have

$$(s+1)\mathbb{Y}(s) = F(s)$$

where

$$\mathcal{L}\{y(t)\} = \mathbb{Y}(s), \qquad \mathcal{L}\{f(t)\} = F(s).$$

Solving for $\mathbb{Y}(s)$ we see that

$$\mathbb{Y}(s) = \underbrace{\frac{1}{s+1}}_{\text{transfer function}=G(s)} F(s)$$

or

$$y(s) = \mathcal{L}^{-1}\{G(s)F(s)\}$$

where $G(s)$ is the *transfer function* for this ODE; here $G(s) = \dfrac{1}{s+1}$.

It would be very useful to have a formula for the inverse transform of the product of two functions. In fact, such a formula exists and is usually known as Borel's Theorem.

Definition 10.1. Define the *convolution* of two functions, $f(t)$ and $g(t)$ defined for $t \geq 0$ as

$$f(t) * g(t) \equiv \int_0^t f(t-x)g(x)dx$$

Example 10.1. Compute the convolution $t * e^{-t}$.

Solution: We just use the definition of the convolution,

$$\begin{aligned} t * e^{-t} &= \int_0^t (t-x)e^{-x}dx = \int_0^t te^{-x} - xe^{-x}dx \\ &= (-te^{-x} + xe^{-x} + e^{-x})|_0^t \\ &= -te^{-t} + t + te^{-t} - 0 + e^{-t} - 1 \\ &= e^{-t} + t - 1 \end{aligned}$$

■

We now make an observation; note that

$$\mathcal{L}\{t\} = \frac{1}{s^2}\,, \qquad \mathcal{L}\{e^{-t}\} = \frac{1}{(s+1)}$$

and

$$\begin{aligned} \mathcal{L}\{t * e^{-t}\} &= \mathcal{L}\{e^{-t} + t - 1\} \\ &= \frac{1}{s^2} + \frac{1}{s+1} - \frac{1}{s} \\ &= \frac{1}{s^2}\frac{1}{(s+1)} \\ &= \mathcal{L}\{t\} \cdot \mathcal{L}\{e^{-t}\}\,. \end{aligned}$$

This is an example of:

Theorem 10.1 (Borel's Theorem)**.** *Suppose that*

$$w(t) = u(t) * v(t)$$

then

$$W(s) = U(s)V(s)$$

where

$$\mathcal{L}\{u(t)\} = U(s), \quad \mathcal{L}\{v(t)\} = V(s), \quad \mathcal{L}\{w(t)\} = W(s)\ .$$

In words: *The Laplace transform of a convolution is the product of the Laplace transforms*. We'll prove this in the next section, but for now let us use the theorem to derive some results.

First, back to our original example; we found that if we Laplace transformed the ODE

$$y' + y = f(t) \qquad y(0) = 0$$

that

$$\mathbb{Y}(s) = F(s) \cdot \frac{1}{s+1}\ ,$$

but now, we can evaluate the inverse transform.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{F(s) \cdot \frac{1}{s+1}\right\} \\ &= f(t) * e^{-t} \\ &= \int_0^t f(x)e^{-(t-x)}dx \end{aligned}$$

So

$$y(t) = e^{-t}\int_0^t e^x f(x)\ dx$$

which is exactly the result one can find using the method of integrating factors.

Here is another example:

Example 10.2. Show $f(t) * g(t) = g(t) * f(t)$.

Solution 1: Use the definition of the convolution.

$$\begin{aligned} f(t) * g(t) &= \int_0^t f(t-x)g(x)dx \qquad u = t - x \quad du = -dx \\ &= \int_{u=t}^{u=0} f(u)g(t-u)(-du) = \int_0^t f(u)g(t-u)du \\ &= g(t) * f(t) \end{aligned}$$

Solution 2: Use Borel's theorem.

$$\begin{aligned} f(t) * g(t) &= \mathcal{L}^{-1}\{F(s)G(s)\} \\ &= \mathcal{L}^{-1}\{G(s)F(s)\} \\ &= g(t) * f(t) \end{aligned}$$

which on the surface seems rather shallow, but since a convolution transforms to a product of Laplace transforms, and multiplication is commutative, $f(t) * g(t)$ must equal $g(t) * f(t)$ if Borel's theorem is true.

■

10.2 A proof of Borel's Theorem

We will now prove:

Theorem 10.2 (Borel's Theorem)**.** *Suppose that*

$$w(t) = u(t) * v(t)$$

then

$$W(s) = U(s)V(s)$$

where

$$\mathcal{L}\{u(t)\} = U(s), \quad \mathcal{L}\{v(t)\} = V(s), \quad \mathcal{L}\{w(t)\} = W(s)\ .$$

Proof: Remember double integrals,

$$\begin{aligned} W(s) = \mathcal{L}\{w(t)\} &= \int_0^\infty e^{-st} u(t) * v(t) dt \\ &= \int_0^\infty e^{-st} \int_0^t u(x)v(t-x)dx\ dt \end{aligned}$$

Exchange the order of integration

$$\begin{aligned} &= \int_0^\infty \int_x^\infty e^{-st} u(x)v(t-x)dt\ dx \\ &= \int_0^\infty u(x) \int_x^\infty e^{-st} v(t-x)dt\ dx \end{aligned}$$

Let $t' = t - x,\ dt' = dt$

$$\begin{aligned} &= \int_0^\infty u(x) \int_0^\infty e^{-s(t'+x)} v(t')dt'\ dx \\ &= \int_0^\infty u(x) e^{-xs} \left[\int_0^\infty e^{-st'} v(t')dt'\right] dx \\ &= U(s)V(s) \end{aligned}$$

which concludes the proof. The real power of this method is for second-order ODEs.

Example 10.3. Solve the following ODE using Laplace transforms and convolutions.

$$y'' + 3y' + 2y = f(t) \qquad y(0) = y'(0) = 0$$

and find the solution explicitly when $f(t) = 1$. ■

Solution: As usual, we Laplace transform and solve for $\mathbb{Y}(s)$

$$(s^2 + 3s + 2)\mathbb{Y}(s) = F(s)$$
$$\mathbb{Y}(s) = \underbrace{\frac{1}{s^2 + 3s + 2}}_{\text{transfer function}} F(s)$$

Now, we need to find the inverse transform of the transfer function; using partial fractions

$$\frac{1}{s^2 + 3s + 2} = \frac{1}{(s+2)(s+1)} = \frac{1}{(s+1)} - \frac{1}{s+2}$$

so

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 3s + 2}\right\} = e^{-t} - e^{-2t} = g(t)$$

and we can write the solution as

$$y(t) = f(t) * g(t) = \int_0^t (e^{-x} - e^{-2x}) f(t - x) dt$$

which we could have found using Variation of Parameters.

We can solve the above convolution integral when $f(t) = 1$ to yield the explicit solution in this case,

$$\begin{aligned} y(t) &= \int_0^t e^{-t} - e^{-2t} \, dt \,, \\ &= -e^{-t} + \left.\frac{e^{-2t}}{2}\right|_0^t \,, \\ &= -e^{-t} + e^{-2t} + 1 - \frac{1}{2} \,. \end{aligned}$$

So

$$y(t) = e^{-2t} - e^{-t} + \frac{1}{2}$$

is the explicit solution.

10.3 An Introduction to the δ-function.

When modelling physical systems it is useful to be able to describe an *impulse*; a nearly instantaneous transfer of a finite amount of momentum. Examples of where this may occur is for a collision of two objects (a mallet striking a croquet ball), or the swallowing of a pill in which a certain amount of medicine is introduced into your body in a moment of time. The idea goes back to the nineteenth century and associated with names like Kirchoff (in the context of electrical circuits), however for physicists it is most strongly associated with Dirac and quantum mechanics.

10.3.1 How do we define a δ-function?

A δ-function models an impulsive forcing; basically the addition of a finite amount of energy to a system in an infinitesimal amount of time. It is defined as the limit of a sequence of functions. A set of functions, $\delta_\epsilon(t)$ is called a δ-sequence if:

(I) Positivity: The function $\delta_\epsilon(t) \geq 0$ for all t. Sometimes this condition is relaxed.

(II) Unit Mass:

$$\int_{-\infty}^{\infty} \delta_\epsilon(t)\, dt = 1.$$

(III) Vanishing Support: As ϵ tends to zero, the function becomes narrower and localized:

$$\lim_{\epsilon\to 0} \delta_\epsilon(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0. \end{cases}$$

Note the support of a function is the set on which the function is non-zero.

The limit of this sequence as $\epsilon \to 0$ is used to define the δ-function. We can define one such sequence using the Heaviside function:

$$\begin{aligned} \delta_\epsilon(t-T) &= \begin{cases} \frac{1}{\epsilon} & T - \frac{\epsilon}{2} \leq t \leq T + \frac{\epsilon}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{1}{\epsilon}\left[H\left(t - \left(T - \frac{\epsilon}{2}\right)\right) - H\left(t - \left(T + \frac{\epsilon}{2}\right)\right)\right] \end{aligned}$$

and now $\lim_{\epsilon\to 0}\delta_\epsilon(t-T)=\delta(t-T)$.

Note the following properties of the δ-function:

(i) Vanishing Support:

$$\delta(t-T)=\begin{cases}0 & t\neq T\\ \infty & t=T\end{cases}$$

(ii) Unit Mass: If we integrate over an interval containing the δ-function, we get its mass, namely unity,

$$\int_a^b \delta(t-T)\,dt=\begin{cases}1 & a<t<b\,,\\ 0 & t<a \text{ or } b<t\,.\end{cases}$$

where we have assumed that $a<b$. If the δ-function falls on the boundary of the integral of integration, traditionally one sets the integral to $\frac{1}{2}$, but this can be the source of confusion sometimes.

(iii) Relationship to the Heaviside Function: From the unit mass property we see that

$$\int_0^t \delta(t'-T)\,dt'=\begin{cases}1 & t>T\\ 0 & t<T\end{cases}=H(t-T)$$

So we write

$$\delta(t-T)=\frac{dH(t-T)}{dt}$$

(iv) Sampling Property: The δ-function samples a function at a point when integrated against it,

$$\int_a^b \delta(t-T)f(t)\,dt=\begin{cases}f(T) & a<t<b\,,\\ 0 & t<a \text{ or } b<t\,.\end{cases}$$

To see this from the limit definition, for our δ-sequence written in terms of the Heaviside function, note that

$$\begin{aligned}\lim_{\epsilon\to 0}\int \delta_\epsilon(t-T)f(t)\,dt &= \lim_{\epsilon\to 0}\int_{T-\epsilon/2}^{T+\epsilon/2}\frac{f(t)}{\epsilon}\,dt\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{T-\epsilon/2}^{T+\epsilon/2} f(t)\,dt\\ &= f(T)\,.\end{aligned}$$

The last step follows from the fact that this integral is measuring the "average value" of $f(t)$ on the interval $T - \frac{\epsilon}{2} \leq t \leq T + \frac{\epsilon}{2}$; as the interval gets smaller, this average value must approach $f(T)$ if the function is continuous.

10.3.2 Laplace Transform of $\delta(t - T)$

If we wish to model impulsive forcing of differential equations, it is useful to be able to compute the Laplace transform of the δ-function. If we assume $T > 0$ then, we can compute the Laplace transform in three different ways:

(i) Using the sampling property:

$$\mathcal{L}\{\delta(t-T)\} = \int_0^\infty e^{-st}\delta(t-T)dt = e^{-sT}$$

(ii) Using the fact that $\delta(t - T)$ is the derivative of the Heaviside function $H(t - T)$:

$$\begin{aligned}\mathcal{L}\{\delta(t-T)\} &= \mathcal{L}\left\{\frac{dH}{dt}(t-T)\right\} \\ &= s\mathcal{L}\{H(t-T)\} - H(0-T) \\ &= s\frac{1}{s}e^{-st} - 0 = e^{-sT}\end{aligned}$$

(iii) Using the limit definition of the δ-function:

$$\begin{aligned}\mathcal{L}\{\delta(t-T)\} &= \lim_{\epsilon\to 0}\mathcal{L}\{\delta_\epsilon(t)\} \\ &= \lim_{\epsilon\to 0}\mathcal{L}\left\{\frac{1}{\epsilon}H\left(t-\left(T-\frac{\epsilon}{2}\right)\right) - \frac{1}{\epsilon}H\left(t-\left(T+\frac{\epsilon}{2}\right)\right)\right\} \\ &= \lim_{\epsilon\to 0}\frac{e^{-s(T-\epsilon/2)} - e^{-s(T+\epsilon/2)}}{s\epsilon} \\ &= \frac{e^{-sT}}{s}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(e^{s\epsilon/2} - e^{-s\epsilon/2}\right) = e^{-sT}\ .\end{aligned}$$

Let us now do an example demonstrating the response to an impulsive forcing:

Example 10.4. A croquet ball is hit at a time $t = T$ with an impulsive force $F_0\,\delta(t-T)$. It feels friction from the grass proportional to its velocity. Compute its velocity as a function of time.

Solution: We first model the problem as a DE; remembering Newton's Law, we see that:

$$Mv' + \alpha v = F_0\,\delta(t-T) \qquad v(0) = 0\ ,$$

or

$$v' + \frac{\alpha}{M} v = \frac{F_0}{M}\,\delta(t-T)\ .$$

A Laplace transform yields

$$\left(s + \frac{\alpha}{M}\right)\mathbb{V} = \frac{F_0}{M}\,e^{-sT}\ .$$

Solving for the transformed velocity, we find

$$\mathbb{V}(s) = \frac{F_0}{M}\frac{1}{\left(s+\frac{\alpha}{M}\right)}\,e^{-sT}\ .$$

An inverse transfer now yields the velocity function,

$$\boxed{v(t) = H(t-T)\frac{F_0}{M}\,e^{-\frac{\alpha}{M}(t-T)}}$$

which is graphed below.

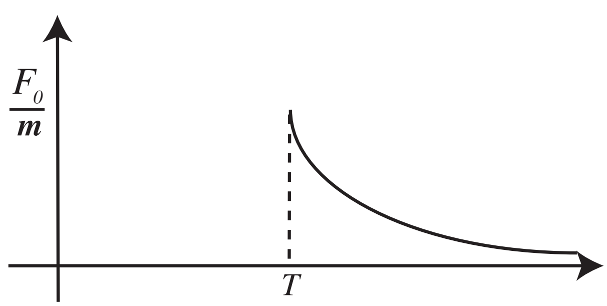

Figure 10.1: A graph of the velocity of the croquet ball; Note that the ball doesn't move before the mallet strike at $t = T$.

■

10.4 Transfer Functions and Green's Functions

We can use the response of a linear differential equation to an impulsive forcing to compute the response to an arbitrary forcing. Consider the Laplace transform of a δ-function applied just after $t = 0$.

$$\begin{aligned}\mathcal{L}\{\delta(t-0^+)\} &= \lim_{\mu\downarrow 0} \mathcal{L}\{\delta(t-\mu)\} \\ &= \lim_{\mu\downarrow 0} e^{-\mu s} \\ &= 1\end{aligned}$$

The *Green's function*, $g(t)$, is defined as the response of a linear differential equation to an impulse at $t = 0^+$. The Laplace transform of the Green's functions is the *transfer function*, $G(s)$.

Example 10.5. Compute the Green's function and transfer function associated with the differential operator

$$P[y] = y'' + 2y' + y.$$

Solution: The Green's function is defined as the response to an impulsive forcing at $t = 0^+$,

$$P[g] = g'' + 2g' + 2g = \delta(t - 0^+)$$

You may Laplace transform the equation to obtain,

$$\begin{aligned}(s^2 + 2s + 2)G(s) = 1 \\ G(s) &= \frac{1}{(s+1)^2}\end{aligned}$$

where $G(s)$ is the transfer function. An inverse transform now yields

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}\ .$$

■

In general the solution to the forced differential equation

$$P[y] = f(t) \qquad\qquad (Ex : P[y] = y'' + 2y' + y)$$

is the convolution of the Green's function, which satisfies

$$P[g] = \delta(t - 0^+)$$

and the forcing function

$$y = g(t) * f(t) = \int_0^t f(t-x)g(x)dx$$

which may be best illustrated by an example.

Example 10.6. Solve for $y(t)$ where $t > 0$

$$y'' + 2y' + y = f(t) \qquad y(0) = 0$$

Solution: First we will solve for $\mathbb{Y}(s) = \mathcal{L}\{y(t)\}$,

$$\mathcal{L}\{y'' + 2y' + y = f(t)\} \Rightarrow [s^2 + 2s + 1]\mathbb{Y}(s) = F(s)$$

$$\mathbb{Y}(s) = \frac{1}{s^2 + 2s + 1}F(s)$$

Now we will use the Green's function; since

$$g'' + 2g' + g = \delta(t - 0^+)$$

$$s^2G + 2sG + G = 1$$

$$G(s) = \frac{1}{s^2 + 2s + 1}$$

we see that

$$\mathbb{Y}(s) = \frac{1}{s^2 + 2s + 1}F(s) = G(s)F(s)$$

that is the Laplace transform of the solution is the product of the transfer function and the Laplace transform of the forcing function. The Green's function is the inverse Laplace transform of the transfer function

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t} \ .$$

So we can see that the solution to the DE, $y(t)$, is just the inverse transform of the product, which by Borel's Theorm is the convolution of the Green's function with the forcing function

$$y(t) = \mathcal{L}^{-1}\{G(s)F(s)\} = g(t) * f(t) = \int_0^t f(x)g(t-x)dx$$

■

These ideas will appear again and again as we look at linear systems with arbitrary forcing.