Eleven

Bromwich's Integral & Inverse Laplace Transforms

OUTLINE OF LECTURE

- The Laplace Transform: Properties and Preliminaries.
- Bromwich's Integrals for Inverse Laplace Transforms
- The Mellin Inversion Formula

11.1 THE LAPLACE TRANSFORM: PROPERTIES AND PRELIMINARIES.

In this Lecture we will discuss an integral transform for computing the inverse of the Laplace transform,

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

to do so, it is first useful to discuss some of the basic properties of the forward transform.

For this discussion, first let us extend the function being transformed to the real line. Let's define

$$\tilde{f}(t) = \begin{cases} f(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Note that this allows us to extend the Laplace transform to an integral over the real line also

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_{-\infty}^\infty e^{-st} \tilde{f}(t)\, dt$$

Second, let us restrict ourselves to functions that grow at worse exponentially. Suppose that

$$|f(t)| \leq Me^{\beta t} \qquad for t > 0$$

for some real constants M and β. then we see that if we think of s as a complex variable, $s = p + iq$ for p and q real that

$$\begin{aligned}
|F(s)| &\leq \int_0^\infty |e^{-st} f(t)| \, dt \\
&\leq M \int_0^\infty |e^{\beta t - (p+iq)t}| \, dt \\
&= M \int_0^\infty e^{(\beta - p)t} \, dt \\
&= \frac{M}{p - \beta} \text{ for } p > \beta.
\end{aligned}$$

that is the Laplace transform, $F(s)$, exists in the right half-plane where $Re\{s\} = p > \beta$. A little more work can show that $F(s)$ is an analytic function in this region also, and in particular has no singularities in this region.

11.2 Bronwich's Integrals for Inverse Laplace Transforms

To compute the inverse Laplace transform we need to define the Mellin Inversion Formula:

Definition 11.1. The Mellin Inversion Formula is defined in terms of the *Bronwich integral*,

$$f(t) = \mathcal{M}\{F(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) \, ds$$

where the contour of integration is the vertical line $Re\{s\} = c$, and c is chosen to lie to the right of the singularities of $F(s)$.

Some notes:

- We showed above that the Laplace transform of functions that grow no worse than exponentially are analytic in a right half-plane, so one can always choose a large enough value of c so that the contour lies to the right of any singularities if $F(s)$ is the Laplace transform of a function.

- This integral is (usually) evaluated by closing the contour and using the residue theorem.
- We will argue below that this integral is in fact the inverse Laplace transform; that is that $\mathcal{M}\{F(s)\} = \mathcal{L}^{-1}\{F(s)\}$.

We will give a series of examples illustrating how to use the inverse transform, and then give an outline of a proof of the formula in the next section.

Example 11.1. Compute the inverse Laplace transform of

$$F(s) = \frac{1}{s+1}$$

via Bromwich's integral. ■

Solution: We wish to compute

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s+1}\, ds$$

where we take $c > -1$ so as to be to the right of the singularity at $s = -1$. We want to use the residue theorem to evaluate these integrals. We will do this by closing the contour in a large semi-circle in the left half plane when $t > 0$ and showing that the contribution from the semi-circle vanishes as long as the magnitude of F is bounded on this semi-circle. Similarly, when $t < 0$, we can close the contour in the right half-plane and in this case we find $f(t) = 0$.

For $t > 0$, we close the contour in the left half plane and apply the residue theorem to yield

$$\begin{aligned} f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)\, ds \qquad &= \frac{1}{2\pi i}\oint e^{zt}F(z)\, dz \\ &= \frac{1}{2\pi i}\oint \frac{e^{zt}}{z+1}\, dz \\ &= \text{Res}[\frac{e^z t}{z+1}; z=-1] \\ &= e^{-t} \end{aligned}$$

so

$$f(t) = e^{-t} \qquad t > 0.$$

For $t < 0$, we will close the contour in the right half plane, but there are no singularities enclosed in the contour (remember, they are all to the left of the Bromwich coutour). Therefore $f(t)$ vanishes for $t < 0$. In summary,

$$f(t) = \begin{cases} e^{-t} & t > 0, \\ 0 & t < 0 \end{cases}$$

11.2.1 Closing the Bromwich contour

We wish to show that for $t > 0$ that we can close the Bromwich contour via a large semi-circle in the left half-plane that does not contribute to teh integral in the limit of large radius, Consider

$$J(t) = \int_C e^{zt} F(z)\, dz \qquad\qquad t > 0$$

and let C be the contour $z = Re^{i\theta} + c$ for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Define $I(z)$ as the integrand of the above integral; then

$$I(z) = e^{R\cos\theta t + ct} e^{iR\sin\theta t} F(Re^{i\theta} + c)$$

and

$$\begin{aligned} |I(z)| &= e^{R\cos\theta t + ct} |F(Re^{i\theta} + c)| \,. \\ &\leq Me^{ct} e^{Rt\cos\theta} \end{aligned}$$

where M is the maximum value of $F(z)$ on the semi-circle.
Now

$$\begin{aligned} |J(t)| &\leq Me^{ct} \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}\, d\theta \\ &= 2Me^{ct} \int_0^{\pi/2} e^{-Rt\sin\theta}\, d\theta \\ &= 2Me^{ct} \int_0^1 \frac{e^{-Rtu}}{\sqrt{1-u^2}}\, du \\ &\leq \frac{4Me^{ct}}{Rt} \end{aligned}$$

which vanishes as R tends to infinity; note that the bound on the integral in the last step depends on t being positive. Therefore, the integral vanishes on the semi-circular contour in the left half plane.

11.3 More Examples of Inverse Transforms

Example 11.2. Compute the inverse Laplace transform of

$$F(s) = \frac{1}{s^n}$$

via Bromwich's integral.

Solution: We wish to compute

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s^n}\, ds$$

where we take $c > 0$ so as to be to the right of the singularity at $s = 0$. We want to use the residue theorem to evaluate these integrals. We will do this by closing the contour in a large semi-circle in the left half plane when $t > 0$ and showing that the contribution from the semi-circle vanishes as long as the magnitude of F is bounded on this semi-circle. Similarly, when $t < 0$, we can close the contour in the right half-plane and in this case we find $f(t) = 0$.

For $t > 0$, we close the contour in the left half plane and apply the residue theorem to yield

$$\begin{aligned} f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)\, ds \qquad &= \frac{1}{2\pi i}\oint e^{zt}F(z)\, dz \\ = \frac{1}{2\pi i}\oint \frac{e^{zt}}{z^n}\, dz \\ = \text{Res}[\frac{e^z t}{z^n}; z = 0] \end{aligned}$$

Note

$$\frac{e^{zt}}{z^n} = \frac{1 + zt + (zt)^2/2! + \ldots}{z^n} = \frac{1}{z^n} + \frac{t}{z^{n-1}} + \frac{t^2/2!}{z^{n-2}} + \ldots \frac{t^{n-1}/(n-1)!}{z} \ldots$$

so

$$\text{Res}\left[\frac{e^{zt}}{z^n}; z = 0\right] = \frac{t^{n-1}}{(n-1)!}.$$

and

$$f(t) = \frac{t^{n-1}}{(n-1)!} \qquad t > 0.$$

For $t < 0$, we will close the contour in the right half plane, but there are no singularities enclosed in the contour (remember, they are all to the left of the Bromwich coutour). Therefore $f(t)$ vanishes for $t < 0$. In summary,

$$f(t) = \begin{cases} \frac{t^{n-1}}{(n-1)!} & t > 0, \\ 0 & t < 0. \end{cases}$$

■

Example 11.3. Compute the inverse Laplace transform of

$$F(s) = \frac{1}{s^4 - 1}$$

via Bromwich's integral.

Solution: We wish to compute

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s^4 - 1}\, ds$$

where we take $c > 1$ so as to be to the right of the singularities.

We want to use the residue theorem to evaluate these integrals. We will do this by closing the contour in a large semi-circle in the left half plane when $t > 0$ and showing that the contribution from the semi-circle vanishes as long as the magnitude of F is bounded on this semi-circle. Similarly, when $t < 0$, we can close the contour in the right half-plane and in this case we find $f(t) = 0$. If

$$F(s) = \frac{1}{s^4 - 1}$$

then there are simple poles at the four zeroes of the denominator,

$$s^4 - 1 = (s-1)(s+1)(s+i)(s-i)$$

that is where $s = -1, 1, i, -i$.

We can use residue calculus is used to evalulate this integral. The vertical line $\text{Re}(z) = c$ must be closed either to the right or to the left. We will close the curve to the right or left using a semicircle of radius R (and letting $R \to \infty$), depending on the values of t.

To determine which curve to use, we analyze the integrand. Writing $s = x + iy$, the magnitude of the integrand becomes

$$\left|\frac{e^{st}}{s^4-1}\right| = \left|\frac{e^{(x+iy)t}}{s^4-1}\right| = \left|\frac{e^{xt}e^{iyt}}{s^4-1}\right| = \frac{e^{xt}}{|s^4-1|}.$$

Therefore, we see that if $t > 0$, we want to use the use the left semicircle (in which $x < 0$), so that the integrand is exponential small as Re$(z) \to -\infty$ and the integral over the semicircle vanishes as $R \to \infty$. We see that

$$\begin{aligned}
f(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s^4-1}\, ds \\
&= \text{Res}\left[\frac{e^{st}}{s^4-1}; s=1\right] + \text{Res}\left[\frac{e^{st}}{s^4-1}; s=-1\right] \\
&\qquad + \text{Res}\left[\frac{e^{st}}{s^4-1}; s=i\right] + \text{Res}\left[\frac{e^{st}}{s^4-1}; s=-i\right] \\
&= \lim_{s\to 1}\,(s-1)\frac{e^{st}}{s^4-1} + \lim_{s\to -1}\,(s+1)\frac{e^{st}}{s^4-1} + \lim_{s\to i}\,(s-i)\frac{e^{st}}{s^4-1} + \lim_{s\to -i}\,(s+i)\frac{e^{st}}{s^4-1} \\
&= \left.\frac{e^{st}}{4s^3}\right|_{s=1} + \left.\frac{e^{st}}{4s^3}\right|_{s=-1} + \left.\frac{e^{st}}{4s^3}\right|_{s=i} + \left.\frac{e^{st}}{4s^3}\right|_{s=-i} \\
&= \frac{e^t}{4} - \frac{e^{-t}}{4} - \frac{e^{it}}{4i} + \frac{e^{-it}}{4i} \\
&= \frac{1}{2}\sinh t - \frac{1}{2}\sin t \qquad for \quad t > 0.
\end{aligned}$$

Likewise, if $t < 0$, then we want to the use the right semicircle. However, because the vertical line is chosen to the right of all singularities of $F(s)$, Cauchy's Theorem tells us that the integral is zero for $t < 0$.

To summarize

$$\boxed{f(t) = \begin{cases} \frac{1}{2}\sinh t - \frac{1}{2}\sin t & t > 0, \\ 0 & t \le 0. \end{cases}}$$

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Example 11.4. Compute the inverse Laplace transform of

$$F(s) = \frac{e^{-\pi s}}{s^2+1}$$

via Bromwich's integral.

Solution: We wish to compute

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}e^{-\pi s}}{s^2+1}\,ds$$

where we take $c > 0$ so as to be to the right of the singularities at $s = \pm i$.

For what values of t should we use the left semicircle? (Remember that since the vertical line is chosen to the right of all singularities of $F(s)$, Cauchy's Theorem tells us that the integral is zero when we use the right semicircle.) Writing $s = x + iy$, the magnitude of the integrand becomes

$$\left|\frac{e^{s(t-\pi)}}{s^2+1}\right| = \left|\frac{e^{(x+iy)(t-\pi)}}{s^2+1}\right| = \left|\frac{e^{x(t-\pi)}e^{iy(t-\pi)}}{s^2+1}\right| = \frac{e^{x(t-\pi)}}{|s^2+1|}.$$

Therefore, we see that if $t > \pi$, we want to use the use the left semicircle (in which $x < 0$), so that the integral over the semicircle vanishes as $R \to 0$. For $t > \pi$,

$$\begin{aligned}
f(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{s(t-\pi)}}{s^2+1}ds = \text{sum of all residues of } \frac{e^{s(t-\pi)}}{s^2+1}\\
&= \left.\frac{e^{s(t-\pi)}}{2s}\right|_{s=i} + \left.\frac{e^{s(t-\pi)}}{2s}\right|_{s=-i}\\
&= \frac{e^{i(t-\pi)}}{2i} - \frac{e^{-i(t-\pi)}}{2i} = \sin(t-\pi).
\end{aligned}$$

For $t < \pi$, we must choose the right semicircle, so $f(t) = 0$.

To summarize,

$$f(t) = \sin(t-\pi)H(t-\pi).$$

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11.4 Proof of the Mellin Inversion Formula

In this section, we will sketch a proof that the Laplace transform is indeed the inverse of the Mellin Inversion Formula.

Suppose $F(s)$ is known and we use the Mellin inversion formula to constrict a function, $\tilde{f}(t) = \mathcal{M}\{F(s)\}$ – we don't know yet that it is really $f(t)$. Then

$$\tilde{F}(z) = \int_0^\infty e^{-zt}\tilde{f}(t)dt \qquad |\tilde{f}(t)| < \epsilon e^{\gamma x}$$

clearly exist for Re$\{z\} > \gamma$. We want to show $\tilde{F}(z) \equiv F(z)$.
From the Mellin inversion formula, then

$$\tilde{F}(z) = \int_0^\infty e^{-zt} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s)\, dsdt$$

Let's assume we can interchange the orders of integration; then

$$\begin{aligned}\tilde{F}(z) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s) \int_0^\infty e^{-(z-s)t} dtds \\ &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F(s)}{z-s} ds, \qquad Re(z) > \gamma\end{aligned}$$

or

$$\tilde{F}(z) = -\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F(z)}{s-z} ds$$

If $F(s) \to 0$ as $s \to \infty$ for $Re(z) \geq \gamma$ then we can close the contour to the

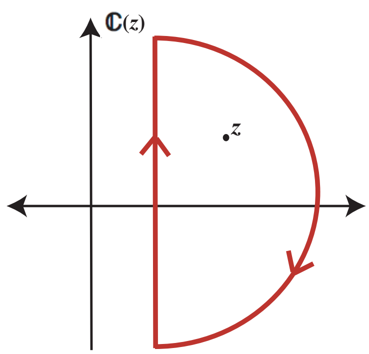

Figure 11.1: The contour for $\tilde{F}(z)$

right. We can also interpret the minus sign as saying that we should traverse the contour in a counter-clockwise sense. Consequently, we see that

$$\tilde{F}(z) = \frac{1}{2\pi i} \oint_C \frac{F(s)}{s-z} ds$$

or

$$\tilde{F}(z) \equiv F(z)$$

by the Cauchy integral formula.