Two

Complex Functions

CHAPTER OUTLINE

- Complex functions.
- Real and imaginary components of a complex function.
- Multi-valued functions.
- Exponential & logarithm functions.
- The derivative of a complex function.

2.1 COMPLEX FUNCTIONS

A complex function is a map from a complex number to another complex number. For example if we define

$$f(z) = z^3$$

then

$$f(1+i) = (1+i)^3 = -2 + 2i.$$

It is useful for visualizing a complex function to break it into real and imaginary components.

2.1.1 Real & Imaginary Components

One way to visualize $f(z)$ is to look at its real and imaginary parts.

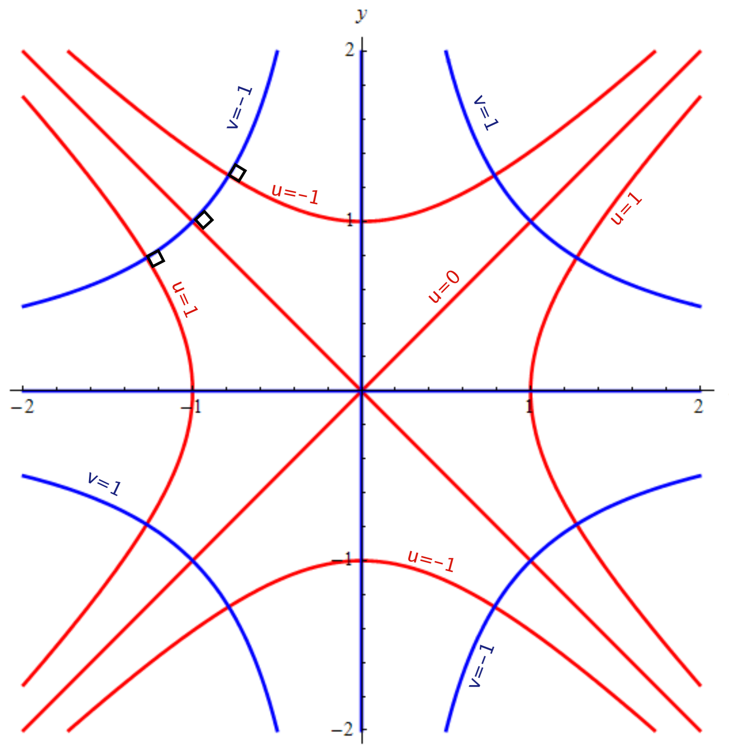


Figure 2.1: Real and imaginary parts of the function $f(z) = z^2$. We found that $f(z) = u(x, y) + iv(x, y)$ where $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. The level curves of $u(x, y)$ are drawn in red and the level curves of $v(x, y)$ in blue. The level curves are mutually orthogonal in this case.

Example 2.1. Consider the function $f(z) = z^2$. Show that if $z = x + iy$ that the real and imaginary parts of $f(z)$ are quadratic functions of x and y.
Solution: Let $z = x + iy$ and define $f(z) = u + iv$ where u and v are real functions. Then

$$\begin{aligned} f(z) &= u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy \\ \text{Re}[f] &= u(x, y) = x^2 - y^2 \\ \text{Im}[f] &= v(x, y) = 2xy \end{aligned}$$

In general, every complex function $f(z)$ can be written in terms of two real functions of two variables, $f(z) = u(x, y) + iv(x, y)$. ■

2.1.2 Multi-valued Functions

We've seen previously that when we take an nth root of a complex number we generally get n answers. This leads to the idea of a "multi-valued" function which is a bit of an oxymoron as, by definition, functions should return a single value. Let's look at an example and then show how to define a function as single-valued.

Example 2.2. Consider $f(z) = \sqrt{z}$. The function $f(z)$ is "multi-valued" and associates two values to (almost) every number. For example

$$f(i) = \sqrt{i} = \pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right).$$

How can we make this choice unique? ■

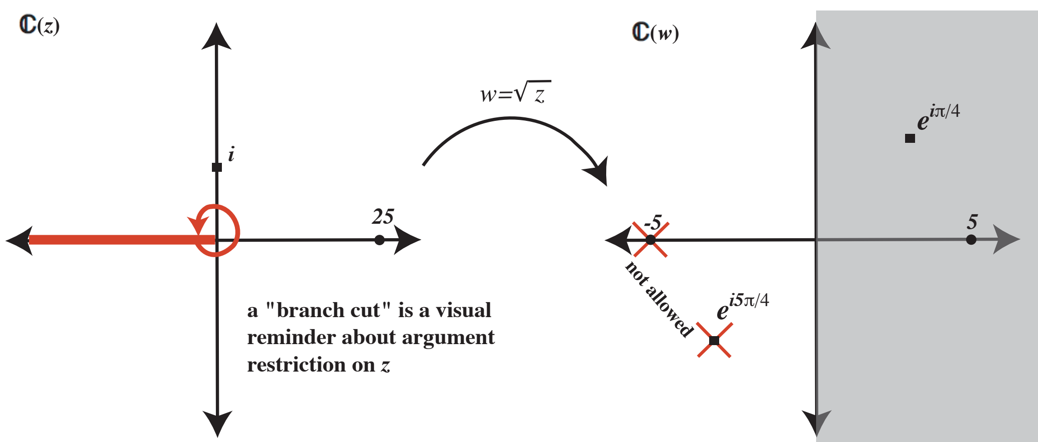


Figure 2.2: Note $f(z) = \sqrt{z}$ maps the complex plane to the right half plane.

To do analysis on these types of functions, we usually have to make a decision about which values we want. We do this by placing restrictions on the *argument* of $f(z)$ to specify a unique value. For example, we can define a unique value for the argument by specifying that it is in a given range. This *principle value* of the argument, $Arg(z)$ is defined as

$$Arg(z) = arg(z) \quad \text{where } arg(z) \in (-\pi, \pi]$$

If $\theta = Arg(z)$ then

$$w = \sqrt{z} = \left(\sqrt{re^{i\theta}}\right) = \left(\sqrt{r}e^{i\theta/2}\right)$$

where we see that $\frac{\theta}{2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Note that along the negative real axis the value of z jumps discontinuously. This is known as a *branch cut*.

2.1.3 Exponential & Logarithm Functions

Using Euler's Formula, we can define the exponential function. Consider

$$w = f(z) = e^z = \exp(z)$$

If $\quad z = x + iy, \quad$ then

$$\begin{aligned} e^z = e^{x+iy} &= e^x e^{iy} \\ &= e^x(\cos y + i \sin y) \end{aligned}$$

So

$$\begin{aligned} u(x, y) &= \text{Re}[f(z)] = e^x \cos y \\ v(x, y) &= \text{Im}[f(z)] = e^x \sin y \end{aligned}$$

Note the answer is defined for all z.

2.1.4 Logarithm Function

To define

$$w = \log z$$

Think of a logarithm as the inverse of the exponential function. If $w = \log z$ then $z = e^w$. Let $w = u + iv$ and $z = x + iy$, then

$$\log z = \log(x + iy) = w = u + iv \Leftrightarrow z = x + iy = e^w = e^{u+iv} = e^u e^{iv}$$

Now we can identify e^u as the magnitude of z and v as the argument of z, so

$$e^u = |x + iy| = \sqrt{x^2 + y^2} \Rightarrow u = \ln\sqrt{x^2 + y^2}$$

and

$$v = \arg(x + iy) = \arg(z)$$

from which we conclude

$$\boxed{\log(z) = \ln|z| + i\arg(z)}$$

Example 2.3. Compute $\log(10),\ \log(-10)$ and $\log(1+2i)$
Solution: Using the formula above, we see that

$$\log(10) = \ln|10| + i\arg(10) = \ln 10 + i(2n\pi), \quad n \in \mathbb{Z}$$

$$\log(-10) = \ln|-10| + i\arg(-10) = \ln 10 + i(\pi + 2n\pi), \quad n \in \mathbb{Z}$$

$$\log(1+2i) = \ln|1+2i| + i\arg(1+2i) = \ln\sqrt{5} + i\left[\tan^{-1}\left(\frac{2}{1}\right) + 2n\pi\right], \quad n \in \mathbb{Z}$$

■

We can define a unique value of $\log(z)$ by choosing the principal value for the $\arg$ function. Remember that $\text{Arg}(z)$, the principle value of $\arg(z)$, is defined as

$$\text{Arg}(z) = \arg(z) \qquad \text{where } z \in (-\pi, \pi]$$

then we define $\text{Log}(z)$, the principle value of the logarithm as

$$\text{Log}(z) = \ln|z| + i\,\text{Arg}(z)$$

Notice the difference between $\arg(z)$ and $\text{Arg}(z)$, and $\log(z)$ and $\text{Log}(z)$.

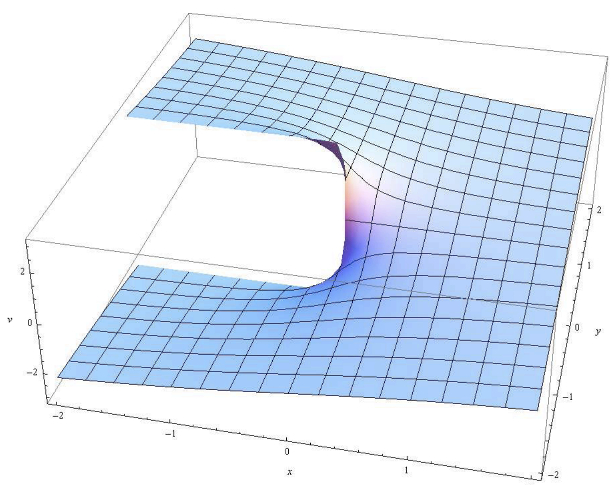


Figure 2.3: The imaginary part of the logarithm, $\text{Im}[\text{Log}(z)] = \text{Arg}(z)$. Note the branch cut along the negative real axis

2.2 The Derivative of a Complex Function

The derivative of a complex function is similar to that of a regular real-valued function. There are some important differences to understand however.

Definition 2.1. We define the derivative of a complex function, $f(z)$, at a point z_0 as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

when the limit exists and is independent of the path along which Δz tends to zero in the complex plane. If $f'(z_0)$ exists, then we say $f(z)$ is *differentiable* at $z = z_0$.

Example 2.4. Compute the derivative of $f(z) = z^2$ from the definition above.

Solution: We see that

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{2z\Delta z + \Delta z^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} 2z + \Delta z \\ &= 2z \end{aligned}$$

which is a relief, because it agrees with what we expected from calculus for a single real variable!! ■

We will spend most our time considering functions that are differentiable for almost all points in the plane. We have some nomenclature to describe these functions:

Definition 2.2. If $f(z)$ is differentiable at $z = z_0$ and in a small neighborhood about z_0, then f is *analytic at* $z = z_0$.

Definition 2.3. If $f(z)$ is analytic for all z, then $f(z)$ is **entire**.

In fact, functions like z^n for n a positive integer, $\sin z$, $\cos z$ and e^z are entire functions, and their derivatives are what you would expect:

$$\frac{d}{dz}(e^z) = e^z \qquad \frac{d}{dz}(\sin z) = \cos z$$

$$\frac{d}{dz}(z^n) = nz^{n-1} \qquad \frac{d}{dz}(\cos z) = -\sin z$$

$$(n = 1, 2, \ldots)$$

Also, for example

$$\frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2} \leftrightarrow \frac{1}{z} \text{ is analytic everywhere except at } z = 0$$

Finally, the usual rules for differentiable function apply.

$$[f+g]' = f' + g' \qquad \& \qquad [fg]' = fg' + f'g$$

etc ...

We end with an example of how things can go wrong:

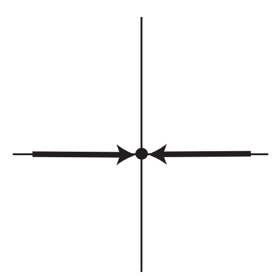
Example 2.5. A non-analytic function. Consider

$$f(z) = |z|^2$$

Compute $f'(z)$.

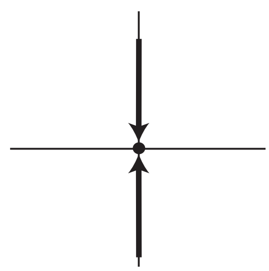
Solution: Remember, $f'(z) = \lim_{\Delta z \to 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z}$
Notice $\Delta z = 0$ means that Δz can approach 0 in any direction in the complex plane. The limit must be the same regardless of direction for the limit to exist.

(a) *Case 1:* approach 0 horizontally
Let $\Delta z = \Delta x \in \mathbb{R}$, $z = x + iy$



$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} &= \lim_{\Delta x \to 0} \frac{|x+iy+\Delta x|^2 - |x+iy|^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + y^2 - (x^2+y^2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\
&= 2x
\end{aligned}$$

(b) *Case 2:* approach 0 vertically
Let $\Delta z = i\Delta y, \;\; \Delta y \in \mathbb{R}, \;\; z = x+iy$



$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} &= \lim_{\Delta y \to 0} \frac{|x+iy+i\Delta y|^2 - |x+iy|^2}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{x^2 + (y+\Delta y)^2 - (x^2+y^2)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{y^2 + 2y\Delta y + (\Delta y)^2 - y^2}{i\Delta y} \\
&= \frac{2y}{i}
\end{aligned}$$

We just computed the derivative limit in two directions and got different answers (for $z \neq 0$), so $f(z)$ is not differentiable. It turns out that $f(z)$ is differentiable at $z = 0$ only, but showing that requires a little more work. We'll save that for another time.

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