Three

Analyticity and The Cauchy Riemann Equations

CHAPTER OUTLINE

- The Cauchy-Riemann Equations.
- Consequences of the Cauchy-Riemann (CR) equations.
- Determining region of analyticity.

3.1 CAUCHY-RIEMANN EQUATIONS

Let's say $f(z)$ is an analytic function,

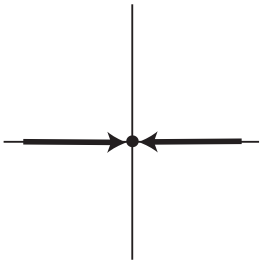
$$f(z) = u(x, y) + iv(x, y).$$

What can we say about the real and imaginary parts, $u(x, y)$ and $v(x, y)$? It turns out that they must satisfy two partial differential equations called the *Cauchy-Riemann Equations*. Let's derive them below.

3.1.1 Derivation of the Cauchy-Riemann Equations

If $f(z)$ is analytic at z_0, then f is differentiable at z_0 and the derivative limit exists and values match if we approach $\Delta z \to 0$ horizontally or vertically. Let's compute these two limits.

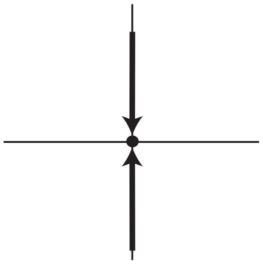
(a) *Case 1:* Let $\Delta z \to 0$ approach horizontally (parallel to the real axis). Let $z_0 = x + iy$ and $\Delta z = \Delta x$ where Δx is real, and let $\Delta x \to 0$.



Then, we compute

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \\
&= \lim_{\Delta x\to 0} \frac{(u(x+\Delta x,y)+iv(x+\Delta x,y))-(u(x,y)+iv(x,y))}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{u(x+\Delta x,y)-u(x,y)}{\Delta x} + i\frac{v(x+\Delta x,y)-v(x,y)}{\Delta x} \\
&= \frac{\partial u}{\partial x} + i\ \frac{\partial v}{\partial x}
\end{aligned}$$

(b) *Case 2:* Let $\Delta z \to 0$ approach vertically (parallel to the imaginary axis).
Let $z_0 = x+iy$ and $\Delta z = i\Delta y$ where Δy is real, and let $\Delta y \to 0$.



$$\begin{aligned}
f'(z_0) &= \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \\
&= \lim_{\Delta y\to 0} \frac{(u(x,y+\Delta y)+iv(x,y+\Delta y))-(u(x,y)+iv(x,y))}{i\Delta y} \\
&= \lim_{\Delta y\to 0} \frac{u(x,y+\Delta y)-u(x,y)}{i\Delta y} + i\frac{v(x,y+\Delta y)-v(x,y)}{i\Delta y} \\
&= -i\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}
\end{aligned}$$

If the limit exist, the answers in cases (a) and (b) must be the same,

$$f'(z_0) = \frac{\partial u}{\partial x} + i\ \frac{\partial v}{\partial x} = -i\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Equating the real and imaginary parts of each side, we get the **Cauchy-Riemann equations**,

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\ .}$$

It turns out that $f(z)$ is analytic in a region D, if and only if the Cauchy-Riemann equations hold in D. If $f(z)$ is analytic then the derivative can be written in terms of the partial derivatives of u and v with respect to x or y,

$$f'(z) = \frac{\partial u}{\partial x} + i\ \frac{\partial v}{\partial x} = -i\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

where the Cauchy-Riemann equation guarantees the expressions are equivalent.

3.1.2 Consequences of the Cauchy-Riemann (CR) equations

The Cauchy-Riemann equations imply a great deal of structure for the real and imaginary parts of an analytic function. We list some properties below:

(a) The real and imaginary parts of analytic functions are *harmonic*, that is they satisfy Laplace's equation. Remember from physics that the Laplace's equation for $\phi(x, y)$ is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$$

This equations relates to fluid dynamics, the wave equation, the diffusion equation, and the electrostatic potential to name a few places. We prove this using the CR equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Substituting into $\nabla^2 u$, we see that

$$\begin{aligned}\nabla^2 u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) \\ &= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0.\end{aligned}$$

where we have assumed that u and v have continuous second partial derivatives which guarantees that the mixed partials are equal.

(b) The CR equations show that level curves of $u(x, y)$ and $v(x, y)$ of an analytic function are mutually orthogonal. A normal vector to the u level curve is the gradient,

$$\nabla u = \langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \rangle.$$

Similarly, a normal vector to the v level curve is

$$\nabla v = \langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \rangle.$$

Taking the dot product of the two normal vectors yields

$$\begin{aligned}\nabla u \cdot \nabla v &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \\ &= \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \\ &= 0\end{aligned}$$

where we have used the CR equations to eliminate the partial derivatives of u. So the normals to the level curves are orthogonal and level curves are perpendicular.

(c) The real and imaginary parts of an analytic function are linked together by the CR equations; each determines the other up to an additive constant and one can "recover" the real part from the imaginary part and vica versa. This is best demonstrated by an example:

Exercise 3.1. For $f(z) = z^2$, the real and imaginary parts are

$$u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy$$

(a) Show that u and v satisfy the CR equations.

(b) Verify that the level curves of u and v are orthogonal.

(c) Suppose we only knew $u(x,y) = x^2 - y^2$, can we recover $f(z)$?

Solution:

(a) Evaluating the partial derivatives, we see that

$$u_x = 2x\ , \quad u_y = -2y\ , \qquad v_x = 2y\ , \quad v_y = 2x\ .$$

Clearly the Cauchy-Riemann equations, $u_x = v_y$ and $u_y = -v_x$, are satisfied.

(b) We take a slightly different tack from the proof above. Remember the gradient of a function is normal to its level curves. Consequently, the level curves are orthogonal if the gradients are perpendicular,

$$\nabla u \cdot \nabla v = u_x \cdot v_x + u_y \cdot v_y = (2x)(2y) + (-2y)(2x) = 0\ .$$

You can check that this is equivalent to the condition on the tangents above.

(c) If $u(x,y)$ is the real part of an analytic function, we can reconstruct $v(x,y)$, the real part, using the Cauchy-Riemann equations.

$$\begin{aligned} u_x &= 2x = v_y \implies v(x,y) = 2xy + \psi(x) \\ u_y &= -2y = -v_x = -2y - \psi'(x) \implies \psi'(x) = 0 \\ &\implies \psi(x) = C \\ &\implies v(x,y) = 2xy + C \end{aligned}$$

Knowing u and v, we can even reconstruct the complex function that they represent.

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) \\ &= \left(x^2 - y^2\right) + i\,(2xy + C) \\ &= (x + iy)^2 + iC \\ &= z^2 + iC \end{aligned}$$

where C is a real constant.

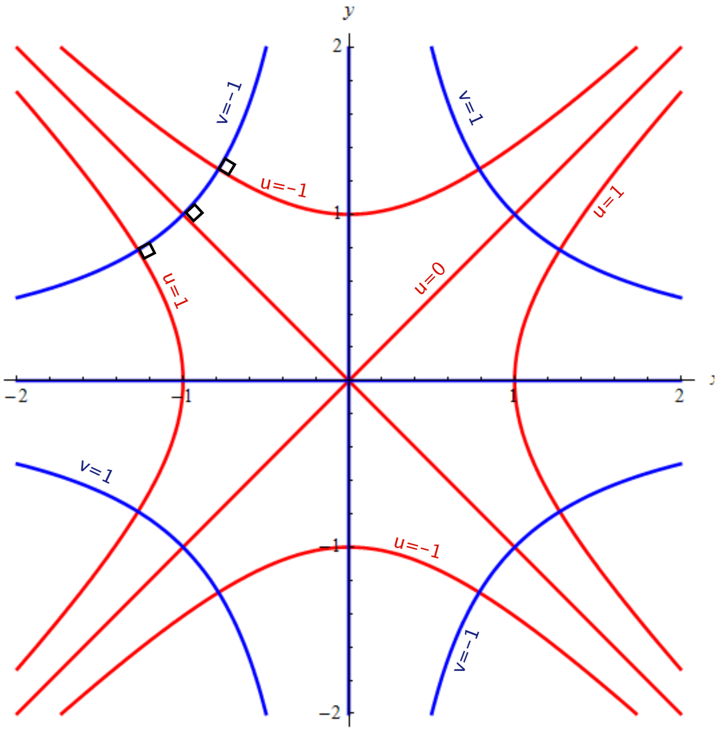


Figure 3.1: The level curves of $u(x,y)$ and $v(x,y)$ in the problem above. Note they are perpendicular

Exercise 3.2. Consider $u(x,y) = 3x^2y + Ay^3$. For what value of A is $u(x,y)$ the real part of an analytic function $f(z)$? For this value of A find $f(z)$.
Solution: If $u(x,y)$ is the real part of a harmonic function, then it is harmonic. That is

$$0 = u_{xx} + u_{yy} = (3x^2y + Ay^3)_{xx} + (3x^2y + Ay^3)_{yy} = 6y + 6Ay = 0$$

from which we determine that $\boxed{A = -1.}$
Now, from the Cauchy-Riemann equations, if $u(x,y) = 3x^2y - y^3$ then

$$\begin{aligned}
&u_x = 6xy = v_y \implies v(x,y) = 3xy^2 + \psi(x)\\
&u_y = 3x^2 - 3y^2 = -v_x = -3y^2 - \psi'(x) \implies \psi'(x) = -3x^2\\
&\quad \implies \psi(x) = -x^3 + C\\
&\quad \implies v(x,y) = 3xy^2 - x^3 + C
\end{aligned}$$

Knowing u and v, we can even reconstruct the complex function that they represent.

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) \\ &= \left(3x^2y - y^3\right) + i\left(3xy^2 - x^3 + C\right) \\ &= -i(x+iy)^3 + iC \\ &= -iz^3 + iC \end{aligned}$$

where C is a real constant.

3.1.3 Determining region of analyticity

One way to determining where a function is differentiable is to use the Cauchy-Riemann equations. If the CR equations are true and f is continuous, then f is differentiable. If this holds in a region D, then f is analytic in D. However, that is time consuming. However, one can build up a library of known analytic functions which allows one to determine the region of analyticity by inspection.

Guidelines for determining analyticity:

- Identify pieces of the function that whose regions of analyticity you know. (For example, polynomials, sine, cosine, and exponential functions are *entire*.)
- Remember that products and compositions of analytic functions are analytic also. So $f(z) = z^2 \sin z$ and $g(z) = \sin(z^2)$ are entire also.
- Identify problem points – division by zero, or discontinuities of the function, etc ...
- Watch out for branch cuts. (Multiple-valued functions, like roots and log, will require a branch cut to be single-valued.)
- Complex conjugate function and magnitude operation typically make the function not analytic everywhere, so $f(z) = |z|^2 = z\bar{z}$ and $g(z) = \sin(\bar{z})$ are not analytic anywhere.

Example 3.1. Determine where these functions are analytic.

(a) The function $a(z) = \sin z + \frac{1}{z-1}$.

Solution: By looking at each piece of this function, we can determine analyticity. Note $\sin z$ is entire (analytic everywhere), but $\frac{1}{z-1}$ is not defined at $z = 1 \Rightarrow$ not differentiable at $z = 1$.

$\boxed{\text{So } a(z) \text{ is analytic everywhere except at } z = 1.}$

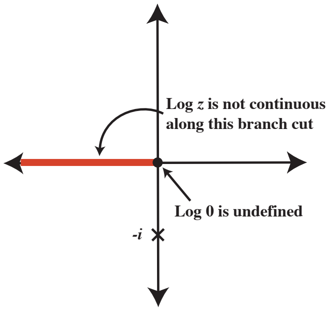


Figure 3.2: Determining where $d(z) = \sin\left(\dfrac{1}{z+i}\right) \text{Log}(z)$ is analytic.

(b) The function $b(z) = \dfrac{\sin\left(z^2\right)}{z^2+4}$.

Solution: The denominator of this fraction is zero at $z = \pm 2i$.

$\boxed{\text{So } b(z) \text{ is analytic everywhere except } z = \pm 2i.}$

(c) The function $c(z) = \csc z = \dfrac{1}{\sin z}$.

Solution: Note $c(z)$ is undefined where $\sin z = 0$, which is $z = n\pi,\ \ n \in \mathbb{Z}$.

$\boxed{\text{So } c(z) \text{ is analytic everywhere } z \neq n\pi,\ \ n \in \mathbb{Z}.}$

(d) The function $d(z) = \sin\left(\dfrac{1}{z+i}\right) \text{Log}(z)$.

Solution: Note $d(z)$ is not defined at $z = -i$ and $\text{Log}(z)$ has a branch cut along the negative real axis (see the figure above).

$\boxed{\text{So } d(z) \text{ is analytic everywhere except at } z = -i \text{ and along negative real axis.}}$

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