Four

Complex Integration

CHAPTER OUTLINE

- Complex integration via Parameterization.
- The fundamental integration theorems.
- The Cauchy's Integral Formula.

4.1 COMPLEX INTEGRATION VIA PARAMETERIZATION

You have seen line integrals in multivariable calculus and the physical concept of work, which is force integrated over a displacement, in mechanics. Integrating a function along a curve in the complex plane can be accomplished in a fashion analogous to a line integral, namely by parameterizing the contour of integration. Let us do a few examples.

Example 4.1. Compute the integral

$$I = \int |z|^2 \, dz.$$

along the curves:

(a) C: The quarter circle with radius 2, traversed from 2 to $2i$.

(b) D: The line segment from 2 to $2i$.

Solution: The basic idea here is that first you parametrize the curve for z as a function of the parameter, then you substitute in and calculate the integral. So:

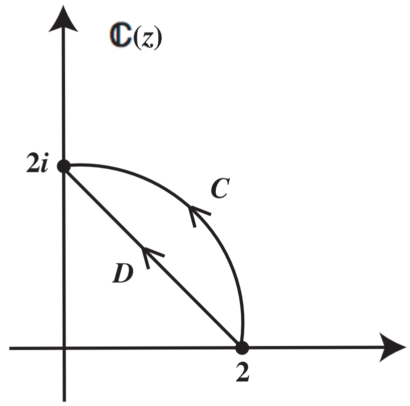


Figure 4.1: Two curves for integration in example 4.1.

(a) One parameterization of C is $z = 2e^{i\theta}$ where θ goes from 0 to $\frac{\pi}{2}$. Along C, $z = 2e^{i\theta}$ and on the curve parameterized by θ,

$$dz = \frac{dz}{d\theta}\, d\theta$$

so $dz = 2ie^{i\theta} d\theta$ for our particular example. Substituting into the integral, we see that

$$\begin{aligned}
\int_C |z|^2 \, dz &= \int_0^{\pi/2} |2e^{i\theta}|^2 \;\; 2e^{i\theta} \, i \, d\theta \\
&= \int_0^{\pi/2} 4 \cdot 2 \cdot e^{i\theta} \cdot i \, d\theta = [8e^{i\theta}]_0^{\pi/2} \\
&= 8e^{i\pi/2} - 8e^{i \cdot 0} \\
&= 8i - 8
\end{aligned}$$

(b) First we need to parameterize the line. It is useful to observe that a line between two points z_0 and z_1 in the complex plane can be parametrized as

$$z(s) = z_0 + (z_1 - z_0)s \qquad 0 \le s \le 1.$$

So a parameterization of D is $z = 2 + (2i - 2)s$ where s increases from 0 to 1. Along D, $dz = (2i - 2)ds$ and

$$\begin{aligned}
\int_D |z|^2 dz &= \int_0^1 |2 + (2i-2)s|^2 \, (2i-2)ds \\
&= \int_0^1 \left[(2-2s)^2 + (2s)^2\right](2i-2)ds \\
&= (2i-2)\int_0^1 [4 - 8s + 8s^2]ds \\
&= (2i-2)\left[4 - 4 + \frac{8}{3}\right] = \frac{16}{3}i - \frac{16}{3}
\end{aligned}$$

Note the answer is different for the two contours above ■

Example 4.2. Now compute the integral

$$I = \int z^2 \, dz$$

along the same two curves as in the example above.
Solution: We can use the same parametrizations as above:

(a) Along C, $z = 2e^{i\theta}$ and $dz = 2ie^{i\theta}d\theta$ so,

$$\begin{aligned}
\int_C z^2 \, dz &= \int_0^{\pi/2} (2e^{i\theta})^2 \;\, 2e^{i\theta} \, i \, d\theta \\
&= 8i \int_0^{\pi/2} e^{3i\theta} \, d\theta \\
&= \frac{8i}{3i}[e^{3i\theta}]_0^{\pi/2} \\
&= \frac{8}{3}\left[e^{3i\pi/2} - e^{i \cdot 0}\right] = \frac{8}{3}(-i-1) \\
&= -\frac{8}{3}(1+i)
\end{aligned}$$

(b) Along D, $\quad dz = (2i - 2)ds.$

$$
\begin{aligned}
\int_D z^2 dz &= \int_0^1 (2+(2i-2)s)^2 (2i-2)ds \\
&= (2i-2)\int_0^1 \left[(2-2s)+(2is)]^2\right] ds \\
&= (2i-2)\int_0^1 (2-2s)^2 + 2(2-2s)(2is) + (2is)^2 \, ds \\
&= (2i-2)\int_0^1 (4-8s) + 8i(s-s^2)ds \\
&= (2i-2)\left[(4s-4s^2) + 8i\left(\frac{s^2}{2}-\frac{s^3}{3}\right)\right]_0^1 \\
&= (2i-2)\left[(4-4) + 8i\left(\frac{1}{2}-\frac{1}{3}\right)\right] = (2i-2)(8i)\frac{1}{6} \\
&= -\frac{8}{3}(1+i)
\end{aligned}
$$

■

Note the when we integrate a non-analytic function, the answers are different but when we integrate an analytic function the answers are the same. In fact, this is an example of the *path independence* of the integrals of analytic functions. We'll learn more about this soon.

4.2 The Fundamental Integration Theorems

Let $f(z)$ be analytic in a simply-connected (basically, no holes) region of the complex plane. The following statements are true.

I. Cauchy-Goursat Integral Theorem:

Theorem 4.1. *For any closed, simple (doesn't cross itself) path C inside the region of analyticity,*

$$\oint_C f(z)dz = 0$$

So, for example

$$\oint_C e^{e^{\sin z}} dz = 0$$

for any closed path C. This theorem is often called *Cauchy's Theorem*.

II. Path Independence Property:

Theorem 4.2. *For any two paths C_1 and C_2 inside the region of analyticity with the same start and end points,*

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

that is the integral is independent of the path taken between the two points.

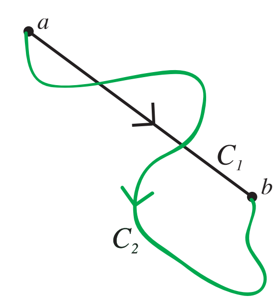


Figure 4.2: Two paths with the same start and end points.

III. Fundamental Theorem of Calculus for Analytic Functions:

Theorem 4.3. *If $f(z) = F'(z)$ then*

$$\int_a^b f(z)dz = F(b) - F(a)$$

where the integral is taken along any contour connecting a and b in the region of analyticity.

We will now outline the proof of these three theorems. We will begin by proving the Fundamental Theorem of Calculus.
Proof: We need to remember some facts about line integrals, in particular:

$$\int_{(x_0,y_0)}^{(x_1,y_1)} \nabla W \cdot d\vec{x} = \int W_x dx + W_y dy = \int dW = W|_{(x_1,y_1)} - W|_{(x_0,y_0)}$$

which is basically the Work-Energy theorem. where W is the *potential* then $-\nabla W$ is the associated force; the line integral is the *work* done along the curve which is just the change in the potential.

Also, remember the Cauchy-Riemann equations; if $F(z)$ is an analytic function and $f(z) = F'(z)$ is its derivative then

$$f(z) = F'(z) == U_x + i\,V_x = V_y - i\,U_y$$

Next we write $dz = dx + i\,dy$. Substituting into the contour integral yields

$$\begin{aligned}\int_a^b f(z)dz &= \int_a^b (U_x + i\,V_x)(dx + i\,dy) \\ &= \int_a^b U_x dx - V_x dy + i\int_a^b V_x dx + U_x dy \\ &= \int_a^b U_x dx + U_y dy + i\int_a^b V_x dx + V_y dx = \int_a^b d(U + i\,V) \\ &= U + i\,V|_{z=a}^{z=b} = F(b) - F(a).\end{aligned}$$

Where the answer is independent of the path of integration!![1]

This proves the Fundamental Theorem of Calculus for Analytic Functions and since we haven't specified the path of integration, it proves Path Independence property of these integrals also. To prove the Cauchy-Goursat Theorem, simply note that if the points a and b are the same, that is, the path C is closed, that $F(b) = F(a)$ and the integral vanishes.

Example 4.3. Compute the integral

$$I = \int_0^{i\pi} \cos z dz.$$

Since $\cos z$ is analytic everywhere, the path of integration does not matter. We can compute

$$\begin{aligned}\int_0^{i\pi} \cos z\; dz = \sin z|_0^{i\pi} = \sin i\pi - 0 &= \frac{e^{i(i\pi)} - e^{-i(i\pi)}}{2i} \\ &= i\frac{(e^{\pi} - e^{-\pi})}{2} = i\;\sinh\pi\;.\end{aligned}$$

■

[1]Some people find this confusing because we are using differentials ($dz = dx + i\,dy$); we could parameterize the path by writing $z(t) = x(t)i\,y(t)$ and then write $dz = \frac{dz}{dt} = \left[\frac{dx}{dt} + i\frac{dy}{dt}\right] dt$ and then the result follows from the chain rule, an exercise we leave for the reader.

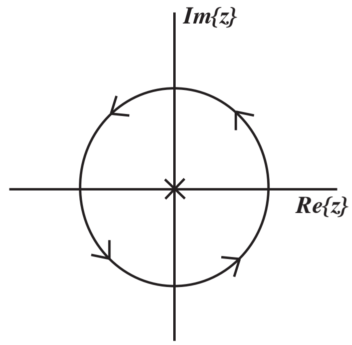


Figure 4.3: The contour of integration in example 4.4, namely the unit circle $|z| = 1$ traversed in a counter-clockwise sense.

Example 4.4. Evaluate the integral

$$\oint_C \frac{1}{z}\, dz,$$

where C is a unit circle traversed in a counter-clockwise sense.

Solution: We parameterize the unit circle

$$z = e^{i\theta} \quad 0 \le \theta \le 2\pi \qquad \Rightarrow \qquad dz = ie^{i\theta} dz$$

and substitute to obtain

$$\begin{aligned}\oint_C \frac{1}{z}\, dz &= \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}} d\theta \\ &= \int_0^{2\pi} i d\theta = 2\pi i.\end{aligned}$$

A good question to ask is why doesn't the integral vanish? The answer is because the singularity at the origin is in the interior of C. Somehow the integral *smells the singularity,* ■

This is an example of the *Cauchy Integral Formula* which allows us to "count" the poles in the interior of a contour via an integral. Cauchy's Theorem has a wide range of uses; one is that it allows us to consider integrals over complicated curves.

The next example is very important for understanding the integral of analytic functions around closed contours.

Example 4.5. Calculate $I = \oint_C (z-a)^n dz$ for any integer n, where C is a circle of radius ρ about a (counterclockwise).

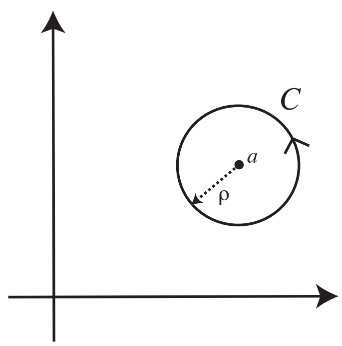


Figure 4.4: The contour for Example 4.5.

Solution: One way to evaluate the integral is to parameterize C by setting $z = a + \rho e^{i\theta}$ where θ increases from 0 to 2π. Then $dz = \rho i e^{i\theta} d\theta$ and we can reduce the problem to an integral in θ.

$$I = \oint_C (z-a)^n dz = \int_0^{2\pi} \left(a + \rho e^{i\theta} - a\right)^n \rho i e^{i\theta} d\theta = \int_0^{2\pi} \rho^{n+1} i e^{i(n+1)\theta} d\theta$$

This integral breaks into two cases:
If $n \neq -1$, then

$$\begin{aligned} I &= \left[\frac{\rho^{n+1} i e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} \\ &= \frac{\rho^{n+1}}{n+1} \left[\underbrace{e^{i(n+1)2\pi}}_{=1,\ \text{since}\, n \in \mathbb{Z}} - e^0\right] = 0 \end{aligned}$$

If $n = -1$, then

$$I = \int_0^{2\pi} \rho^0 i e^0 d\theta = [i\theta]_0^{2\pi} = 2\pi i$$

This yields our final answer:

$$\oint_C (z-a)^n dz = \begin{cases} 0, & \text{if } n \in \mathbb{Z},\ \ n \neq -1 \\ 2\pi i, & \text{if } n \in \mathbb{Z},\ \ n = -1 \end{cases}$$

■

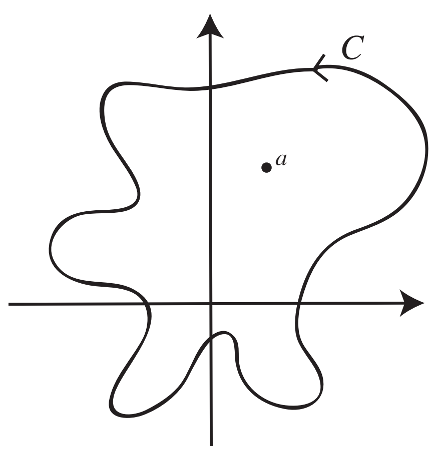
Note that if n is a positive integer, that $f(z) = (z-a)^n$ is analytic and the fact that the integral vanishes follows from the Cauchy-Goursat Theorem

4.3 Cauchy's Integral Formula

Let's revisit the warm-up problem; we will compute

$$\oint_C (z-a)^n dz, \quad n \in \mathbb{Z},$$

where now C is a simple (doesn't intersect itself) closed curve containing $z = a$, traversed counter-clockwise.



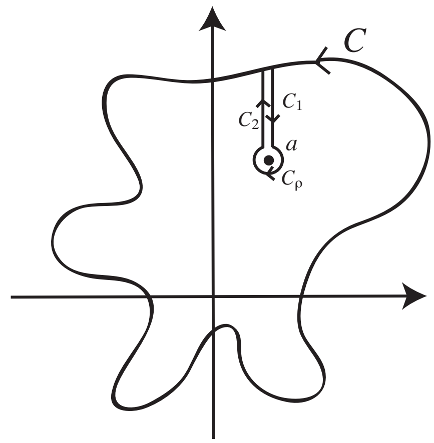
A simple closed curve, C, containing $z = a$.

We'll show that the answer is the same as in our warm-up problem.

Let's do some surgery on C. Consider

$$D = C \bigcup C_1 \bigcup C_\rho \bigcup C_2,$$

where C_ρ is the circle centered at $z = a$ of radius ρ traversed clockwise.



The interior of D no longer contains $z = a$ and by *Cauchy's Theorem:*

$$\oint_D (z-a)^n dz = 0$$

as the interior of D is simply-connected and $(z-a)^n$ is analytic inside D. Note also that

$$0 = \oint_D (z-a)^n dz = \int_C + \int_{C_1} + \int_{C_\rho} + \int_{C_2} (z-a)^n dz$$

Now let C_1 & C_2 get so close they lie on top of each other. Then,

$$C_1 \to -C_2 \Rightarrow \int_{C_1} (z-a)^n dz \approx -\int_{C_2} (z-a)^n dz.$$

Also in the limit as C_1 approached C_2, C_ρ approached a circle of radius ρ traversed *clockwise*. As such, the answer is the negative of the answer of the example computed at the beginning of the chapter,

$$\oint_{C_\rho} (z-a)^n dz = \begin{cases} 0, & \text{if } n \in \mathbb{Z},\ \ n \neq -1 \\ -2\pi i, & \text{if } n \in \mathbb{Z},\ \ n = -1 \end{cases}$$

So, as the integrals along C_1 and C_2 cancel, we see that

$$\oint_C (z-a)^n dz = -\oint_{C_\rho} (z-a)^n dz = \begin{cases} 0, & \text{if } n \in \mathbb{Z},\ \ n \neq -1 \\ 2\pi i, & \text{if } n \in \mathbb{Z},\ \ n = -1 \end{cases}$$

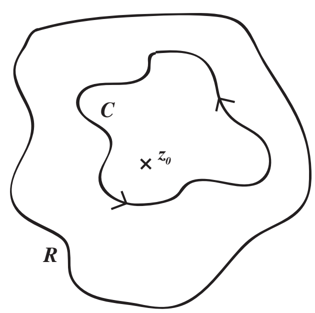
Consequently the integral around any closed contour C that encloses $z = a$ is the same. Cauchy's Theorem basically allows us to *deform* C into a circle of radius ρ, a much simpler curve for which we know the answer. Since C was quite arbitrary, one can see the power of Cauchy's Theorem.

Note: Cauchy's Theorem says that the integral around a simple closed curve of a function analytic in the curve's interior vanishes. The converse is not true. For instance, look at the warm-up exercise, for $n = -2, -3, -4, \dots$.

This previous example motivates a big result in complex integration:

Theorem 4.4 (Cauchy's Integral Formula)**.** *Let $f(z)$ be analytic in a simply-connected region, R and let C be a simple, closed, positively-oriented (counterclockwise) curve in the region. Then for any point z_0 inside C,*

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz.$$



Proof: Let

$$I = \oint_C \frac{f(z)}{z - z_0} dz = \underbrace{\oint_C \frac{f(z) - f(z_0)}{z - z_0} dz}_{I_1} + \underbrace{\oint_C \frac{f(z_0)}{z - z_0} dz}_{I_2}$$

For I_1, we claim the function

$$F(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0 \\ f'(z_0) & \text{if } z = z_0 \end{cases}$$

is analytic. This follows from L'Hôpital's Rule,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0),$$

and some details about continuity which we have omitted (trust me, it works - if you have any questions look at any complex variables textbook). So I_1 vanishes.

For I_2, the integrand $\dfrac{f(z_0)}{z-z_0}$ is analytic except at $z = z_0$.
Consequently, we can deform the contour to a circle of radius ρ centered at $z = z_0$.
Now $dz = z_0 + \rho e^{i\theta} d\theta \quad 0 \le \theta \le 2\pi$

$$\begin{aligned} I_2 &= \oint_C \frac{f(z_0)}{z-z_0} dz = f(z_0) \oint_C \frac{dz}{z-z_0} \\ &= \int_0^{2\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}} d\theta = 2\pi i\, f(z_0). \end{aligned}$$

As a bonus, we can derive a contour integral for the higher derivatives of f at z_0.

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{\overbrace{f(z_0+\Delta z)}^{\text{apply CIF}} - \overbrace{f(z_0)}^{\text{apply CIF}}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-(z_0+\Delta z)} dz - \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0} dz\right] \\ &= \frac{1}{2\pi i}\oint_C f(z) \lim_{\Delta z \to 0}\left[\frac{1}{z-(z_0+\Delta z)} - \frac{1}{z-z_0}\right]\frac{1}{\Delta z} \\ &= \frac{1}{2\pi i}\oint_C f(z) \frac{d}{dz_0}\left[\frac{1}{z-z_0}\right] dz \\ &= \frac{1}{2\pi i}\oint_C f(z) \frac{1}{(z-z_0)^2} dz \end{aligned}$$

So,

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2} dz \tag{4.1}$$

and inductively one can show that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

where

$$f^{(n)}(z_0) = \frac{d^n}{dz^n}(f(z))\big|_{z=z_0}.$$

From this generalization, we can see that when a function is analytic at $z = z_0$, it is actually infinitely differentiable there. This follows from the fact that if $f(z)$ is bounded near z_0 (from continuity), these integrals exist and therefore the derivatives of the function are bounded. A little more hard work shows that a function analytic at $z = z_0$ also has a convergent power series at this point.

Let us examine this point in a bit more detail.