Five

Taylor's Theorem and Classifying Isolated Singularities

CHAPTER OUTLINE OF LECTURE

- Classifying Isolated Singularities
  - Taylor series
  - Isolated singularities
  - Laurent series

In this section we will generalize Taylor's series

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2!} f''(z_0)(z - z_0)^2 + \dots$$

from a function of a real variable to a function of a complex variable. We will also define it's generalization to a *Laurent series* which includes negative powers.

5.1 TAYLOR SERIES

Another big result from complex analysis that we will not prove is:

Theorem 5.1 (Taylor's Theorem). *$f(z)$ is analytic at $z = z_0$ if and only if it has a convergent Taylor series about $z = z_0$,*

$$\begin{aligned} f(z) &= f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2!} f''(z_0)(z - z_0)^2 + \ldots \\ &= \sum_{n=0}^{\infty} \frac{f^(n)(z_0)}{n!}(z - z_0)^n \end{aligned}$$

Furthermore, the radius of convergence of the series is the distance from z_0 to the nearest singularity (any place where the function is not analytic).

We can apply this theorem to some classical examples.

Example 5.1. The Taylor series for e^z, $\sin z$, and $\cos z$ about $z = 0$ are

$$\begin{aligned} e^z &= 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \ldots \\ \sin z &= z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 + \ldots \\ \cos z &= 1 - \frac{1}{2!} z^2 + \frac{1}{4!} z^4 + \ldots \end{aligned}$$

Note e^z, $\sin z$, and $\cos z$ are analytic at $z = 0$, so they have convergent power series expansions for all z. The above functions converge for all z because these functions are entire. ■

Example 5.2. A geometric series. Consider

$$f(z) = \frac{1}{1 + z^2}.$$

The Taylor series expansion for $f(z)$ about $z = 0$ is

$$\frac{1}{1 + z^2} = 1 - z^2 + z^4 - z^6 + \ldots = \sum_{n=0}^{\infty} z^{2n}(-1)^n$$

where we have used the formula for the expansion of a geometric series,

$$1 + A + A^2 + A^3 + \ldots = \frac{1}{1 - A}$$

We can ask what the radius of convergence of this series is. The *ratio test* can be used to compute the radius of convergence.

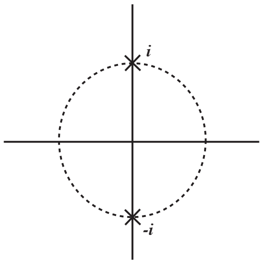

Figure 5.1: Singularities of $f(z) = \frac{1}{1+z^2}$. The Taylor series of $f(z)$ at $z = 0$ converges for $|z| < 1$ because this is the largest disc around $z = 0$ for which $f(z)$ is analytic.

Ratio Test:

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ the series diverges.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ the series converges.

Apply the ratio test, we see that

$$\lim_{n\to\infty} \left|\frac{z^{2n+2}(-1)^{n+1}}{z^{2n}(-1)^n}\right| = \lim_{n\to\infty} |z^2| = |z^2| < 1$$

The region $|z^2| < 1$ is the interior of the unit disc. The function

$$f(z) = \frac{1}{1+z^2}$$

has singularities at $\pm i$. Consequently the unit disc is also the largest circle at the origin in which the function is analytic, as predicted by Taylor's Theorem above. ■

Exercise 5.1. What is the power series expansion for $f(z) = \dfrac{1}{z}$ about $z = 1$?

Solution: One solution method is to use Taylor's formula; note that

$$f'(z) = -\frac{1}{z^2} \quad f''(z) = \frac{2}{z^3} \quad f'''(z) = -\frac{3!}{z^4} \quad \ldots \quad f^{(n)}(z) = (-1)^n \frac{n!}{z^{n+1}}$$

from which we see that

$$\begin{aligned} f(z) &= f(1) + f'(1)(z-1) + \frac{1}{2!}f''(1)(z-1)^2 + \frac{1}{3!}f'''(1)(z-1)^3 + \dots \\ &= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \dots \end{aligned}$$

Another method uses the formula for the sum of a geometric series:

$$\frac{1}{1-A} = 1 + A + A^2 + A^3 + \dots$$

which converges for $|A| < 1$. Using this we again see

$$\frac{1}{z} = \frac{1}{1+(z-1)} = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \dots$$

which converges for $|z-1| < 1$.

5.2 ISOLATED SINGULARITIES

Analytic functions have very nice properties (infinitely differentiable, Cauchy-Riemann equations hold, etc.). But it turns out that the singularities (places where a function is not analytic) of complex functions are also really useful and contain lots of information.

Definition 5.1. A singularity of $f(z)$ at $z = z_0$ is *isolated* if $f(z)$ is analytic in a punctured disc centered at $z = z_0$; i.e., analytic in $0 < |z - z_0| < \epsilon$, for some $\epsilon > 0$.

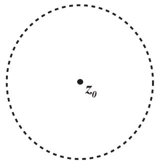

Figure 5.2: An isolated singularity at $z = z_0$.

Let's classify some singularities!

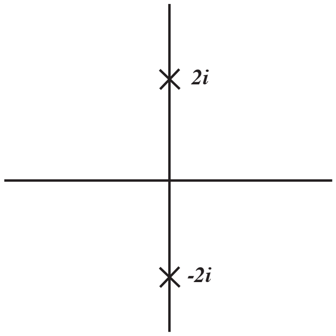

Figure 5.3: The function $f(z) = \frac{1}{z^2+4}$ has isolated singularities at $z = \pm 2i$.

(a) The function

$$f(z) = \frac{1}{z^2+4}$$

is analytic everywhere except for two isolated singgulaitries at $z = \pm 2i$.

(b) The function $g(z) = \text{Log}(z)$ is analytic everywhere except along the negative real axis. This branch cut is not isolated.

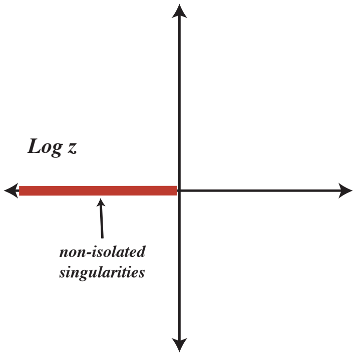

Figure 5.4: The function $g(z) = \text{Log}(z)$ has non-isolated singularities.

Big idea: There are only three kinds of isolated singularities.

Suppose $f(z)$ has an isolated singularity at $z = z_0$. The singularity must be:

(a) A **removable singularity**, which mean $\lim_{z \to z_0} f(z)$ exists.

(b) A **pole**, which means $\lim_{z \to z_0} |f(z)| = \infty$, but for some positive integer n and non-zero constant c, $\lim_{z \to z_0} (z - z_0)^n f(z) = c$. The number n is called the *order* of the pole.

(c) An **essential singularity**, if neither (a) or (b).

Example 5.3. The function

$$f(z) = \frac{1}{z}$$

has a pole at $z = 0$. To see this note that

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{1}{z}$$

does not exist because we get $+\infty$ if $z \to 0$ along the real positive axis, and we get $-\infty$ if $z \to 0$ along the negative real axis. But,

$$\lim_{z \to 0} z f(z) = \lim_{z \to 0} z \cdot \frac{1}{z} = 1$$

Therefore $f(z)$ has a *pole of order 1* at $z = 0$. ■

Example 5.4. The function

$$g(z) = \frac{z - 1}{z^2 - 1}$$

has isolated singularities at $z = \pm 1$. Note that

$$\lim_{z \to 1} g(z) = \lim_{z \to 1} \frac{z - 1}{(z - 1)(z + 1)} = \frac{1}{2}$$

so we can *remove* the singularity at $z = 1$ by defining

$$g(z) = \frac{1}{z + 1}$$

which only changes the value at $z = 1$ from undefined to $g(1) = 1/2$. Note that at $z = -1$

$$\lim_{z\to -1}(z+1)g(z) = \lim_{z\to -1}\frac{(z+1)}{z^2-1} = \lim_{z\to -1}\frac{1}{z^{-}1} = -\frac{1}{2}$$

which means $z = -1$ is a *pole of order one.* ■

In general, functions with a *removable singularity* at some point $z = z_0$ can be redefined at that point so the singularity goes away; that is the redefined function is now analytic at z_0.

Example 5.5. $h(z) = \dfrac{\sin z}{z}$ has a possible singularity at $z = 0$. But

$$\lim_{z\to 0} h(z) = \lim_{z\to 0}\frac{\sin z}{z} \overset{\text{l'H}}{=} \lim_{z\to 0}\frac{\cos z}{1} = 1$$

so we can define

$$h(z) = \begin{cases} 1 & \text{if } z = 0 \\ \dfrac{\sin z}{z} & \text{if } z \neq 0 \end{cases}$$

to clarify that the function is well-behaved at $z = 0$. Thus, this is another example of a removable singularity. ■

Example 5.6. The function $q(z) = e^{1/z}$ has an essential singularity at $z = 0$. To see this, we show that the function

$$\lim_{z\to 0} z^n f(z) = \lim_{z\to 0} z^n e^{1/z}$$

does not exist for n a non-negative integer. Consider

$$\begin{aligned}
\lim_{z\to 0}\left|z^n \exp\left(\frac{1}{z}\right)\right| &= \lim_{z\to 0}|z|^n\left|\exp\left(\frac{1}{z}\right)\right| \\
&= \lim_{r\to 0^+} r^n\left|\exp\left(\frac{1}{re^{i\theta}}\right)\right| \qquad \text{let } z = re^{i\theta} \\
&= \lim_{r\to 0^+} r^n\left|\exp\left(\frac{1}{r}e^{-i\theta}\right)\right| \\
&= \lim_{r\to 0^+} r^n\left|\exp\left(\frac{1}{r}\cos(\theta) - i\frac{1}{r}\sin(\theta)\right)\right| \\
&= \lim_{r\to 0^+} r^n\left|\exp\left(\frac{1}{r}\cos(\theta)\right)\underbrace{\exp\left(-\frac{i}{r}\sin(\theta)\right)}_{|e^{i\alpha}|=1 \text{ when } \alpha \text{ is real}}\right| \\
&= \lim_{r\to 0^+} r^n\left|\exp\left(\frac{1}{r}\cos\theta\right)\right| \\
&= \lim_{r\to 0^+} r^n \exp\left(\frac{\cos\theta}{r}\right) \\
&= \begin{cases} +\infty & \text{if } \cos\theta > 0 \\ 0 & \text{if } \cos\theta < 0 \end{cases}
\end{aligned}$$

so the limit doesn't exist and the isolated singularity must be essential. ■

5.3 LAURENT SERIES

The Laurent series generalizes the Taylor series by allowing for negative exponents.

$$\text{Taylor Series: } \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{Laurent Series: } \sum_{n=-k}^{\infty} a_n(z-z_0)^n$$

where $-k$ is a negative integer or possibly negative infinity. Functions with isolated singularities always have a Laurent Series.

Theorem 5.2 (Laurent Series). *Suppose $f(z)$ is analytic in the punctured disc $0 < |z-z_0| < R$; that is $f(z)$ has an isolated singularity at $z = z_0$. Then $f(z)$ has a Laurent series at $z = z_0$,*

$$f(z) = \sum_{n=-k}^{\infty} a_n(z-z_0)^n$$

where $a_{-k} \neq 0$, which converges in the punctured disc. Moreover,

- *If $k = 0$ then $f(z)$ has a removable singularity,*
- *if k is a positive integer then $f(z)$ has a pole of order k, and*
- *if $k = -\infty$ then $f(z)$ has an essential singularity at $z = z_0$.*

Exercise 5.2. Suppose we have a function $f(z)$ with a pole at $z = z_0$. Previously we defined the *order* of a pole as a positive integer n such that $\lim_{z\to z_0}(z - z_0)^n f(z) = c$ for some non-zero constant c. Alternatively, we could define the order of a pole as the least nonnegative integer k such that

$$f(z) = \sum_{n=-k}^{\infty} a_n (z - z_0)^n$$

Show that these definitions are equivalent.

In general, the best way to find the Laurent series is to use the techniques we developed for Taylor series.

Exercise 5.3. Find the Laurent series for

$$h(z) = \frac{\sin z}{z}$$

at $z = 0$ and show it has a removable singularity.

Solution: Consider the Taylor expansion of $\sin z$,

$$\begin{aligned} &= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right] \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots \end{aligned}$$

which is valid for $z \neq 0$. We clearly see $\lim_{z\to 0} h(z) = 1$ is well-defined. Since there are no terms with negative exponents then this is a removable singularity.

Exercise 5.4. Show

$$f(z) = \frac{1}{z}$$

has a pole at $z = 0$.

Solution: The Laurent series expansion for $f(z)$ at $z = 0$ is just

$$f(z) = \frac{1}{z}$$

Since it has one term with a negative exponent of -1, it is *a pole of order one*.

Exercise 5.5. Show $q(z) = e^{1/z}$ has an essential singularity at $z = 0$.

Solution: we use the Taylor series expansion for the exponential function,

$$= 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

Since there are an infinite number of terms with negative exponents, it an *essential singularity*.

Exercise 5.6. Show that $a(z) = \dfrac{\cos z}{z^2}$ has an isolated singularity at $z = 0$.

Solution: The Taylor series expansion of $\cos z$ is useful here,

$$\begin{aligned}\frac{\cos z}{z^2} &= \frac{1}{z^2}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right] \\ &= \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \dots\end{aligned}$$

Since there are a finite number of terms with negative exponents, this is a pole, specifically a *a pole of order two*.

Example 5.7. Consider

$$b(z) = \frac{\sin z}{z^2(z-\pi)}$$

(a) Find locations of isolated singularities.

(b) Find the first three non-zero terms of the Laurent expansion for the function about each point.

(c) Determine what type of singularities the function has.

Solution:

(a) Note that $b(z)$ is analytic everywhere except possibly at $z = 0$ and $z = \pi$.

(b) <u>For $z = 0$</u>:

$$\begin{aligned}\sin z &= z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \dots \\ \frac{1}{z-\pi} &= -\frac{1}{\pi} \cdot \frac{1}{1 - \frac{z}{\pi}} = -\frac{1}{\pi}\left[1 + \frac{z}{\pi} + \frac{z^2}{\pi^2} + \frac{z^3}{\pi^3} + \dots\right]\end{aligned}$$

Combining these we see

$$\begin{aligned}
\frac{\sin z}{z^2(z-\pi)} &= \frac{1}{z^2}\Big(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \dots\Big)\Big(-\frac{1}{\pi}\Big[1 + \frac{z}{\pi} + \frac{z^2}{\pi^2} + \dots\Big]\Big) \\
&= -\frac{1}{\pi}\left[\frac{1}{z} + \frac{1}{\pi} + z\left(\frac{1}{\pi^2} - \frac{1}{3!}\right) + \dots\right] \\
&= -\frac{1}{\pi}\frac{1}{z} - \frac{1}{\pi^2} + \left[\frac{1}{6\pi} - \frac{1}{\pi^3}\right]z\cdots
\end{aligned}$$

<u>For $z = \pi$</u>:
Note that

$$\begin{aligned}
\sin z &= \sin[(z-\pi)+\pi] \\
&= -\sin(z-\pi) \\
&= -(z-\pi) + \frac{1}{3!}(z-\pi)^3 - \frac{1}{5!}(z-\pi)^5 + \dots
\end{aligned}$$

We use a trick to expand $1/z^2$ at $z = \pi$. First we compute that

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{\pi + (z-\pi)} \\
&= \frac{1}{\pi}\cdot\frac{1}{1 + \frac{(z-\pi)}{\pi}} \\
&= \frac{1}{\pi}\left[1 - \frac{z-\pi}{\pi} + \left(\frac{z-\pi}{\pi}\right)^2 - \left(\frac{z-\pi}{\pi}\right)^3 + \dots\right]
\end{aligned}$$

Then, as

$$-\frac{d}{dz}\left(\frac{1}{z}\right) = \frac{1}{z^2}$$

we see that

$$\begin{aligned}
\frac{1}{z^2} &= -\frac{d}{dz}\left(\frac{1}{z}\right) \\
&= -\frac{d}{dz}\left[\frac{1}{\pi} - \frac{z-\pi}{\pi^2} + \frac{(z-\pi)^2}{\pi^3} - \frac{(z-\pi)^3}{\pi^4} + \dots\right] \\
&= \left[\frac{1}{\pi^2} - \frac{2(z-\pi)}{\pi^2} + \frac{3(z-\pi)^2}{\pi^4}\cdots\right]
\end{aligned}$$

Combining these we see

$$\begin{aligned}\frac{\sin z}{z^2(z-\pi)} &= \frac{1}{z-\pi}\left[-(z-\pi)+\frac{1}{3!}(z-\pi)^3\cdots\right]\cdot\left[\frac{1}{\pi^2}-\frac{2(z-\pi)}{\pi^2}+\frac{3(z-\pi)^2}{\pi^4}\cdots\right]\\ &= -\frac{1}{\pi^2}+\frac{2}{\pi^3}(z-\pi)+\left[\frac{1}{6\pi^2}-\frac{3}{\pi^4}\right](z-\pi)^2\cdots\end{aligned}$$

(c) For $z=0$, we have one term with a negative exponent, so it is a pole of order 1. For $z=\pi$, there are no terms with negative exponents, so it is a removable singularity, and

$$\lim_{z\to\pi} b(z) = -\frac{1}{\pi^2}.$$

■

In the next lecture, we will show how to use the Laurent series to evaluate closed contour integrals for functions with only isolated singularities.