Six

The Residue Calculus

CHAPTER OUTLINE

- Evaluating Residues
- The Residue Theorem

The residue calculus is a tool for evaluating integrals around closed contour of functions which are analytic except for isolated singularities. One finds that the integral is a sum of contributions, called *residues*, from each singularity.

6.1 EVALUATING RESIDUES

We begin by defining the *residue* of an isolated singularity.

Definition 6.1. If $f(z)$ has an isolated singularity at $z = z_0$ then the *residue* of $f(z)$ is the coefficient of the $(z - z_0)^{-1}$ term in its Laurent series expansion at that point. Thai is, if the Laurent series at $z = z_0$ is

$$\sum_{n=-k}^{\infty} a_n (z - z_0)^n$$

then the residue of $f(z)$ at $z = z_0$ is

$$\text{Res}[f(z); z = z_0] \equiv a_{-1}$$

Let's do some exercises in identifying singularities and calculating residues.

Example 6.1. Identify the singularities and calculate the residues of the following functions:

(a) The function $a(z) = z^{-10}$ at $z = 0$.

Solution: This function is differentiable everywhere except at $z = 0$, where it is not defined. Therefore, it is analytic everywhere except at $z = 0$ where it has an isolated singularity. Note that

$$a(z) = \frac{1}{z^{10}} = \underbrace{\frac{1}{z^{10}} + \frac{0}{z^9} + \frac{0}{z^8} + \ldots + \frac{0}{z} + 0 + 0z + 0z^2 + \ldots}_{\text{series about } z=0}$$

If we look at the coefficient of the term with z^{-1}, we can see that the residue for the function is

$$\text{Res}\left[\frac{1}{z^{10}}; z = 0\right] = 0.$$

(b) The function $b(z) = \frac{1}{(z-2)}$ at $z = 2$.

Solution: the function $b(z)$ is analytic everywhere except at $z = 2$ it has a simple pole (i. e. a pole of order 1). It's Laurent series again has only one term and

$$\text{Res}\left[\frac{1}{z-2}; z = 2\right] = 1\ .$$

(c) The function $c(z) = \sin(1/z)$ at $z = 0$.

Solution: Note $\sin(z)$ is entire, so $c(z)$ is analytic everywhere except at $z = 0$ where it has an essential singularity. Since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$$

we see

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} \ldots.$$

This function has an essential singularity at $z = 0$ because it has an infinite number of terms with negative exponents. The coefficient of the term with z^{-1} is 1, so the residue is

$$\text{Res}\left[\sin\left(\frac{1}{z}\right); z = 0\right] = 1$$

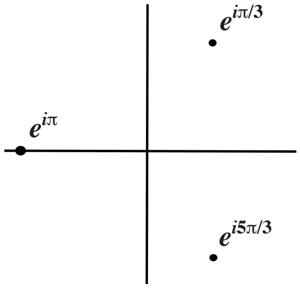

Figure 6.1: The singularities of $d(z) = \frac{1}{z^3+1}$ are three simple poles at $z = -1, e^{-i\pi/3}$, and $e^{i\pi/3}$.

(d) The function $d(z) = \frac{1}{z^3+1}$ has three isolated singularities. Find the residues at each singularity.

Solution: This function is analytic everywhere except where the denominator $z^3 + 1$ is zero. We begin by noting that

$$z^3 + 1 = 0 \Rightarrow z^3 = -1 = \cdots e^{-3i\pi}, e^{-i\pi}, e^{i\pi}, e^{3i\pi} \cdots$$

So that

$$z = \cdots e^{-i\pi}, e^{-i\pi/3}, e^{i\pi/3}, e^{i\pi} \cdots$$

from which we quickly deduce that there are three zeroes at $z = -1, e^{-i\pi/3}$, and $e^{i\pi/3}$. We can now factor the denominator as

$$z^3 + 1 = (z+1)(z - e^{-i\pi/3})(z - e^{i\pi/3})$$

and in fact, we will see that $d(z)$ has three simple poles at these locations.

Let's say we want the residue at $z = e^{i\pi/3}$. To analyze the function at $z = e^{i\pi/3}$, we note that

$$\frac{1}{z^3+1} = \frac{1}{(z - e^{i\pi/3}) \cdot (z - e^{-i\pi/3})(z+1)}$$

Let's define

$$R(z) = \frac{1}{(z - e^{-i\pi/3})(z+1)},$$

then

$$\frac{1}{z^3+1} = \frac{R(z)}{z - e^{i\pi/3}}$$

Observe that $R(z)$ is analytic at $z = e^{i\pi/3}$, so it has a Taylor series expansion there.

$$R(z) = R\left(e^{i\pi/3}\right) + R'\left(e^{i\pi/3}\right)\left(z - e^{i\pi/3}\right) + \frac{1}{2!}R''\left(e^{i\pi/3}\right)\left(z - e^{i\pi/3}\right)^2 + \dots$$

and $d(z)$ can now be expanded as

$$d(z) = \frac{R(z)}{z - e^{i\pi/3}} + R'\left(e^{i\pi/3}\right) + \frac{1}{2!}R''\left(e^{i\pi/3}\right)\left(z - e^{i\pi/3}\right) + \dots$$

And we now see that there is a simple pole (that is a pole of order one) at $z = e^{i\pi/3}$ and the associated residue is given by

$$\begin{aligned}
\text{Res}\left[d(z); z = e^{i\pi/3}\right] &= R\left(e^{i\pi/3}\right) \\
&= \frac{1}{\left(e^{i\pi/3} - e^{-i\pi/3}\right)\left(e^{i\pi/3} + 1\right)} \\
&= \frac{1}{e^{i\pi/6}\left(e^{i\pi/3} - e^{-i\pi/3}\right)\left(e^{i\pi/6} + e^{-i\pi/6}\right)} \\
&= \frac{e^{-i\pi/6}}{\left(2i\sin(\pi/3)\right)\left(2\cos(\pi/6)\right)} \\
&= \frac{-ie^{-i\pi/6}}{\left(2\cdot(\sqrt{3}/2)\right)\left(2\cdot(\sqrt{3}/2)\right)} \\
&= \frac{-i}{3}e^{-i\pi/6} \\
&= \frac{-i}{3}[\cos(\pi/6) - i\sin(\pi/6)] \\
&= \frac{-i}{3}\left[\frac{\sqrt{3}}{2} - i\frac{1}{2}\right] \\
&= -\frac{1}{6} - i\frac{\sqrt{3}}{6}
\end{aligned}$$

where we used the identities

$$e^{iz} = \cos z + i\sin z \qquad e^{iz} + e^{-iz} = 2i\cos z \qquad e^{iz} - e^{-iz} = 2i\sin z.$$

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Example 6.2. For the function

$$f(z) = \frac{\cos z}{z^2 + z^4}$$

locate the isolated singularities, characterize them and calculate their residues.

Solution: Note that $\cos z$ is an entire function, and the denominator can be factored as $z^2(z^2+1)$, so there are isolated singularities at $z = 0, \pm i$.

(a) At $z = 0$, the Laurent Series for $f(z)$ looks like

$$f(z) = \frac{1}{z^2}\left(\frac{\cos z}{1+z^2}\right) = \frac{1}{z^2}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right)\left(1 - z^2 + z^4 - \ldots\right)$$
$$= \frac{1}{z^2} + \frac{0}{z} + \ldots$$

So $f(z)$ has a 2^{nd} order pole at $z = 0$ & the residue there is 0.

(b) At $z = \pm i$, Laurent series for $f(z)$ looks like

$$f(z) = \frac{1}{z \mp i} \cdot \underbrace{\frac{\cos z}{z^2(z \pm i)}}_{\text{analytic @ } z=\pm i}$$

So $f(z)$ has a 1^{st}-order pole ("simple pole") at $z = \pm i$,

$$\text{Res} = \left[\frac{\cos z}{z^2(z \pm i)}\right]_{z=\pm i} = \frac{\cos \pm i}{(-1)(\pm 2i)} = \frac{\cosh 1}{\mp 2i}$$

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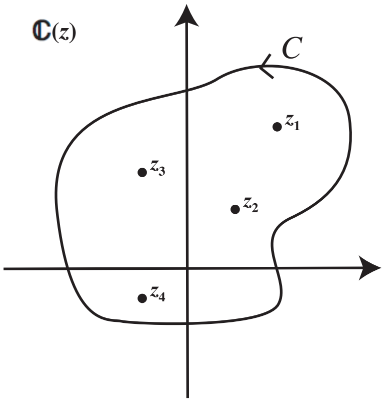
6.2 THE RESIDUE THEOREM

The residue theorem allows us to calculate the integral around a closed curve of a function that is analytic except at a finite number of isolated singularities.

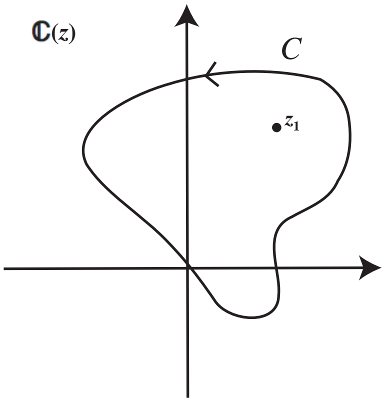
Theorem 6.1 (Residue Theorem). *Let $f(z)$ be analytic in and on a simple, closed, positively-oriented curve C except for possibly finitely many isolated singularities inside C. Then,*

$$\oint_C f(z)dz = 2\pi i \sum_{n=1}^{N} \mathit{Res}[f; z = z_n],$$

where $z_1, z_2 \cdots z_N$ are the locations of the isolated singularities inside C.



Sketchy Proof: Suppose $f(z)$ is analytic on C except for one isolated singularity inside C, called z_1.



Expand $f(z)$ in a Laurent series about $z = z_1$.

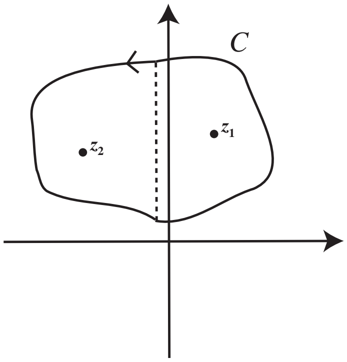
$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_1)^k$$

Then, as

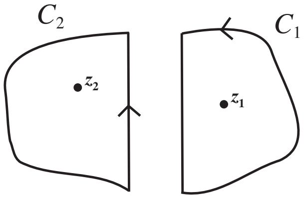
$$\oint_C (z-z_1)^k dz = \begin{cases} 0 & \text{if } k \in \mathbb{Z},\ k \neq -1 \\ 2\pi i & \text{if } k \in \mathbb{Z},\ k = -1 \end{cases}$$

we see that

$$\begin{aligned} \oint_C f(z)dz &= \oint_C \sum_{k=-\infty}^{\infty} a_k (z-z_1)^k dz \\ &= \sum_{k=-\infty}^{\infty} a_k \oint_C (z-z_1)^k dz \\ &= 2\pi i\, a_{-1} \\ &= 2\pi i\ \text{Res}[f; z=z_1] \end{aligned}$$



But what do we do if we have two singularities? We can break the contour into two closed loops, one which contains each singularity. Note the “up" and “down" portion of the contours cancel.



Now

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz = \text{Res}[f(z); z_1] + \text{Res}[f(z); z_2]$$

We will leave it as a challenge to you to think about how to generalize this to a functions with n singularities.

So to reiterate and present the theorem in a new way,

$$\boxed{\textit{Residue Theorem: } \oint_C f(z)dz = 2\pi i \cdot \sum \text{residues of } f \text{ inside } C}$$

Example 6.3. Evaluate

$$(a) \quad I = \oint_C \frac{dz}{2z^2 + 5z + 2} \qquad (b) \quad J = \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$$

where C is the unit circle traversed in a counter-clockwise sense.

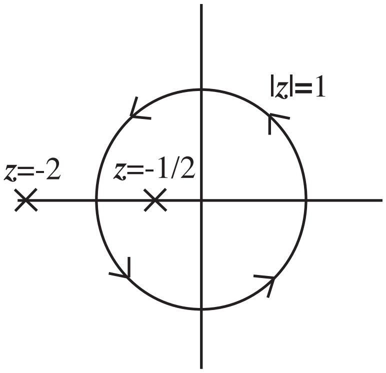


Figure 6.2: The contour of integration for Example 6.3; note one singularity is in the circle and the other is outside.

Solution: We factor the denominator

$$(2z^2 + 5z + 2) = (2z + 1)(z + 2)$$

so there are poles at $z = -\frac{1}{2}$ and $z = -2$. By the Residue Theorem

$$I = \oint_C \frac{dz}{2z^2 + 5z + 2} = 2\pi i \operatorname{Res}\left[\frac{1}{2z^2 + 5z + 2}; z = -\frac{1}{2}\right]$$

Because the pole at $z = -2$ is outside the contour of integration. But

$$\begin{aligned}
\frac{1}{2z^2 + 5z + 2} &= \frac{1}{(2z + 1)}\frac{1}{(z + 2)} \\
&= \frac{1}{2}\frac{1}{(z + \frac{1}{2})}\frac{1}{[(z + \frac{1}{2}) + \frac{3}{2}]} \\
&= \frac{1}{2}\frac{1}{(z + \frac{1}{2})}\left[\frac{2}{3} \cdot \frac{1}{1 + \frac{2}{3}\left(z + \frac{1}{2}\right)}\right] \\
&= \frac{1}{3}\frac{1}{(z + \frac{1}{2})}\left[1 - \frac{2}{3}\left(z + \frac{1}{2}\right) + \dots\right]
\end{aligned}$$

So

$$\operatorname{Res}\left[\frac{1}{2z^2 + 5z + 2}; z = -\frac{1}{2}\right] = \frac{1}{3}$$

and

$$I = \oint_C \frac{dz}{2z^2 + 5z + 2} = \frac{2\pi i}{3}$$

For part (b), let $z = e^{i\theta}$ where $0 \le \theta \le 2\pi$, so $dz = ie^{i\theta} d\theta$.

$$\begin{aligned}
I &= \int_0^{2\pi} \frac{ie^{i\theta} d\theta}{2\left(e^{i\theta}\right)^2 + 5e^{i\theta} + 2} \\
&= i\int_0^{2\pi} \frac{d\theta}{2\left(e^{i\theta}\right) + 5 + 2e^{-i\theta}} \\
&= i\int_0^{2\pi} \frac{d\theta}{5 + 2\left(e^{i\theta} + e^{-i\theta}\right)} \\
&= i\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}
\end{aligned}$$

So we see

$$I = iJ = \frac{2\pi i}{3} \Rightarrow J = \frac{2\pi}{3}$$

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We demonstrate how to evaluate many more integrals in the next chapter.