Seven

Evaluation of Integrals

7.1 Evaluation of Integrals

Remember from the previous lecture:
Residue Theorem: (in a nutshell)

$$\boxed{\oint_C f(z)dz = 2\pi i \cdot \sum \text{residues of } f \text{ inside } C}$$

We will now use this to do some examples.

Example 7.1. Compute the integrals:

$$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\, dx \;,$$
$$J = \int_{0}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx \;.$$

Solution: The integral I can be evaluated by relating it to an integral over a closed contour, where the closure is done through a semi-circular contour at"infinity." Define the function

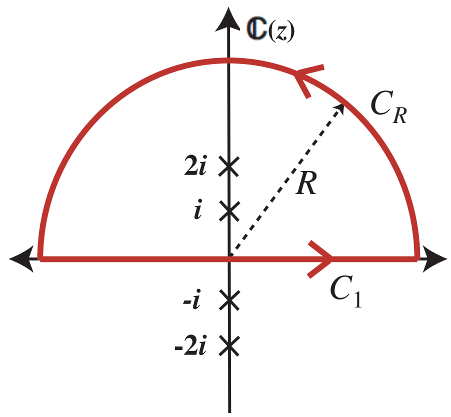
$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$$

and consider the integral

$$\oint_C f(z)\, dz$$

where C is a closed semi-circular contour in the upper half-plane.

Define the contour C,



where $C = C_1 \bigcup C_R$, C_1 is a portion of the real axis and C_R is a semi-circle of radius R.

Note that $f(z)$ has singularities at $\pm i, \pm 2i$ but only the ones at i and $2i$ are inside C. By the residue theorem,

$$\oint_C \frac{z^2}{(z^2+1)(z^2+4)} dz = 2\pi i \cdot (\text{Res}\,[f(z); z=i] + \text{Res}\,[f(z); z=2i])$$

The denominator $(z^2+1)(z^2+4) = (z-i)(z+i)(z-2i)(z+2i)$ has simple zeroes at $\pm i, \pm 2i$, so $f(z)$ has simple poles (poles of order one) at the same points. Evaluating the residues

$$\begin{aligned}
\text{Res}\,[f(z); z=i] &= \lim_{z\to i} \, (z-i)f(z) \\
&= \lim_{z\to i} \frac{z^2(z-i)}{(z-i)(z+i)(z^2+4)} \\
&= \lim_{z\to i} \frac{z^2}{(z+i)(z^2+4)} \\
&= \frac{(i)^2}{(i+i)((i)^2+4)} \\
&= \frac{-1}{(2i)(3)} \\
&= \frac{i}{6}
\end{aligned}$$

Likewise,

$$\begin{aligned}\text{Res}\left[\frac{z^2}{(z^2+1)(z^2+4)};\ z=2i\right] &= \lim_{z\to 2i}\ (z-2i)\cdot\frac{z^2}{(z^2+1)(z+2i)(z-2i)}\\ &= \left[\frac{z^2}{(z^2+1)(z+2i)}\right]_{z=2i}\\ &= -\frac{i}{3}\end{aligned}$$

$$\begin{aligned}\therefore \oint_C \frac{z^2}{(z^2+1)(z^2+4)}dz &= 2\pi i\cdot(\text{Res}\,[f(z);z=i]+\text{Res}\,[f(z);z=2i])\\ &= 2\pi i\left(\frac{i}{6}-\frac{i}{3}\right)=\frac{\pi}{3}\end{aligned}$$

Now we need to relate this to I.

$$\oint_C = \int_{C_1} + \int_{C_R} \qquad \text{as } R\to\infty$$

Along C_1, parameterize as $z=x$, x goes from $-R$ to R.

$$\int_{C_1}\frac{z^2}{(z^2+1)(z^2+4)}\,dz = \int_{-R}^{R}\frac{x^2}{(x^2+1)(x^2+4)}\,dx$$

and as R tends to infinity, we see that

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{(x^2+1)(x^2+4)}\,dx = I$$

Along C_R we wish to show that the integral goes to zero as $R\to\infty$. That is

$$\boxed{\text{Claim:}\qquad \lim_{R\to\infty}\int_{C_R}\frac{z^2}{(z^2+1)(z^2+4)}\,dz=0}$$

we will establish this result below; for the moment, let's continue with our calculation.

Summarizing, we have established with the residue theorem that

$$\begin{aligned}\frac{\pi}{3} &= \oint_C\frac{z^2}{(z^2+1)(z^2+4)}\,dz\\ &= \int_{C_1}+\int_{C_R}\frac{z^2}{(z^2+1)(z^2+4)}\,dz\end{aligned}$$

and also

$$\lim_{R\to\infty} \int_{C_1} + \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\, dz = I + 0\,.$$

Since the value of the integral around C is independent of R, we conclude that

$$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{\pi}{3}\,.$$

Notice this integrand is an even function, so

$$J = \frac{1}{2}\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\, dx = \frac{1}{2} I = \frac{\pi}{6}\,.$$

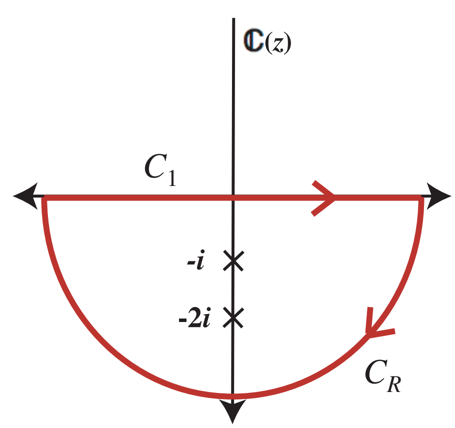
Salient features of this residue example:

(a) To use the residue calculus to evaluate a real integral, first think of a related integral over a closed path in the complex plane (because the Residue Theorem only applies to closed-path complex integrals).

(b) Evaluate the complex integral using the Residue Theorem. In other words, calculate residues for all the isolated singularities inside the closed path.

(c) Figure out how the real integral and the complex integral relate to each other. Often, you will have to use some sort of bounding argument to show that certain parts of the complex integral are negligible.

Note: In this problem, we could have used a different contour,



and we would get the same answer. The only difference in the method would be that we use the poles at $-i$ and $-2i$, and we take into consideration that this path is negatively oriented. ■ We now will

establish the result above that the integral along C_R vanishes as $R \to \infty$.

7.1.1 Establishing Bounds on Integrals

In the last example, we made a claim that:

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$$

for a particular function

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)} \, .$$

Here we will show that this is true for a large class of function $f(z)$, including the one above. The first result we will establish is

Lemma 7.1. *For a complex function, $f(z)$, and a contour C*

$$\left| \int_C f(z) dz \right| \leq \max_{z\in C} |f(z)| \cdot [length\ of\ C]$$

assuming the integrals exist.

Proof: The triangle inequality for integrals states that

$$\left| \int_C f(z) dz \right| \leq \int_C |f(z)|\, dz$$

which is basically the continuous version of the statement that: *The sum of the absolute values is less than the absolute value of the sum.* Also, clearly

$$|f(z)| \leq \max_{z\in C} |f(z)|$$

for z on C. Combining these two results yields

$$\left| \int_C f(z) dz \right| \leq \int_C |f(z)|\, dz \leq \int_C \max_{z\in C} |f(z)|\, dz = \max_{z\in C} |f(z)| \cdot [\text{length of } C]$$

where the last statement follows because the integrand is constant.

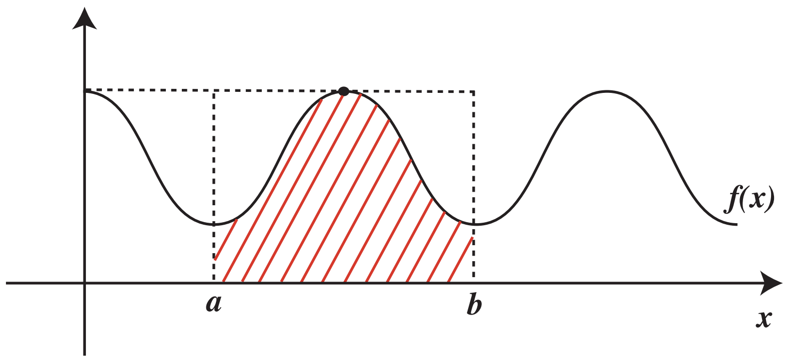


Figure 7.1: The real version of our lemma states that $\left|\int_a^b f(x)dx\right| \leq \max_{a\leq x\leq b}|f|\cdot|b-a|$ which is easy to see from the above picture.

Finally, we can use this to prove the theorem:

Theorem 7.1. *Suppose C_R is an arc of a circle of a circle of radius R centered at the origin and spanning an angle Δ and $f(z)$ is a continuous and differentiable function such that*

$$\lim_{R\to\infty} R|f(Re^{i\theta})| = 0$$

for any angle θ in the arc. Then

$$\lim_{R\to\infty}\int_{C_R} f(z)\, dz = 0.$$

Proof: We use the Lemma above to bound the integral. Clearly

$$\left|\int_{C_R} f(z)\, dz\right| \leq \max_{z\in C}|f(z)|\cdot R\Delta$$

where Δ is the angle spanned by the arc c_R, so $R\Delta$ is its length. Now, if $f(z)$ is continuous and the arc is closed (that is contains its endpoints, then the

$$\lim_{R\to\infty} R|f(Re^{i\theta})| = 0$$

guarantees

$$\lim_{R\to\infty} R\cdot\max_{z\in C}|f(Re^{i\theta})| = 0.$$

For the analysts among you this follows from the fact that a differentiable function on a closed interval is uniformly continuous. Taking the limit as $R \to \infty$ in the inequality above proves the theorem.

In our example above we need to show that

$$\lim_{R\to\infty} R\left|\frac{z^2}{(z^2+1)(z^2+4)}\right| = \lim_{R\to\infty} \frac{R^3}{|R^4e^{4i\theta}+5R^2e^{2i\theta}+5|} = 0$$

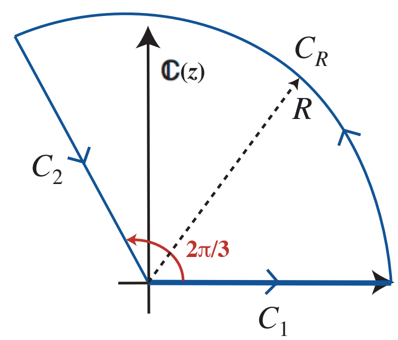
which is true as the R^4 dominates in the denominator at large R.

7.1.2 Two more examples

Example 7.2. Evaluate the integral:

$$A = \int_0^\infty \frac{1}{1+x^3}\, dx$$

Solution: It turns out one can evaluate this integral using standard means, but the residue calculus gives us a new way to evaluate this integral. Our first goal is relate the integral to one around a closed contour.



Let $C = C_1 \bigcup C_R \bigcup C_2$ and consider $B = \oint_C \frac{1}{z^3+1} dz$

The integrand $\frac{1}{z^3+1}$ has 3 poles at $e^{i\pi/3}, -1,\ e^{-i\pi/3}$ but only $e^{i\pi/3}$ is inside C. So, by the residue theorem

$$\begin{aligned} B = \oint_C \frac{1}{z^3+1} dz &= 2\pi i \cdot \text{Res}\left[\frac{1}{z^3+1}; e^{i\pi/3}\right] \\ &= 2\pi i \cdot \lim_{z\to e^{i\pi/3}} \frac{z-e^{i\pi/3}}{z^3+1} \\ &\overset{\text{l'H}}{=} 2\pi i \cdot \frac{1}{3(e^{i\pi/3})^2} \\ &= \frac{2\pi i}{3} e^{-2\pi i/3}\,. \end{aligned}$$

We need to figure out how B is related to A. Note

$$B = \oint_C \frac{1}{z^3+1}\, dz = \int_{C_1} \frac{1}{z^3+1}\, dz + \int_{C_R} \frac{1}{z^3+1}\, dz + \int_{C_2} \frac{1}{z^3+1}\, dz$$

We claim that as
$$\lim_{R\to\infty}\int_{C_R}\frac{1}{z^3+1}\,dz\to 0$$
which follows because
$$\lim_{R\to\infty} R\cdot\left|\frac{1}{(Re^{i\theta})^3+1}\right| = 0$$
as the R^3 dominates the denominator.
Parameterize C_1: Let $z=x$ where x goes from 0 to R. Then $dz=dx$ and
$$\int_{C_1}\frac{dz}{z^3+1}=\int_0^R\frac{dx}{x^3+1}\to A\quad\text{as } R\to\infty.$$
Parameterize C_2: Let $z=re^{i2\pi/3}$ so $dz=dr\,e^{i2\pi/3}$ and remember that r goes from R to 0, so
$$\begin{aligned}\int_{C_2}\frac{dz}{z^3+1}&=\int_R^0\frac{dr\,e^{2\pi i/3}}{\left(re^{i2\pi/3}\right)^3+1}\\&=-e^{2\pi i/3}\int_0^R\frac{dr}{r^3e^{2\pi i}+1}\\&=-e^{2\pi i/3}\int_0^R\frac{dr}{r^3+1}\to -e^{i2\pi/3}A\text{ as } R\to\infty\end{aligned}$$
Putting it all together, we see that
$$\begin{aligned}\frac{2\pi i}{3}e^{-2\pi i/3}=B&=\int_{C_1}+\int_{C_R}+\int_{C_2}\\&\to A+0-e^{i2\pi/3}A\quad\text{as } R\to\infty\end{aligned}$$
So,
$$\begin{aligned}\frac{2\pi i}{3}e^{-2\pi i/3}&=A\left(1-e^{2\pi i/3}\right)\\A&=\frac{2\pi i}{3}\frac{e^{-2\pi i/3}}{1-e^{2\pi i/3}}\\&=\frac{2\pi i}{3}\frac{e^{-2\pi i/3}}{e^{\pi i/3}\left(e^{-i\pi/3}-e^{i\pi/3}\right)}\\&=\frac{2\pi i}{3}\frac{e^{-i\pi}}{-2i\sin(\pi/3)}\\&=\frac{2\pi i}{3}(-1)\frac{1}{-i\sqrt{3}}\\&=\frac{2\pi\sqrt{3}}{9}\,.\end{aligned}$$

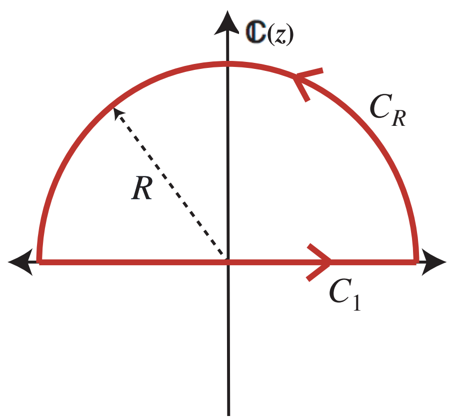
Therefore

$$\int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi\sqrt{3}}{9}$$

So from the residue theorem, we can calculate integrals by hand that we normally would not be able to do. ■

Example 7.3. Compute $Q = \displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx \quad (a > 0)$

The first thing to try is $\displaystyle\oint_C \frac{\cos z}{z^2+a^2} dz$ for $C = C_1 \bigcup C_R$.



But this won't work because $\displaystyle\int_{C_R} \frac{\cos z}{z^2+a^2} dz \nrightarrow 0$ as $R \to \infty$.

$$\begin{aligned} |\cos z| = |\cos(x+iy)| &= |\cos x \cosh y - i \sin x \sinh y| \\ &= \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} \\ &\to \infty \quad \text{as} \quad y \to \infty \end{aligned}$$

Trick: Add zero to Q:

$$\begin{aligned} \text{Let} \quad Q &= \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} + i \underbrace{\int_{-\infty}^{\infty} \frac{\sin x}{x^2+a^2} dx}_{=0,\ \text{because integrand is odd}} \\ &= \int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{x^2+a^2} dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2} dx \end{aligned}$$

So consider $\displaystyle\oint_C \frac{e^{iz}}{z^2+a^2} dz = \int_{C_1} + \int_{C_R}$ as $R \to \infty$, $\displaystyle\int_{C_1} \to Q$.

How does $\int_{C_R}$ behave as $R \to \infty$?

$$\left| \int_{C_R} \frac{e^{iz}}{z^2 + a^2} dz \right| \leq \max_{z \in C_R} \left| \frac{e^{iz}}{z^2 + a^2} \right| \cdot \text{ length of } C_R$$

Along C_R, $z = Re^{i\theta}$ for $\theta \in [0, \pi]$

$z = R\cos\theta + iR\sin\theta$

$$\begin{aligned} \left| \frac{e^{iz}}{z^2 + a^2} \right| &= \left| \frac{\exp(iR\cos\theta - R\sin\theta)}{z^2 + a^2} \right| \\ &= \frac{|\exp(iR\cos\theta)|\,|\exp(-R\sin\theta)|}{|z^2 + a^2|} = \frac{e^{-R\sin\theta}}{|z^2 + a^2|} \end{aligned}$$

$|\exp(iR\cos\theta)| = 1$, because $R, \cos\theta$ are real numbers, and

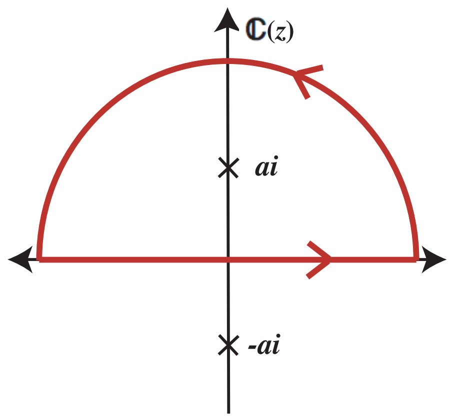
because $\sin\theta \geq 0$, $\dfrac{e^{-R\sin\theta}}{|z^2 + a^2|} \to 0$ as $R \to \infty$.

so $\displaystyle\oint_C \frac{e^{iz}}{z^2 + a^2} dz = \int_{C_1} + \int_{C_R}$
$\to Q + 0$ as $R \to \infty$.

Then, by the Residue Theorem,

$$Q = \oint_C \frac{e^{iz}}{z^2 + a^2} dz = 2\pi i \sum \text{ Residues inside } C.$$

$\dfrac{e^{iz}}{z^2 + a^2} = \dfrac{e^{iz}}{(z + ai)(z - ai)}$ has singularities at $z = \pm ai$.



So,

$$
\begin{aligned}
Q &= 2\pi i \,\mathrm{Res}\left(\frac{e^{iz}}{z^2+a^2}; z = ai\right) \\
&= 2\pi i \,\mathrm{Res}\left(\underbrace{\frac{e^{iz}}{z+ai}}_{D(z)} \cdot \frac{1}{z-ai}; z = ai\right) \\
&= 2\pi i \left[\frac{e^{iz}}{z+ai}\right]_{z=ai} = 2\pi i \frac{e^{-a}}{2ai} \\
\therefore \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx &= \frac{\pi e^{-a}}{a} \qquad (a > 0)
\end{aligned}
$$

■