Eight

An Introduction to Laplace Transforms

CHAPTER OUTLINE

- Definition of the Laplace Transform
- Laplace Transforms of Derivatives
- First Shifting Theorem
- Inversion by Partial Fractions

The Laplace transform is a tool that is particularly useful for solving initial value problems for linear differential equations. It gives us tools for dealing with forcing that occurs as an impulse or that is switched on and off. The basic idea is that one looks at the differential equation in terms of a *transform variable*.

8.1 THE DEFINITION OF THE LAPLACE TRANSFORM

Definition 8.1. The Laplace transform of a function $f(t)$,

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)}$$

maps a function, $f(t)$, to a function of the *transform variable* s. By convention we will use lowercase letters to denote the origin function $[f(t)]$ and uppercase to denote the transformed function $[F(s)]$.

Note that the transform only depends the values of $f(t)$ for $t > 0$ and that $F(s)$ may only be defined for s sufficiently large. we will deal with this technical difficulties later; let's do some examples to get the hang of these transforms.

Example 8.1. Compute the following Laplace transforms of the following functions:

(a) The function $f(t) = 1$.
Solution:

$$\begin{aligned}\mathcal{L}\{1\} &= \int_0^\infty e^{-st}\cdot(1)dt = \int_0^\infty e^{-st}dt \\ &= -\frac{e^{-st}}{s}\bigg|_{t=0}^{t=\infty} \\ &= -\left[\frac{e^{-\infty}}{s} - \frac{e^0}{s}\right] = \frac{1}{s}\end{aligned}$$

which is valid for $s > 0$.

(b) The function $f(t) = t$.
Solution:

$$\begin{aligned}\mathcal{L}\{t\} &= \int_0^\infty e^{-st}\cdot(t)dt = \int_0^\infty te^{-st}dt \\ &= -\frac{e^{-st}(st+1)}{s^2}\bigg|_{t=0}^{t=\infty} \\ &= -\left[\frac{e^{-\infty}}{s^2} - \frac{e^0}{s^2}\right] = \frac{1}{s^2}\end{aligned}$$

Note that again that this is defined when $s > 0$ for which

$$\lim_{t\to\infty}(st+1)e^{-st} = 0$$

because the exponential function decays much faster than the linear function grows.

(c) The function $f(t) = t^n$.
Solution:

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}\cdot(t^n)dt = \int_0^\infty t^n e^{-st}dt$$

Let $v = st$, to give

$$\mathcal{L}\{t^n\} = \frac{1}{s^{n+1}} \int_0^\infty v^n e^{-v} dv = \frac{n!}{s^{n+1}}$$

which again is valid for $s > 0$. we have used the fact that

$$\int_0^\infty v^n e^{-v}\, dv = n!$$

which can be proved via integration by parts and induction.

(d) The function $f(t) = e^{at}$ where a is a constant.
Solution:

$$\begin{aligned}
\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st} \cdot e^{at} dt \\
&= \int_0^\infty e^{-(s-a)t} dt = \frac{-1}{s-a}\, e^{-(s-a)t}\Big|_{t=0}^{t=\infty} \\
&= \frac{1}{-s+a}\left[e^{(-s+a)\infty} - e^{(-s+a)0}\right]
\end{aligned}$$

Only if $s > a$ does $e^{(-s+a)\infty}$ vanish, so we must restrict our answer

$$= \frac{1}{s-a} \qquad (\text{for } s > a)\ .$$

(e) The function $f(t) = \cos(at)$ where a is a constant.
Solution:

$$\begin{aligned}
\mathcal{L}\{\cos(at)\} &= \int_0^\infty e^{-st} \cdot \cos(at)\, dt \\
&= \text{Re}\left\{\int_0^\infty e^{-st} \cdot e^{iat}\, dt\right\} \\
&= \text{Re}\left\{\frac{1}{s-ia}\right\} = \text{Re}\left\{\frac{1}{s^2+a^2} \cdot (s+ia)\right\} \\
&= \frac{s}{s^2+a^2}
\end{aligned}$$

(f) The function $f(t) = \sin(at)$ where a is a constant.

Solution:

$$\begin{aligned}
\mathcal{L}\{\sin(at)\} &= \int_0^\infty e^{-st} \cdot \sin(at)\, dt \\
&= \mathrm{Im}\left\{\int_0^\infty e^{-st} \cdot e^{iat}\, dt\right\} \\
&= \mathrm{Im}\left\{\frac{1}{s-ia}\right\} = \mathrm{Im}\left\{\frac{1}{s^2+a^2} \cdot (s+ia)\right\} \\
&= \frac{a}{s^2+a^2}
\end{aligned}$$

■

One can ask if every function $f(t)$ for $t > 0$ has a Laplace transform? The answer is no, but there is a very large class of functions for which the Laplace transform exist, namely those functions that grow no worse than exponentially.

Theorem 8.1. *Suppose that $f(t)$ is a piecewise continuous function such that*

$$|f(t)| \leq M e^{\beta t} \qquad \textit{for} \quad t > 0$$

for some real constants M and β. Then the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists in the half-plane Re$\{s\} > \beta$.

Remark. We've extended the Laplace transform $F(s)$ to a function of the complex variable s. In fact the much stronger statement that $F(s)$ is analytic in the half-plane R$e\{s\} > \beta$ is true!!

Proof. As the Laplace transform is defined as

$$\mathcal{L}\{f(t)\} \equiv \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

It suffices to show that the function $e^{-st}f(t)$ is absolutely integrable for R$e\{s\} > \beta$. Think of s as a complex variable, $s = p + iq$ for p and q real that

$$\begin{aligned}
|F(s)| &\leq \int_0^\infty |e^{-st} f(t)|\, dt \\
&\leq M \int_0^\infty |e^{\beta t-(p+iq)t}|\, dt \\
&= M \int_0^\infty e^{(\beta-p)t}\, dt \\
&= \frac{M}{p-\beta} \text{ for } p > \beta.
\end{aligned}$$

which shows that the integrand $e^{-st}f(t)$ is absolutely integrable in the right half-plane where $\text{Re}\{s\} = p > \beta$. If the function is absolutely integrable, then the function is integrable also (that is the infinite integral converges to a finite value) in this half plane also. □

A little more work can show that $F(s)$ is an analytic function in this region also, and in particular has no singularities in this region.

We can also derive properties of the Laplace transform that help expand the number of functions whose transforms we can find. The most important is *linearity*.

Example 8.2. *Linearity:* Show the Laplace Transform is a *linear operator*, that is

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} .$$

Solution: The proof follows directly from the definition,

$$\begin{aligned}\mathcal{L}\{af(t) + bg(t)\} &= \int_0^\infty e^{-st} \cdot [af(t) + bg(t)]dt \\ &= a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt \\ &= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} .\end{aligned}$$

Hence we've proved the linearity of the Laplace transform. ■

In practice one constructs a table of common Laplace transforms and properties of Laplace transforms and looks up transforms as opposed to re-evaluating them from definition.

8.2 LAPLACE TRANSFORMS OF DERIVATIVES

We can compute the Laplace transform of the derivative of a function in terms of the Laplace transform of the function.

Theorem 8.2. *Suppose the Laplace transform of $y(t)$ and $y'(t)$ exist. If $\mathcal{L}\{y(t)\} = \mathbb{Y}(s)$, then*

$$\mathcal{L}\{y'(t)\} = s\mathbb{Y}(s) - y(0).$$

Proof. We use integration by parts:

$$\mathcal{L}\{y'(t)\} = \int_0^\infty e^{-st}y'(t)\,dt.$$

Let

$$\begin{aligned} y(t) &= u & y'(t)dt &= du \\ e^{-st} &= v & -se^{-st}\,dt &= dv. \end{aligned}$$

So

$$\mathcal{L}\{y'(t)\} = y(t)e^{-st}\Big[_{t=0}^{\infty} - \int_0^{\infty} (-s)e^{-st} y(t)\,dt.$$

As $y(t)$ is integrable, we assume that $y(t)e^{-st} \to 0$ as $t \to \infty$ and the first term becomes $-y(0)$ the second term we identify as $s\mathbb{Y}(s)$ from which we conclude that

$$\mathcal{L}\{y'(t)\} = s\mathbb{Y}(s) - y(0)$$

as desired. □

This can be very useful, especially when it comes to ordinary differential equations.

Example 8.3. Use the derivative property of the Laplace transform to solve the following ODE

$$y' + y = 1, \qquad y(0) = 0.$$

Solution: We begin by transforming both sides of the differential equation,

$$\begin{aligned} \mathcal{L}\{y' + y = 1\} &\Rightarrow \mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{1\} \\ &\Rightarrow s\mathbb{Y} + \underbrace{y(0)}_{=0} + \mathbb{Y} = \frac{1}{s} \\ &\Rightarrow (s+1)\mathbb{Y} = \frac{1}{s} \\ &\Rightarrow \mathbb{Y} = \frac{1}{s(s+1)} \qquad \text{(apply partial fractions)} \\ &\Rightarrow \mathbb{Y} = \frac{1}{s} - \frac{1}{s+1} \end{aligned}$$

we now need to apply an inverse Laplace transform to get a function in terms of t. From our examples earlier, we can see that the inverse is

$$\begin{aligned} \mathcal{L}^{-1}\{\mathbb{Y}\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ y(t) &= 1 - e^{-t} \end{aligned}$$

So we can see that the Laplace transform is a very useful tool in evaluating ordinary differential equations. ■

The observant reader will ask if perhaps we have pulled a quick-one here. In particular, one should ask:

> *Is the Laplace transform of a function unique?*

The answer is yes, if we restrict the type of functions we are look at. For example if $f(t)$ and $g(t)$ are continuous and have the same Laplace transform, then $f(t) = g(t)$ for $t > 0$. We can even say that if $f(t)$ and $g(t)$ are both continuous at $t = t_0 > 0$ and have the same Laplace transform, then $f(t_0) = g(t_0)$ for $t > 0$.

Consequently, if we know the Laplace transform of a continuous function (like the solution to most differential equations), then we can match it up with the appropriate transform in our table and find the inverse to obtain a unique of t. We say this symbolically by writing $\mathcal{L}^{-1}\{\mathbb{Y}(s)\} = y(t)$.

8.2.1 Higher derivatives

What about second order derivatives? We can use the first derivative to derive the formula,

$$\begin{aligned}\mathcal{L}\{y''(t)\} &= \mathcal{L}\{(y'(t))'\} = s\mathcal{L}\{y'(t)\} - y'(0)\\ &= s[s\mathbb{Y}(s) - y(0)] - y'(0)\\ &= s^2\mathbb{Y}(s) - sy(0) - y'(0)\ .\end{aligned}$$

We can continue this process for higher-order derivatives,

$$\begin{aligned}\mathcal{L}\{y^{(n)}(t)\} &= \mathcal{L}\{\left(y^{(n-1)}(t)\right)'\} = s\mathcal{L}\{y^{(n-1)}(t)\} - y^{(n-1)}(0)\\ &= s[s\mathcal{L}\{\left(y^{(n-2)}(t)\right)'\} - y^{(n-1)}(0)] - y^{(n-2)}(0)\\ &\vdots\\ &= s^n\mathbb{Y}(s) - \sum_{k=1}^{n} s^{k-1} f^{(n-k)}(0)\ .\end{aligned}$$

So we can use the Laplace transform to evaluate any order derivative.

Example 8.4. Using the Laplace transform, solve the following ODE.

$$y'' + \omega^2 y = 0 \qquad y(0) = A \quad y'(0) = B$$

where A and B are constants.

$$\begin{aligned}
\mathcal{L}\{y''(t)+\omega^2 y(t)=0\} &\to \mathcal{L}\{y''(t)\}+\omega^2\mathcal{L}\{y(t)\}=\mathcal{L}\{0\} \\
&\to s^2\mathbb{Y}(s)-s\underbrace{y(0)}_{=A}-\underbrace{y'(0)}_{=B}+\omega^2\mathbb{Y}(s) \\
&\to (s^2+\omega^2)\mathbb{Y}(s)-sA-B=0 \\
&\to \mathbb{Y}(s)=A\frac{s}{s^2+\omega^2}+B\frac{1}{s^2+\omega^2} \\
&\to \mathbb{Y}(s)=A\frac{s}{s^2+\omega^2}+\frac{B}{\omega}\frac{\omega}{s^2+\omega^2} \\
\mathcal{L}^{-1}\{\mathbb{Y}(s)\} &= A\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\}+\frac{B}{\omega}\,\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} \\
y(t) &= A\cos(\omega t)+\frac{B}{\omega}\sin(\omega t)
\end{aligned}$$

■

Again, we are assuming the Laplace transform has a unique inverse. If we know a function in terms of s, then we can match it up with the appropriate transform in our table and move the opposite way to obtain a function in terms of t.

8.3 FIRST SHIFTING THEOREM

It turns out that the Laplace transform of an exponential times a functions shifts the transform by a constant amount.

Theorem 8.3. *Suppose* $\mathcal{L}\{f(t)\}=F(s)$*, then*

$$\mathcal{L}\{e^{-at}f(t)\}=F(s+a).$$

This is known as the *First Shifting Theorem*; we'll talk about the Second Shifting Theorem in the next chapter.

Proof. The theorem follows quickly from the definition of the Laplace transform,

$$\int_0^\infty e^{-st}e^{-at}f(t)dt=\int_0^\infty e^{-(s+a)t}dt=F(s+a)\ .$$

□

This observation proves useful for solving certain ODE problems.

Example 8.5. Solve the following ODE:

$$y'(t) + y(t) = e^{-bt} \qquad y(0) = 0$$

Make sure you consider both cases when $b \neq 1$ and $b = 1$.
Solution: Laplace transform both sides of the ODE

$$\mathcal{L}\{y'(t)\} + \mathcal{L}\{y(t)\} = s\mathbb{Y}(s) + \underbrace{y(0)}_{=0} + \mathbb{Y}(s) = \frac{1}{s+b}$$

Now solve for $\mathbb{Y}(s)$

$$\mathbb{Y}(s) = \frac{1}{(s+1)(s+b)} \qquad b \neq 1$$

If $b \neq 1$, we can use partial fractions

$$\begin{aligned} \mathbb{Y}(s) &= \frac{1}{b-1}\left[\frac{1}{s+1} - \frac{1}{s+b}\right] \\ \mathcal{L}^{-1}\{\mathbb{Y}(s)\} &= \frac{1}{b-1}\left[\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+b}\right\}\right] \\ y(t) &= \frac{1}{b-1}\left[e^{-t} - e^{-bt}\right] \end{aligned}$$

If $b = 1$, we use the first shifting theorem

$$\mathbb{Y}(s) = \frac{1}{(s+1)^2}$$

we can find the inverse transform by using the shifting theorem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} \\ &= e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \\ &= te^{-t} \end{aligned}$$

■

Example 8.6. Solve the following ODE:

$$y''(t) + 2y'(t) + 2y(t) = 0 \qquad y(0) = 1 \quad y'(0) = 0$$

Solution: Laplace Transform the ODE,

$$\begin{aligned} \mathcal{L}\{y''(t)\} + 2\mathcal{L}\{y'(t)\} + 2\mathcal{L}\{y(t)\} &= \mathcal{L}\{0\} \\ s^2\mathbb{Y}(s) - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0} + 2s\mathbb{Y}(s) - 2\underbrace{y(0)}_{=1} + 2Y(s) &= 0 \\ s^2\mathbb{Y}(s) - s + 2s\mathbb{Y}(s) - 2 + 2\mathbb{Y}(s) &= 0 \\ (s^2 + 2s + 2)\mathbb{Y}(s) &= s + 2 \end{aligned}$$

Solving for $\mathbb{Y}(s)$ yields

$$\begin{aligned} \mathbb{Y}(s) &= \frac{s+2}{s^2+2s+2} \\ &= \frac{s+2}{(s+1)^2+1} \\ &= \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} \end{aligned}$$

Again, we use the first shifting theorem,

$$\begin{aligned} &= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} \\ &= e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ y(t) &= e^{-t}\cos t + e^{-t}\sin t \end{aligned}$$

■

8.4 INVERSION BY PARTIAL FRACTIONS

We can use some of our complex variable techniques to find partial fraction expansions.

Theorem 8.4. *Suppose $F(s) = q(s)/p(s)$ where p and q are polynomials, the degree of q is less than the degree of p, and p has distinct roots $s_1, s_2, \dots, s_n$, then*

$$F(s) = \frac{q(s)}{p(s)} = \frac{a_1}{s-s_1} + \frac{a_2}{s-s_2} + \cdots + \frac{a_n}{s-s_n}$$

where $a_j = \textit{Res}[F(s); s_j] = \lim_{s \to s_j} (s - s_j) \dfrac{q(s)}{p(s)}$.

Proof. For example, to compute the value of a_1, multiply by $s - s_1$

$$(s - s_1)F(s) = a_1 + a_2 \frac{(s - s_1)}{(s - s_2)} + \cdots + a_n \frac{(s - s_1)}{(s - s_n)}$$

as $s \to s_1$

$$\lim_{s \to s_1} (s - s_1)F(s) = \lim_{s \to s_1} \frac{q(s)}{p(s)} = a_1$$

and the result follows. □

Example 8.7. Compute the following inverse transforms using partial fractions:

(a) The function

$$\mathbb{Y}(s) = \frac{1}{(s+1)(s+2)}.$$

Solution: Using partial fractions, we see that

$$\frac{1}{(s+1)(s+2)} = \frac{a_1}{s+1} + \frac{a_2}{s+2}$$

and since a_1 and a_2 are the residues at $s = 1$ and 2 respectively,

$$a_1 = \frac{1}{-1+2} = 1 \quad a_2 = \frac{1}{-2+1} = -1$$

so

$$\mathcal{L}^{-1}\left\{\mathbb{Y}(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} = e^{-t} - e^{-2t}$$

(b) The function

$$\mathbb{Y}(s) = \frac{1}{(s+1)(s^2+1)}.$$

Solution: Using partial fractions, we see that

$$\frac{1}{(s+1)(s^2+1)} = \frac{a_1}{s+2} + \frac{a_2}{s+i} + \frac{a_3}{s-i}$$

and again using residue methods,

$$\begin{aligned}
a_1 &= \frac{1}{5} \\
a_2 &= \lim_{s\to -i} \frac{s+i}{(s+2)(s^2+1)} = \frac{1}{(2-i)(-2i)} = \frac{-1}{2+4i} \\
&= \frac{4i-2}{20} = -\frac{1}{10} + \frac{i}{5} \\
a_3 &= \lim_{s\to i} \frac{s-i}{(s+2)(s^2+1)} = -\frac{1}{10} - \frac{i}{5}
\end{aligned}$$

so

$$\begin{aligned}
\mathbb{Y}(s) &= \frac{1}{10}\left[\frac{2}{s+2} + \frac{-1+2i}{s+i} + \frac{-1-2i}{s-i}\right] \\
&= \frac{1}{10}\left[\frac{2}{s+2} + \frac{4-2s}{s^2+1}\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\mathbb{Y}(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{5}\frac{1}{(s+2)} + \frac{1}{5}\frac{2-s}{(s^2+1)}\right\} \\
&= \frac{1}{5}e^{-2t} - \frac{1}{5}\cos t + \frac{2}{5}\sin t
\end{aligned}$$

(c) The function

$$\mathbb{Y}(s) = \frac{1}{(s+1)^2}\frac{1}{(s+2)}$$

Solution: Again, using partial fractions, we see that

$$\frac{1}{(s+1)^2}\frac{1}{(s+2)} = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s+2}\ ,$$

and cross multiplying implies that

$$1 = A(s+2) + B(s+1)(s+2) + C(s+1)^2\ .$$

Substituting $s=-2$ implies $C=1$ and substituting $s=-1$ implies $A=1$. Also, from the coefficient of s^2, we see that $B+C=0$ which implies $B=-1$. This yields the partial fraction expansion

$$\frac{1}{(s+1)^2(s+2)} = \frac{1}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s+2}\ .$$

Taking the inverse transform yields

$$\begin{aligned}\mathcal{L}^{-1}\{\mathbb{Y}(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s+2}\right\} \\ &= te^{-t} - e^{-t} + e^{-2t}\ .\end{aligned}$$

■

We end with an example where we use partial fractions to solve an initial value problem for inhomogeneous second-order ODE.

Example 8.8. Solve the ODE:

$$y''(t) + 2y'(t) = 8t\ , \qquad y(0) = y'(0) = 0\ .$$

Solution: We transform the ODE to yield

$$\begin{aligned}\mathcal{L}\{y''(t)\} + 2\mathcal{L}\{y'(t)\} &= 8\mathcal{L}\{t\}\ , \\ s^2\mathbb{Y}(s) - sy(0) - y'(0) + 2s\mathbb{Y}(s) + 2y(0) &= \frac{8}{s^2}\ .\end{aligned}$$

Applying the initial conditions $y(0) = y'(0) = 0$ and solving for $\mathbb{Y}(s)$ we find that

$$\mathbb{Y}(s) = \frac{8}{s^3(s+2)}\ ,$$

which we will invert using partial fractions. The proper expansion is

$$\frac{8}{s^3(s+2)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+2}$$

and cross-multiplying yields

$$8 = A(s+2) + Bs(s+2) + Cs^2(s+2) + Ds^3\ .$$

Substituting $s = -2$ yields $D = -1$, while $s = 0$ yields $A = 4$. Substituting these results and expanding yields

$$\begin{aligned}8 &= 4(s+2) + Bs(s+2) + Cs^2(s+2) - s^3\ , \\ 8 &= 8 + (4+2B)s + (B+2C)s^2 + (C-1)s^3\end{aligned}$$

from which we quickly see that $B = -2$ and $C = 1$.

Putting it all together

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\mathbb{Y}(s)\right\} \ , \\ &= \mathcal{L}^{-1}\left\{\frac{4}{s^3} - \frac{2}{s^2} + \frac{1}{s} - \frac{1}{s+2}\right\} \ , \\ &= 2t^2 - 2t + 1 - e^{-2t} \ . \end{aligned}$$

So

$$\boxed{y(t) = 2t^2 - 2t + 1 - e^{-2t} \ .}$$

■