Nine

Heaviside Functions and the Second Shifting Theorem

CHAPTER OUTLINE

- Heaviside Functions
- Second Shifting Theorem
- An example of a DE with a switched input.

9.1 HEAVISIDE FUNCTIONS

Remember that the Heaviside function is defined as

$$H(t-t_0)=\begin{cases}0 & t<t_0\ ,\\ 1 & t>t_0\ .\end{cases}$$

The Heaviside function is essentially a switch; it turns on when $t=t_0$. We can compute its Laplace transform; since $H(t-t_0)=0$ for $t<t_0$, we can just evaluate the integral from t_0 to ∞.

$$\mathcal{L}\{H(t-t_0\}=\int_{t_0}^{\infty}e^{-st}dt=\left.\frac{e^{-st}}{s}\right|_{t=t_0}^{t=\infty}=\frac{e^{-st_0}}{s},\qquad t_0>0\ .$$

If we have a function $f(t)$ defined for $t\geq 0$, we can use the Heaviside function to create a new function whose start time is “delayed" to a time t_0,

Example 9.1. Suppose $f(t) = t^2$ for $t > 0$. Define a function that vanishes for $t < t_0$ and looks like $f(t)$ delayed to a start time of t_0.
Solution: Let

$$g(t) = \underbrace{H(t - t_0)f(t - t_0)}_{f(t) \text{ delayed until } t_0}$$

then

$$g(t) = \begin{cases} 0, & t < t_0 \\ (t - t_0)^2, & t > t_0 \end{cases}$$

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We can compute the Laplace transform of this shifted function in terms of the Laplace transform of $f(t)$; this leads us to our second shifting theorem for Laplace Transforms.

9.2 SECOND SHIFTING THEOREM

The first shifting theorem showed that multiplying a function by e^{-at} shifted the Laplace transform by a distance a in the transform variable s. The second shifting theorem shows that shifting the original function by a distance t_0 multiplies the Laplace transform by an exponential; the result has a nice sort of symmetry to it.

Theorem 9.1 (Second Shifting)**.** *If* $t_0 > 0$ *and* $\mathcal{L}\{y(t)\} = \mathbb{Y}(s)$ *then*

$$\mathcal{L}\{H(t - t_0)y(t - t_0)\} = e^{-st_0}\mathbb{Y}(s).$$

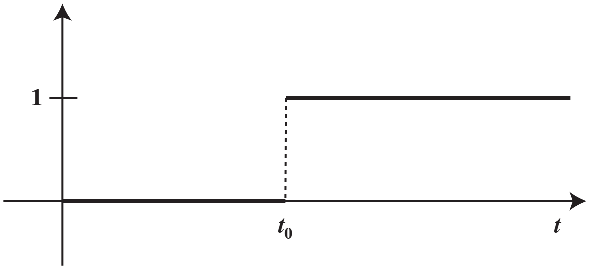

Figure 9.1: A graph of the Heaviside function, $H(t - t_0)$.

Proof. A simple calculation shows that:

$$\begin{aligned}\mathcal{L}\{H(t-t_0)y(t-t_0)\} &= \int_{t_0}^{\infty} e^{-st}y(t-t_0)dt \\ t' = t - t_0, &\qquad dt' = dt \\ &= \int_0^{\infty} e^{-s(t'+t_0)}y(t')dt' \\ &= e^{-st_0}\int_0^{\infty} e^{-st'}y(t')dt' = e^{-st_0}\mathbb{Y}(s)\end{aligned}$$

as advertised above. □

Example 9.2. Let us compute the Laplace transform of the shifted function in Example 1.

$$\mathcal{L}\{t^2\} = \frac{1}{s^2} \quad \Rightarrow \quad \mathcal{L}\{H(t-t_0)(t-t_0)^2\} = \frac{e^{-st_0}}{s^2}, \quad t_0 > 0.$$

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Example 9.3. Compute the inverse Laplace transform of $\frac{e^{-s}}{s^2+4}$ We know that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$$

So by the shifting theorem

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+4}\right\} = \underbrace{\frac{1}{2}\sin 2(t-1)}_{f(t) \text{ delayed until } t=1} \cdot H(t-1)$$

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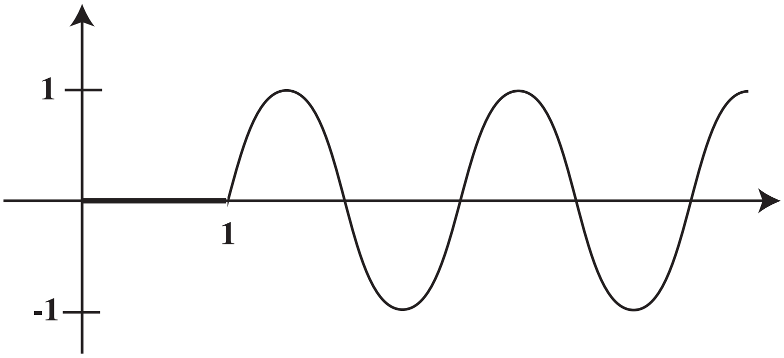


Figure 9.2: The sine wave $f(t)$ is “delayed" until $t = 1$

9.3 RESPONSE TO A SWITCHED INPUT

We can use the Heaviside function to describe an input that is switched on and off in a differential equation. This is one of the reasons the function is a favorite of electrical and systems engineers.

Exercise 9.1. Drug delivery
A patient receives morphine through an IV at the rate of $Q(t)$ ml/hr where

$$Q(t) = \begin{cases} 1 & 1 < t < 2 \\ 0 & 0 < t < 1 \text{ or } t > 2 \end{cases}$$

He excretes the drug at a rate proportional to the amount of morphine in the body. If the patient is initally drug-free, model the level of morphine in his body as a function of time.

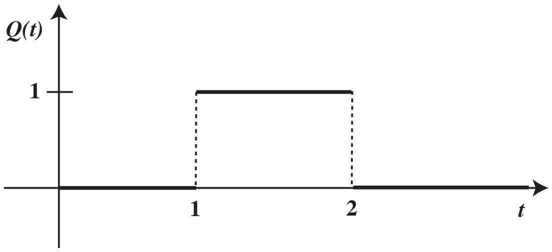


Figure 9.3: A graph of the morphine delivery rate, $Q(t)$.

Solution: Let $y(t) =$ amount of morphine in patient (ml) at time t. Then we can model the injection and excretion of morphine,

$$y'(t) = \underbrace{Q(t)}_{\text{input}} - \underbrace{ky(t)}_{\text{excretion}}, \qquad y(0) = 0$$

where k is the excretion rate in (hr)$^{-1}$ - remember that rate constants always have units of inverse time. Since the IV injection rate is $Q(t) = H(t-1) - H(t-2)$, we see that the governing equation can be written as

$$y' + ky = H(t-1) - H(t-2), \qquad y(0) = 0$$

which we can solve via a Laplace transform,

$$\begin{aligned}
\mathcal{L}\{y' + ky\} &= \mathcal{L}\{H(t-1) - H(t-2)\} \\
\mathcal{L}\{y'\} + k\mathcal{L}\{y\} &= \mathcal{L}\{H(t-1)\} - \mathcal{L}\{H(t-2)\} \\
s\mathbb{Y}(s) - \underbrace{y(0)}_{=0} + k\mathbb{Y}(s) &= \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \\
(s+k)\mathbb{Y}(s) &= \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \\
\mathbb{Y}(s) &= \frac{e^{-s}}{s(s+k)} - \frac{e^{-2s}}{s(s+k)}
\end{aligned}$$

We need to invert the transform to find $y(t)$, but first we need to decompose the function $1/s(s+k)$ via partial fractions,

$$\begin{aligned}
\frac{1}{s(s+k)} &= \frac{A}{s} + \frac{B}{s+k} \\
1 &= A(s+k) + Bs \\
Ak = 1 \quad \Rightarrow A = \frac{1}{k} \qquad & A = -B \quad \Rightarrow B = -\frac{1}{k} \\
\therefore \frac{1}{s(s+k)} &= \frac{1}{k}\left[\frac{1}{s} - \frac{1}{s+k}\right]
\end{aligned}$$

Now we can take the inverse transform pretty easily.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s(s+k)}\right\} &= \frac{1}{k}\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+k}\right\} \\
&= \frac{1}{k}\left(1 - e^{-kt}\right)
\end{aligned}$$

So

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+k)}\right\} = H(t-1)\left[\frac{1}{k}\left(1-e^{-k(t-1)}\right)\right]$$
$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+k)}\right\} = H(t-2)\left[\frac{1}{k}\left(1-e^{-k(t-2)}\right)\right]$$

and putting it all together yields

$$y(t) = H(t-1)\left(\frac{1-e^{-k(t-1)}}{k}\right) - H(t-2)\left(\frac{1-e^{-k(t-2)}}{k}\right) .$$

To graph $y(t)$, first graph $\left(1-e^{-kt}\right)/k$:

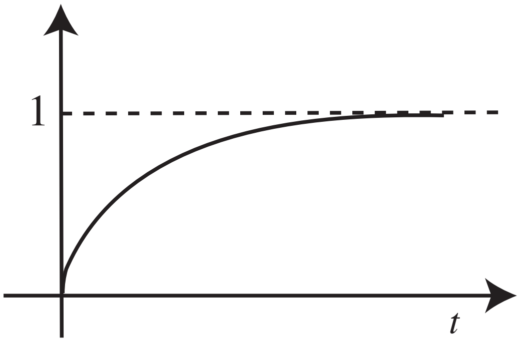
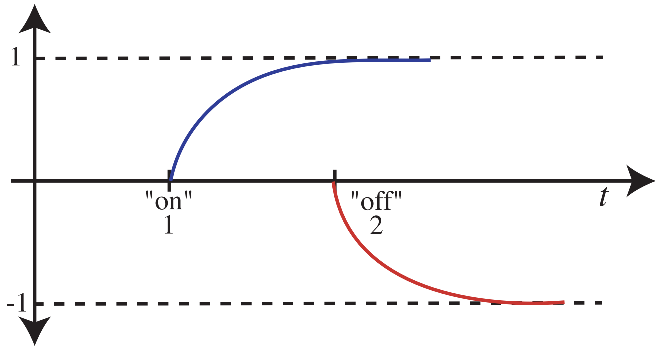


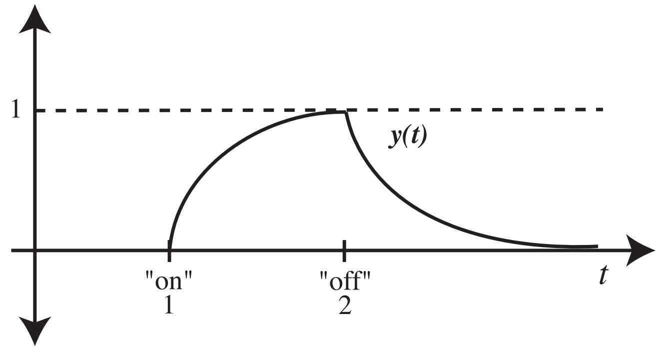
Figure 9.4: A graph of $\left(1-e^{-kt}\right)/k$ which would be the response to a constant unit input rate of morphine to the body.

This would be the response of the body to a constant drip, $Q(t) = 1$. The function approaches $1/k$ at large times which can be understood at the concentration where the flux in equals the rate excreted, that is $Q(t) = ky(t)$.

Now graph the function delayed one and two units in time:



The blue line is the graph of $H(t-1)[1-e^{-k(t-1)}]/k$ and the red line is the graph of $-H(t-2)[1-e^{-k(t-2)}]/k$. Now, graph the sum of the two functions: You can see that the function increases towards $1/k$ and then



decreases down to zero again.