An Introduction to Partial Differential Equations in the Undergraduate Curriculum

Andrew J. Bernoff

LECTURE 1
What is a Partial Differential Equation?

1.1. Outline of Lecture

- What is a Partial Differential Equation?
- Classifying PDE's: Order, Linear vs. Nonlinear
- Homogeneous PDE's and Superposition
- The Transport Equation

1.2. What is a Partial Differential Equation?

You've probably all seen an ordinary differential equation (ODE); for example the pendulum equation,

$$\frac{d^2\Theta}{dt^2} + \frac{g}{L}\sin\Theta = 0, \tag{1.1}$$

describes the angle, Θ, a pendulum makes with the vertical as a function of time, t. Here g and L are constants (the acceleration due to gravity and length of the pendulum respectively), t is the **independent variable** and Θ is the **dependent variable**. This is an ODE because there is only one independent variable, here t which represents time.

A partial differential equation (PDE) relates the partial derivatives of a function of two or more independent variables together. For example, Laplace's equation for $\Phi(x, y)$,

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{1.2}$$

arises in many places in mathematics and physics. For simplicity, we will use subscript notation for partial derivatives, so this equation can also be written $\Phi_{xx} + \Phi_{yy} = 0$.

We say a function is a **solution** to a PDE if it satisfy the equation and any side conditions given. Mathematicians are often interested in if a solution **exists** and when it is **unique**.

Exercise 1. Show that $\Phi_1 = x$ and $\Phi_2 = x^2 - y^2$ are solutions to Laplace's equation (1.2). How can you combine them to create a new solution?

Exercise 2. Show that

$$Z(x, y) = \ln\left(\frac{\sin(y)}{\sin(x)}\right)$$

is a solution to the **minimal surface equation**,

$$(1 + Z_y^2)Z_{xx} - 2Z_xZ_yZ_{xy} + (1 + Z_x^2)Z_{yy} = 0, \tag{1.3}$$

in the region $0 < x < \pi$, $0 < y < \pi$. What happens on the boundary of this region? Suppose we consider a constant multiple of $Z(x, y)$ – is it still a solution of the PDE?

1.3. Classifying PDE's: Order, Linear vs. Nonlinear

When studying ODEs we classify them in an attempt to group similar equations which might share certain properties, such as methods of solution. We classify PDE's in a similar way. The **order** of the differential equation is the highest partial derivative that appears in the equation. So, for example Laplace's Equation (1.2) is second-order.

Some other examples are the **convection equation** for $u(x,t)$,

$$u_t + Cu_x = 0, \tag{1.4}$$

which is first-order. Here C is the wave speed. The **minimal surface equation**,

$$(1 + Z_y^2)Z_{xx} - 2Z_xZ_yZ_{xy} + (1 + Z_x^2)Z_{yy} = 0, \tag{1.5}$$

describes an area minimizing surface, $Z(x,y)$, and is second-order. Finally, the **Korteweg-deVries equation** (sometimes called **KdV**),

$$h_t + 6hh_x = h_{xxx} \tag{1.6}$$

is a model of the amplitude of a wave, $h(x,t)$, on the surface of a fluid and is third-order.

We also define **linear** PDE's as equations for which the dependent variable (and its derivatives) appear in terms with degree at most one. Anything else is called **nonlinear**. So, for example, the most general first-order linear PDE for $u(x,t)$ would be

$$a(x,t)u_t + b(x,t)u_x + c(x,t)u = d(x,t), \tag{1.7}$$

where a, b, c and d are known functions (called coefficients).

Exercise 3. Which of Laplace's equation (1.2), the convection equation (1.4), the minimal surface equation (1.5) and the Korteweg-deVries equation (1.6) are linear?

Exercise 4. Write down the most general constant coefficient linear second-order equation for $\Phi(x,y)$.

1.4. Homogeneous PDE's and Superposition

Linear equations can further be classified as **homogeneous** for which the dependent variable (and it derivatives) appear in terms with degree **exactly** one, and non-homogeneous which may contain terms which only depend on the independent variable. So, the convection equation

$$u_t + cu_x = 0$$

is homogeneous, but its cousin, the general first-order linear PDE for $u(x,t)$, is non-homogeneous

$$a(x,t)u_t + b(x,t)u_x + c(x,t)u = d(x,t),$$

unless $d(x,t) = 0$.

Because partial differentiation is distributive, you can quickly convince yourself that if two solutions, say u_1 and u_2, satisfy a linear homogeneous PDE, that any linear combination of them

$$u = c_1 u_1 + c_2 u_2 \tag{1.8}$$

is also a solution. So, for example, since

$$\Phi_1 = x^2 - y^2 \qquad \Phi_2 = x$$

both satisfy Laplace's equation, $\Phi_{xx} + \Phi_{yy} = 0$, so does any linear combination of them

$$\Phi = c_1\Phi_1 + c_2\Phi_2 = c_1(x^2 - y^2) + c_2 x.$$

This property is extremely useful for constructing solutions which satisfy certain initial conditions and boundary conditions.

1.5. The Transport Equation

One of the driving motivations for studying PDE's is to describe the physical world around us. We can use a **flux argument** to derive equations describing the evolution of a **density**, which is just a fancy word describing the concentration of something (mass in a region, heat in a metal bar, traffic on a highway) per unit volume.

Consider a one-dimensional freeway and let $\rho(x,t)$ be the density of cars per unit length on the freeway.

Figure 1.1: Flux argument for cars on a freeway.
(draw your own figure).

Then the mass of cars in the region $a < x < b$ is given by

$$M = \int_a^b \rho(x,t)\ dx\ . \tag{1.9}$$

Now suppose we are measuring the flux, Q, of cars **into** this region measured in mass/unit time. It can written in terms of the number of cars crossing into the region at $x = a$, called $q(a)$, minus the number of cars that flow out of the region at $x = b$, called $q(b)$,

$$Q = q(a) - q(b). \tag{1.10}$$

Now, by **conservation of mass**, the rate of change of the mass between a and b is given by the flux into the region,

$$\frac{dM}{dt} = Q. \tag{1.11}$$

We can rewrite the flux by a clever application of the fundamental theorem of calculus:

$$Q = q(a,t) - q(b,t) = - \left. q(x,t) \right|_{x=a}^{x=b} = - \int_a^b q_x \; dx \; . \tag{1.12}$$

We can now rewrite the conservation of mass equation as

$$\frac{dM}{dt} = \frac{d}{dt} \int_a^b \rho \; dx = \int_a^b \rho_t \; dx = Q = - \int_a^b \; q_x \; dx, \tag{1.13}$$

or, rearranging

$$\int_a^b \rho_t + q_x \; dx = 0. \tag{1.14}$$

Since this is true for **every** interval $a < x < b$, the integrand must vanish identically. So

$$\rho_t + q_x = 0. \tag{1.15}$$

Equations of this form are called **transport equations** or **conservation laws** – they are a very active area of study in PDE's.

We can propose a simple model for the flux function $q(x,t)$ – suppose we assume the cars are all moving at a constant speed C. Then we can argue that the flux is just equal to the product of the number of cars time the speed they are moving at,

$$q(x,t) = C\rho(x,t). \tag{1.16}$$

Substituting into the transport equation yields

$$\rho_t + C\rho_x = 0, \tag{1.17}$$

which is just the convection equation. If we specify the initial distribution of cars,

$$\rho(x,0) = F(x), \tag{1.18}$$

we can show fairly easily that the solution to the convection equation with this initial condition is just

$$\rho(x,t) = F(x - Ct), \tag{1.19}$$

corresponding to cars moving uniformly to the right.

Physically, we just see the distribution of cars translating to the right with a speed of C.

Figure 1.2: Solution to the convection equation.
(draw your own figure).

To verify this solution let $\xi = x - Ct$, and look for a solution $F(\xi)$. Then, by the chain rule

$$F_t = F_\xi \xi_t = -CF_\xi \qquad F_x = F_\xi \xi_x = F_\xi \tag{1.20}$$

Substituting $\rho(x,t) = F(\xi)$ into the convection equation (1.17), we find

$$\rho_t + C\rho_x = F_t + CF_x = -CF_\xi + CF_\xi = 0. \tag{1.21}$$

Moreover, when $t = 0$, we find $\xi = x$ so that the initial condition $\rho(x,0) = F(x)$ is satisfied also.

1.6. Challenge Problems for Lecture 1

Problem 1. Classify the follow differential equations as ODE's or PDE's, linear or nonlinear, and determine their order. For the linear equations, determine whether or not they are homogeneous.

(a) The **diffusion equation** for $h(x,t)$:
$$h_t = Dh_{xx}$$
(b) The **wave equation** for $w(x,t)$:
$$w_{tt} = c^2 w_{xx}$$
(c) The **thin film equation** for $h(x,t)$:
$$h_t = -(hh_{xxx})_x$$
(d) The **forced harmonic oscillator** for $y(t)$:
$$y_{tt} + \omega^2 y = F\cos(\Omega t)$$
(e) The **Poisson Equation** for the electric potential $\Phi(x,y,z)$:
$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 4\pi\rho(x,y,z)$$
where $\rho(x,y,z)$ is a known charge density.

(f) **Burger's equation** for $h(x,t)$:
$$h_t + hh_x = \nu h_{xx}$$

Problem 2. Suppose when deriving the convection equation, we assumed the speed of the cars was given by βx for $x > 0$.

(a) Explain why the flux function now is given by $q(x,t) = \beta x\rho$ and the associated transport equation is given by
$$\rho_t + (\beta x\rho)_x = 0.$$
(b) Explain why
$$\rho(0,t) = 0, \qquad \rho(x,0) = xe^{-x}$$
correspond to a **boundary condition** of no flux of cars in from the origin and an **initial condition** specifying the distribution of cars at $t = 0$.

(c) Verify that
$$\rho(x,t) = xe^{-(2\beta t + xe^{-\beta t})}$$
is a solution to both the transport equation given in (a) and the initial and boundary conditions given in (b).

Problem 3. Show that the helicoid

$$Z(x,y) = \tan^{-1}(y/x)$$

satisfies the minimal surface equation,

$$(1+Z_y^2)Z_{xx} - 2Z_xZ_yZ_{xy} + (1+Z_x^2)Z_{yy}$$

MAPLE may be helpful with the algebra.

Problem 4. Show that the soliton

$$h(x,t) = 2\alpha^2 \text{sech}\left(\alpha(x-4\alpha^2 t)\right)$$

satisfies the the Korteweg-deVries equation,

$$h_t + 6hh_x = h_{xxx}$$

MAPLE may be helpful with the algebra, in particular if you don't remember your hyperbolic trigonometric identities.