HW#2

$\frac{100}{100}$

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MATH 61 - SECTION 2
PROF. GU
HW #2 - DUE 5/19/04

SECTION 4.2 (7, 10, 20, 22(a), 29, 32)

Identify & determine the nature of the critical points of the given functions:

7) $f(x,y) = xy + \frac{8}{x} + \frac{1}{y}$

$+\frac{15}{15}$ DEB

$\frac{\delta f}{\delta x} = y - \frac{8}{x^2}$ $\qquad \frac{\delta f}{\delta y} = x - \frac{1}{y^2} = 0 \Rightarrow x = \frac{1}{y^2} \Rightarrow$

$y - \frac{8}{x^2} = 0 \Rightarrow y = \frac{8}{x^2}$

$y - \frac{8}{\left(\frac{1}{y^2}\right)^2} = 0 \Rightarrow y - 8y^4 = 0$

$y(1-8y^3) = 0$

when $y = 0$ or

$y^3 = \frac{1}{8} \Rightarrow y = \frac{1}{2}$

but if $y = 0$, x D.N.E.
so $y = \frac{1}{2}$, $x = 4$

CRITICAL POINT: $(4, \frac{1}{2})$

$Hf(a,b) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} \frac{16}{x^3} & 1 \\ 1 & \frac{2}{y^3} \end{bmatrix}$ $\quad Hf(4, \frac{1}{2}) = \begin{bmatrix} \frac{1}{4} & 1 \\ 1 & 16 \end{bmatrix}$

$D = f_{xx} f_{yy} - (f_{xy})^2 = 4 - 1 = 3$

So since $D > 0$ & $f_{xx} > 0$, this is a Minimum.

Great Explanations!

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10) $f(x,y) = (x+y)(1-xy) = x - x^2y + y - xy^2$ HW#2

$\frac{+15}{16}$ JB

$\frac{\delta f}{\delta x} = 1 - 2xy - y^2$

$= 1 - y(2x+y) = 0$

$y(2x+y) = 1$

$2x + y = \frac{1}{y} \Rightarrow 2x = \frac{1}{y} - y$

$2xy = 1 - y^2 \Rightarrow x = \frac{1-y^2}{2y}$

$\frac{\delta f}{\delta y} = -x^2 + 1 - 2xy$

$= 1 - x(x+2y) = 0$

$x(x+2y) = 1$

$x + 2y = \frac{1}{x} \Rightarrow 2y = \frac{1}{x} - x$

$4y^2\left(-\left(\frac{1-y^2}{2y}\right)^2 + 1 - 2\left(\frac{1-y^2}{2y}\right)y\right) = -(1-y^2)^2 + 4y^2 - 4y^2(1-y^2) = 0$

$\Rightarrow -(1^2 - 2y^2 + y^4) + 4y^2 - 4y^2 + 4y^4 = 0 \Rightarrow 2y^2 + 3y^4 - 1 = 0$

$3y^4 + 2y^2 - 1 = 0 \qquad (3y^2-1)(y^2+1) = 0 \qquad y^2 = \frac{1}{3} \text{ or } -1$

Ignoring the complex term, $y = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$

$x = \frac{1 - \frac{1}{3}}{\pm\frac{2}{\sqrt{3}}} = \pm\frac{\frac{2}{3}}{\frac{2}{\sqrt{3}}} = \pm\frac{3}{\sqrt{3}} = \pm\sqrt{3}$

CRITICAL POINTS: $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ OR $\left(\frac{-\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right)$

$Hf(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -2y & -2x-2y \\ -2x-2y & -2x \end{bmatrix}$

$Hf\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) = \begin{bmatrix} -\frac{2\sqrt{3}}{3} & -\frac{4\sqrt{3}}{3} \\ -\frac{4\sqrt{3}}{3} & -\frac{2\sqrt{3}}{3} \end{bmatrix} \qquad Hf\left(\frac{-\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right) = \begin{bmatrix} \frac{2\sqrt{3}}{3} & \frac{4\sqrt{3}}{3} \\ \frac{4\sqrt{3}}{3} & \frac{2\sqrt{3}}{3} \end{bmatrix}$

$D = f_{xx}f_{yy} - (f_{xy})^2 \quad D\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) = \frac{4}{3} - \frac{16}{3} = \frac{-12}{3} = -4 = D\left(\frac{-\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right)$

So both are saddle points

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20) $f(x,y,z) = e^x(x^2 - y^2 - 2z^2)$

$+\frac{15}{15}$ DEB

$\frac{\delta f}{\delta x} = 2xe^x + x^2e^x - e^xy^2 - e^x2z^2$ $\quad \frac{\delta f}{\delta y} = -2ye^x$ $\quad \frac{\delta f}{\delta z} = -4ze^x$

$2xe^x + x^2e^x = 0$ if $x = 0$ or $x = -2$ $\quad = 0$ if $y = 0$ $\quad = 0$ if $z = 0$

$xe^x(2+x)$

$Df(x,y,z) = [2xe^x + x^2e^x \quad -2ye^y - y^2e^x \quad -4ze^x - 2z^2e^x] = [0\ 0\ 0]$

Critical points: $(0,0,0)$, $(-2,0,0)$

$$Hf(x,y,z) = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 4xe^x + 2e^x + x^2e^x - e^xy^2 - e^x2z^2 & -2ye^x & -4ze^x \\ -2ye^x & -2e^x & 0 \\ -4ze^x & 0 & -4e^x \end{bmatrix}$$

$$Hf(0,0,0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \quad Hf(-2,0,0) = \begin{bmatrix} -.271 & 0 & 0 \\ 0 & -.271 & 0 \\ 0 & 0 & -.541 \end{bmatrix}$$

Sequence of principal minors:

$$d_1 = f_{xx} \quad d_2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \quad d_3 = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$$

$d_1(0,0,0) = 2 \quad d_1(-2,0,0) = -.271$

$d_2(0,0,0) = -4 \quad d_2(-2,0,0) = 0.073$

$$d_3(0,0,0) = f_{xx}\begin{vmatrix} f_{yy} & f_{yz} \\ f_{zx} & f_{zy} \end{vmatrix} - f_{xy}\begin{vmatrix} f_{yx} & f_{yz} \\ f_{zx} & f_{zz} \end{vmatrix} + f_{xz}\begin{vmatrix} f_{yx} & f_{yy} \\ f_{zx} & f_{zy} \end{vmatrix}$$

$$= 2(8-0) - 0 - 0 = 16$$

$$d_3(-2,0,0) = f_{xx}\begin{vmatrix} f_{yy} & f_{yz} \\ f_{zx} & f_{zy} \end{vmatrix} = -0.03966$$

For $(0,0,0)$ $d_k > 0$ for k odd & $d_k < 0$ for k even so $(0,0,0)$ is a saddle point.

For $(-2,0,0)$ $d_k < 0$ for k odd & > 0 for k even so $(-2,0,0)$ is a local maximum.

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22) a) Under what conditions on the constant k will the $f(x,y) = kx^2 - 2xy + ky^2$ have a nondegenerate local minimum at (0,0)? What about a local maximum?

$+\frac{15}{15}$ JB

Nondegenerate means that $Hf(a) \neq 0$.

$\frac{\delta f}{\delta x} = 2kx - 2y$ $\frac{\delta f}{\delta y} = 2ky - 2x$

$Hf(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2k & -2 \\ -2 & 2k \end{bmatrix}$ $Hf(0,0) = \begin{bmatrix} 2k & -2 \\ -2 & 2k \end{bmatrix}$

$D = f_{xx}f_{yy} - (f_{xy})^2 = 4k^2 - 4$

The function will have a minimum when $D > 0$ & $f_{xx} > 0$ so when $k > 0$ & $k > 1$

So will have a minimum when $k > 1$

The fn will have a maximum when $D > 0$ & $f_{xx} < 0$

so when $k < -1$

29) What point on the plane $3x - 4y - z = 24$ is closest to the Origin?

$+\frac{20}{20}$ DEB

$d(x,y,z) = \sqrt{x^2+y^2+z^2}$

Consider $D(x,y,z) = d^2(x,y,z) = x^2 + y^2 + z^2$

Minimizing d is equivalent to minimizing D. Good!

(x,y,z) is on the plane: $z = 3x - 4y - 24$

$\underbrace{D(x, y, 3x-4y-24)}_{f(x,y)} = x^2 + y^2 + (3x - 4y - 24)^2$

oh, sorry - I think you want to include D

$= x^2 + y^2 + 9x^2 - 12xy - 72x - 12xy + 16y^2 + 96y - 72x + 96y + 576$

$= 10x^2 + 17y^2 - 24xy - 144x + 192y + 576$

$\frac{\delta f}{\delta y} = 20x - 24y - 144 = 4(5x - 6y - 36) = 0$ $\frac{\delta f}{\delta y} = 34y - 24x + 192 = 2(17y - 12x + 96) = 0$

CRITICAL POINT: $(36/13, -48/13)$

$Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 20 & -24 \\ -24 & 34 \end{bmatrix}$ $D = f_{xx}f_{yy} - (f_{xy})^2 = 680 - 576 = 104 > 0$

$D > 0$ & $f_{xx} > 0$ so this is a minimum. good!

Closest point to origin $= \left(\frac{36}{13}, -\frac{48}{13}\right)$

I'd give z-coordinate, but ok since it's on a plane

(4.)

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32) A metal plate has the shape of the region $x^2+y^2 \le 1$. The plate is heated so that the temperature at any point (x,y) on it is indicated by

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JB

$$T(x,y) = 2x^2 + y^2 - y + 3$$

Find the hottest & coldest points on the plate, & the temperature at each of these points. (Hint: parametrize the boundary of the plate in order to find any critical points there.)

Plate is round: $(\overset{x=}{\cos t}, \overset{y=}{\sin t})$ Boundary when $0 \le t \le 2\pi$

$r = 1$, is greatest

$\frac{\delta T}{\delta x} = 4x \quad \frac{\delta T}{\delta y} = 2y - 1 \quad \frac{\delta T}{\delta x} = 0$ when $x=0$ $\quad \frac{\delta T}{\delta y} = 0$ when $y = \frac{1}{2}$

Critical point: $(0, \frac{1}{2})$ = Local min. b/c $D > 0$ & $f_{xx} > 0$.

Restriction on T to the boundary of the plate.

$T(\cos t, \sin t) \Rightarrow T(x,y) = 2\cos^2 t + \sin^2 t - \sin t + 3$

$$\frac{dT}{dt} = -4\sin t\cos t + 2\sin t\cos t - \cos t = 0$$
$$= -2\sin t\cos t - \cos t = 0$$
$$\Rightarrow \cos t(-2\sin t - 1) = 0 \text{ when } \cos t = 0$$
$$\text{or } -2\sin t = 1 \text{ so when } \sin t = -\tfrac{1}{2}$$

$t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ or $\frac{7\pi}{6}$ or $\frac{11\pi}{6}$ Plug into $(\cos t, \sin t)$

Critical points: $t = \frac{\pi}{2}$ $(0,1)$ & $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$ $\frac{7\pi}{6}$

$\frac{3\pi}{2}$: $(0,-1)$ & $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ $\frac{11\pi}{6}$

$T' = -\sin 2t - \cos t \Rightarrow T'' = -2\cos 2t + \sin t$

$T''(\frac{\pi}{2}) = 2 + 1 = 3$ >0 = min $\quad T''(\frac{3\pi}{2}) = 2 - 1 = 1$ >0 = min

$T''(\frac{7\pi}{6}) = -1 - \frac{1}{2} = -\frac{3}{2}$ <0 = max $\quad T''(\frac{11\pi}{6}) = -1 - \frac{1}{2} = -\frac{3}{2}$ <0 = max

$(0,1)$ is local minimum $\quad (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$ is local min.

$(0,-1)$ is local max $\quad (\frac{\sqrt{3}}{2}, -\frac{1}{2})$ is local max

$T(\frac{\pi}{2}) = 0 + 1 - 1 + 3 = 3 \quad T(\frac{3\pi}{2}) = 0 + 1 + 1 + 3 = 5$

$T(\frac{7\pi}{6}) = \frac{6}{4} + \frac{1}{4} + \frac{1}{2} + 3 = \frac{21}{4} \quad T(\frac{11\pi}{6}) = \frac{6}{4} + \frac{1}{4} + \frac{1}{2} + 3 = \frac{21}{4}$

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t_0	$T(t_0)$
$\frac{\pi}{2}$	3
$\frac{3\pi}{2}$	5
$\frac{7\pi}{6}$	$\frac{21}{4}$
$\frac{11\pi}{6}$	$\frac{21}{4}$

$$T(x,y) = 2x^2 + y^2 - y + 3$$

$$T(0, \tfrac{1}{2}) = \tfrac{1}{4} - \tfrac{1}{2} + 3 = \tfrac{11}{4}$$

Local mins @ $(0,1)$, $(0,-1)$

Another min @ $(0, \frac{1}{2})$

Local max's @ $(\frac{-\sqrt{3}}{2}, \frac{-1}{2})$, $(\frac{\sqrt{3}}{2}, \frac{-1}{2})$

At $(0,1)$, $T=3$

At $(0,-1)$, $T=5$

At $(0,\frac{1}{2})$, $T=\frac{11}{4}$

So minimum temperature is at $(0, \frac{1}{2})$.

At $(\frac{-\sqrt{3}}{2}, \frac{-1}{2})$, $T = \frac{21}{4}$

At $(\frac{\sqrt{3}}{2}, \frac{-1}{2})$, $T = \frac{21}{4}$

So maximum temperature is at $(\frac{\pm\sqrt{3}}{2}, \frac{-1}{2})$