HW 4
5.5 (1, 4, 6, 8, 10, 11, 16 19)

$\frac{100}{100}$!

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Math 61
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5.5 (1)

Let $\vec{T}(u,v) = (3u, -v)$

(a) Write $\vec{T}(u,v)$ as $A\begin{bmatrix}u\\v\end{bmatrix}$ for a suitable matrix A.

(b) Describe the image $D = \vec{T}(D^*)$, where D^* is the unit square $[0,1]\times[0,1]$.

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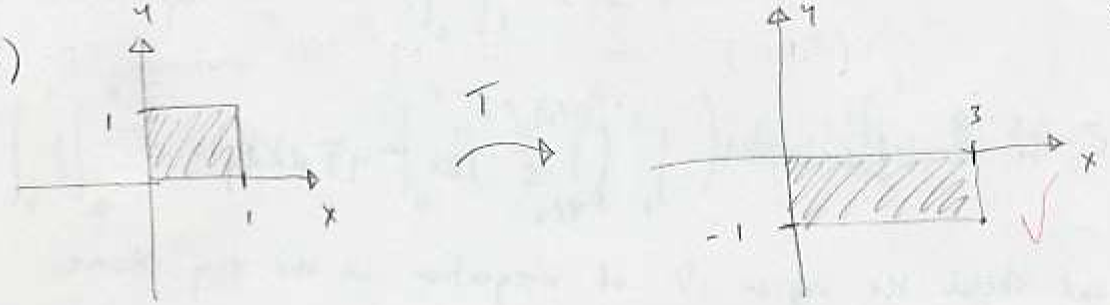
(a) $\vec{T}(u,v) = \underbrace{\begin{bmatrix}3 & 0\\ 0 & -1\end{bmatrix}}_{2\times2}\underbrace{\begin{bmatrix}u\\v\end{bmatrix}}_{2\times1} = \underbrace{\begin{bmatrix}3u\\-v\end{bmatrix}}_{2\times1}$ ✓

(b)

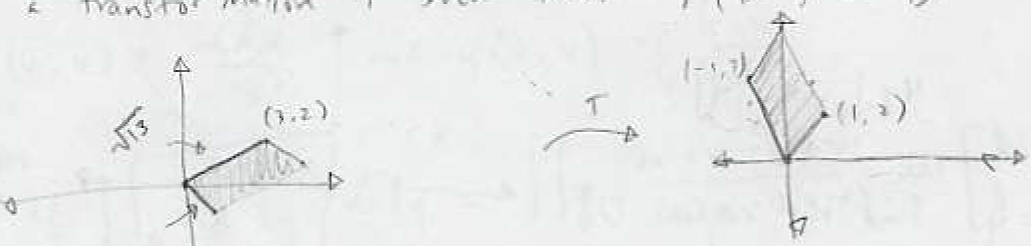


as $\vec{T}(1,1) = (3,-1)$
$T(1,0) = (3,0)$
$T(0,1) = (0,-1)$
$T(0,0) = (0,0)$.

✓

5.5 (4)

If D^* is the parallelogram whose vertices are $(0,0)$, $(-1,3)$, $(1,2)$, & $(0,5)$ and D is the parallelogram whose vertices are $(0,0)$, $(3,2)$, $(1,-1)$, & $(4,1)$ find a transformation $\vec{T}$ such that $T(D^*) = D$



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$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$

$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$

$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0\\5\end{bmatrix} = \begin{bmatrix}4\\1\end{bmatrix}$

$-a + 3b = 3$
$-c + 3d = 2$
$a + 2b = 1$
$c + 2d = -1$
$5b = 4$
$5d = 1$

$b = \frac{4}{5}$
$d = \frac{1}{5}$

$-a + \frac{12}{5} = \frac{15}{5}$
$a = -\frac{3}{5}$
$-c + \frac{3}{5} = \frac{10}{5}$
$c = -\frac{7}{5}$

Let T be defined by $T(u,v) = \begin{bmatrix}-\frac{3}{5} & \frac{4}{5}\\ -\frac{7}{5} & \frac{1}{5}\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}$ ✓

6. Suppose $T(u,v) = (u, uv)$. Explain (perhaps using pictures) how T transforms the unit square $D^* = [0,1] \times [0,1]$. Is T one-one on D^*?

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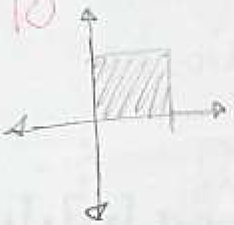
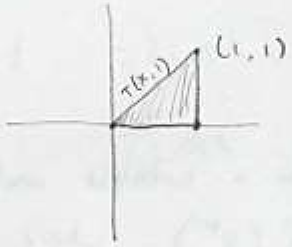


$\xrightarrow{T}$



all points $(1,y)$

$T(1,y) = (1,y)$

So $x=1$ is sent to $x=1$.

$y=1$:

$T(x,1) = (x,x)$

$x=0$:

$T(0,y) = (0,0)$

likewise, $T(x,0) = (x,0) \Rightarrow y=0$ is fine

T is not 1-1 as the entire line $x=0$ gets sent to $(0,0)$. ✓

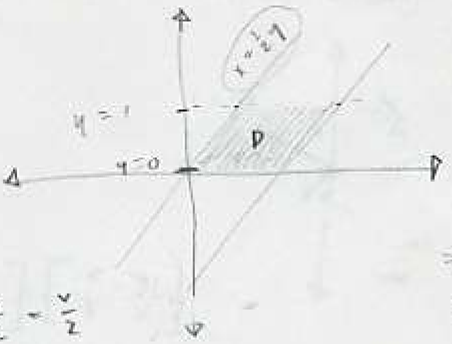
8. This problem concerns the iterated integral $\int_0^1 \int_{y/2}^{(y/2)+2} (2x - y)\,dx\,dy$ $\frac{20}{20}$

(a) Evaluate this integral and sketch the region D of integration in the x-y plane

$$\int_0^1 \int_{y/2}^{(y/2)+2} (2x-y)\,dx\,dy = \int_0^1 \left(x^2 - yx \Big|_{y/2}^{(y/2)+2} \right) dy =$$

$$\int_0^1 \left(\frac{y}{2}+2\right)^2 - y\left(\frac{y}{2}\right) - 2y - \left(\frac{y}{2}\right)^2 + \frac{y^2}{2}\,dy = \int_0^1 \frac{y^2}{4} + \frac{2\cdot 2y}{2} + 4 - \frac{y^2}{2} - 2y - \frac{y^2}{4} + \frac{y^2}{2}$$

$$= \int_0^1 4\,dy = 4y\Big|_0^1 = \boxed{4}$$ ✓



$x = \frac{1}{2}y$

$\Rightarrow y = 2x$

$x = \frac{1}{2}y + 2$

$\Rightarrow y = 2(x-2)$

$= 2x - 4$

b) Let $u = 2x - y$ and $v = y$. Find the region D^* in the u-v plane that corresponds to v.

$D^* = [0,4] \times [0,1]$ ✓

$x = \frac{y}{2} = \frac{v}{2}$

$u = 2\left(\frac{v}{2}\right) - v = 0$

$u = 2\left(\frac{v}{2} + 2\right) - v = 4$

(c) Use the change of variables thm to evaluate the integral by using the substitution $u = 2x - y$, $v = y$

THM: $\iint_D f(x,y)\,dx\,dy = \iint_D f(x(u,v),\ y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

In this case, $\iint_D f(x,y)\,dx\,dy = \int_0^1 \int_{y/2}^{(y/2+2)} 2x - y\ dx\,dy$

$\begin{cases} x(u,v): \\ y(u,v \end{cases}$ $u = 2x - y$ $y = v$

$x = \frac{u+v}{2}$

and $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \det\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = \frac{1}{2}$

So the integral becomes $\int_0^1 \int_0^4 \cancel{2}\left(\frac{u+v}{\cancel{2}}\right) - v\ du\,dv\left(\frac{1}{2}\right)$

$= \int_0^1 \frac{1}{2}\int_0^4 u\,du\,dv = \int_0^1 \frac{1}{2}\left(\frac{u^2}{2}\Big|_0^4\right) dv = \frac{1}{2}\int_0^1 8\ dv = \frac{1}{2}\left(8v\Big|_0^1\right) = \frac{1}{2}(8)$

$= \boxed{4}$ ✓ with above

10. Determine the value of

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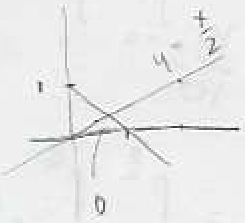
$\iint_D \sqrt{\frac{x+y}{x-2y}}\ dA$, where D is the region in $\mathbb{R}^2$ enclosed by the lines $y = \frac{x}{2}$, $y = 0$, and $x + y = 1$. call $u = x + y$ and $v = x - 2y$ ✓

$x(u,v) = \frac{2u+v}{3}$ and $y(u,v) = \frac{u-v}{3}$



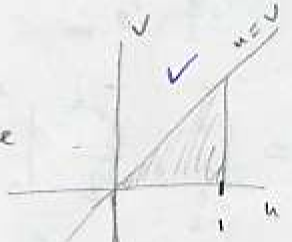
by THM above, $\iint_D \sqrt{\frac{x+y}{x-2y}}\ dx\,dy = \iint_D \frac{2u+v-u-v}{2u+v-2(u-v)} \det\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} du\,dv$

Compute bounds on u and v, considering the boundaries of D.

$x + y = 1 \iff u = 1$

$y = \frac{x}{2} \iff x - 2y = 0 \iff v = 0$ so D^* looks something like

$y = 0 \iff \frac{u-v}{3} = 0 \iff u = v$



and $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = \left|-\frac{2}{9} - \frac{1}{9}\right| = \frac{1}{3}$, giving the resulting integral

$\frac{1}{3}\int_0^1 \int_0^u \sqrt{\frac{u}{v}}\ dv\,du$ ✓ $= \frac{1}{3}\int_0^1 2v^{\frac{1}{2}}\sqrt{u}\Big|_0^u du = \frac{1}{3}\int_0^1 2u^{\frac{1}{2}}u^{\frac{1}{2}} = \frac{1}{3}\int_0^1 2u = \frac{u^2}{3}\Big|_0^1 = \boxed{\frac{1}{3}}$ ✓

11. Evaluate $\iint_D (2x+y)^2 e^{x-y}\,dA$, where D is the region enclosed by $2x+$

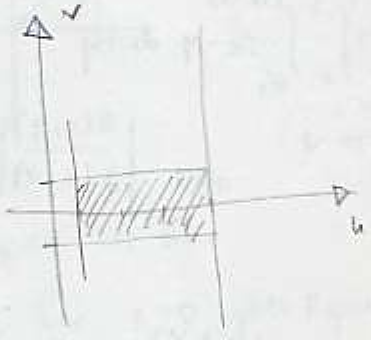
$\frac{10}{10}$ $2x+y=4$, $x-y=-1$, and $x-y=1$.

let $u = 2x+y$ ✓ $\Rightarrow$ $x = \frac{1}{3}u + \frac{1}{3}v$. This gives $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{vmatrix} = \left| -\frac{2}{9} - \frac{1}{9} \right|$

$v = x-y$. $\quad y = \frac{1}{3}u - \frac{2}{3}v$

$= \frac{1}{3}$ ✓

find bounds on u and v:



$2x+y=4 \iff u=4$

$x-y=-1 \iff v=-1$

$x-y=1 \iff v=1$

$2x+y=1 \iff u=1$

this gives the integral $\frac{1}{3}\int_{-1}^{1}\int_{1}^{4} u^2 e^v \,du\,dv$ ✓ $= \frac{1}{3}\int_{-1}^{1} e^v \left(\frac{u^3}{3}\Big|_1^4 \right) dv$

$= \frac{1}{3}\int_{-1}^{1} e^v \left(\frac{64-1}{3} \right) dv = 7\int_{-1}^{1} e^v dv = \boxed{7\left(e - \frac{1}{e}\right)}$ ✓

16. Transform the given integral in Cartesian coordinates to one in polar coordinates, and evaluate the polar integral. $\quad r^2 = x^2+y^2$, $x = \cos\theta$ $\quad y = \sin\theta$

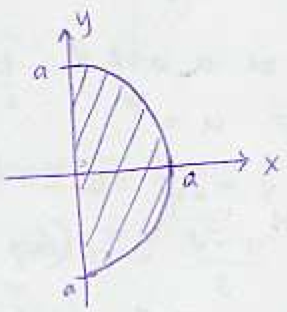
$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} e^{x^2+y^2}\,dx\,dy$ $\quad x = \sqrt{a^2-y^2} \Rightarrow r = a$

$\frac{10}{10}$ $\quad -a = y$ $\Rightarrow$ θ goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$a = y$

and we know $dx\,dy = r\,dr\,d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{a} e^{r^2} r\,dr\,d\theta$ ✓

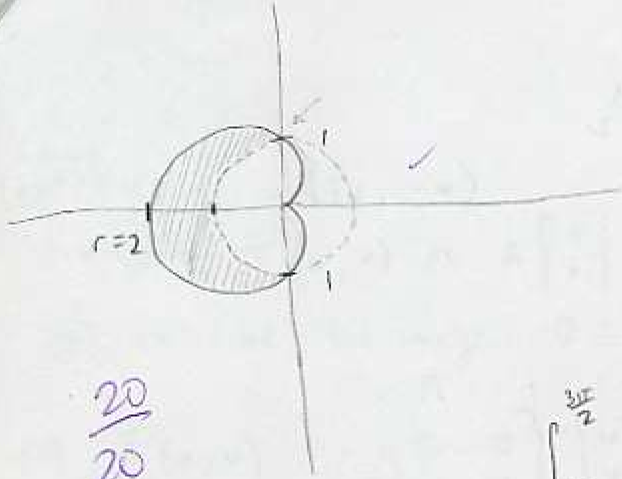


$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{r^2}}{2}\Big|_0^a d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{a^2} - 1\,d\theta$

$= \frac{1}{2}e^{a^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta - \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta$

$= \frac{1}{2}\pi\left(e^{a^2} - 1\right)$ ✓

1. Find the area of the region inside the cardioid $r = 1 - \cos\theta$ and outside the circle $r = 1$.



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r restricted from 1 to $1-\cos\theta$, to be ~~circle = 1~~ outside circle but inside cardioid.

restrict θ from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$

Integrate across $dA = r\,dr\,d\theta$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{1}^{1-\cos\theta} r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{r^2}{2} \Big|_{1}^{1-\cos\theta} \right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1-\cos\theta)^2 - 1 \; d\theta = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cancel{1} - 2\cos\theta + \cos^2\theta - \cancel{1} \; d\theta$$

$$\underset{\text{by TI-89}}{=} \boxed{\frac{\pi}{2} + 4}$$