95/100 EJ NH

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HW 6
multi V
Sect 1

6.3 1, 8, 12, 13, 15, 19 6.4 20, 22

① Consider the line int $\int_C z^2dx + 2ydy + xzdz$

a) Evaluate this integral, where C is the line seg from (0,0,0) to (1,1,1)

line seg can be parameterized by $(x,y,z) = (t,t,t)$

thus, $\int_0^1 t^2 + 2t + t^2\,dt = \frac{t^3}{3} + t^2 + \frac{t^3}{3}\Big|_0^1 = \frac{1}{3} + 1 + \frac{1}{3} = \boxed{\frac{5}{3}}$ ✓

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b) Evaluate this int where C is the path from (0,0,0) to (1,1,1) parametrized by $\vec{x}(t) = (t, t^2, t^3)$, $0 \le t \le 1$.

$\int_0^1 t^6\,d(t) + 2t^2\,d(t^2) + t^4\,d(t^3) = \int_0^1 t^6 + 4t^3 + 3t^6\,dt$

$= \frac{t^7}{7} + t^4 + \frac{3t^7}{7} = \boxed{\frac{11}{7}}$ ✓

c) Is the vector field $F = z^2 i + 2y j + xz K$ conservative? Why or why not?

F is not conservative because path integrals over F should be the same if F is cons but as seen in part a) and b) two path integrals with F ~~can~~ have different value, thus F isn't cons. same endpoints

⑧ determine whether the given vector field $\vec{F}$ is conservative. If it is, find a scalar potential function for $\vec{F}$.

$\vec{F} = (y+z)i + 2z j + (x+y)K$

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$\vec{F}$ is cons if curl $\vec{F} = 0$ thus:

$$\text{curl}\vec{F} = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & 2z & x+y \end{vmatrix} = i(1-2) + j(1-1) + (0-1)K = -1i - K$$

thus the curl $\vec{F} \neq 0$ so $\vec{F}$ is not conservative ✓

⑫ Of the two vector fields $\vec{F} = xy^2z^3 i + 2x^2y j + 3x^2y^2z^2 K$ and $\vec{G} = 2xy i + (x^2+2yz) j + y^2 K$, one is conservative and one is not. Determine which is which, and, for the conservative field find a scalar potential function.

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^3 & 2x^2y & 3x^2y^2z^2 \end{vmatrix} = i(6x^2z^2y - 0) + j(6y^2z^2x - 3xy^2z^2) + K(4xy - 2yxz^3)$$

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$$= 6x^2z^2y\, i + 3y^2z^2x\, j + 4xy - 2yxz^3\, K$$

$\therefore \vec{F}$ is not cons

$$\text{curl } \vec{G} = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2+2yz & y^2 \end{vmatrix} = i(\cancel{2y-2y})^{0} + j(\cancel{0-0})^{0} + K(\cancel{2x-2x})^{0} = 0$$

$\therefore \vec{G}$ is conservative ✓

find scalar potential function for $\vec{G}$

$g_x = 2xy \Rightarrow g = x^2y + i(y,z)$

$g_y = x^2+2yz \Rightarrow g = x^2y + y^2z + h(x,z)$

$g_z = y^2 \Rightarrow g = y^2z + j(x,y)$

✓

therefore $\boxed{g = x^2y + y^2z + C}$

(13) Let $\vec{F} = x^2 i + \cos y \sin z\, j + \sin y \cos z\, k$.

a) Show that $\vec{F}$ is conservative and find a scalar pot function f for $\vec{F}$.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & \cos y \sin z & \sin y \cos z \end{vmatrix} = i\big(\cancelto{0}{\cos y \cos z - \cos y \cos z}\big) + j(0-0) + k(0-0) = 0$$

✓

thus, since $\operatorname{curl}\vec{F} = 0$, $\vec{F}$ is conservative

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$f_x = x^2 \Rightarrow f = \frac{x^3}{3} + g(y,z)$

$f_y = \cos y \sin z \Rightarrow f = \sin y \sin z + h(x,z)$

$f_z = \sin y \cos z \Rightarrow f = \sin y \sin z + i(x,y)$

$\therefore \boxed{f = \frac{x^3}{3} + \sin y \sin z + C}$ ✓

b) Evaluate $\int_x \vec{F} \cdot d\vec{s}$ along the path $x : [0,1] \to \mathbb{R}^3$, $x(t) = (t^2+1, e^t, e^{2t})$.

$$\int_0^1 \vec{F} \cdot (2t, e^t, 2e^t) = \int_0^1 (t^2+1, \cos e^t \sin e^{2t}, \sin e^t \cos e^{2t}) \cdot (2t, e^t, 2e^t)$$

can't do

since cons can use Thm 3.3 when $x=0$ $x(0) = (1,1,1) = A$

when $x=1$ $x(1) = (2, e, e^2) = B$

$$\int_C \vec{F} \cdot ds = f(B) - f(A) = \frac{8}{3} + \sin(e)\sin(e^2) + C + \frac{1}{3} - \sin 1 \sin 1 - C$$

$$= \frac{7}{3} + \sin(e)\sin(e^2) - \sin(1)^2$$ ✓

Show that the line int are path ind, and eval them along given curve and by Thm 3.3.

(15) $\int_C \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$; C is the semicircular arc of $x^2+y^2=4$ from $(2,0)$ to $(-2,0)$

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix} \quad i(0-0) + j(0-0) + K\left(\frac{-yx}{(y^2+x^2)^{3/2}} + \frac{yx}{(y^2+x^2)^{3/2}}\right)$$
$$= 0$$

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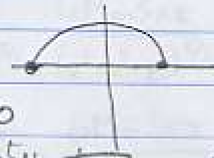
thus $\vec{F}$ is conservative and thus the line integral is path independent ✓

parameterize curve $x(t) = (2\cos t, 2\sin t)$



$$\int_0^{\pi} \frac{2\cos t(-2\sin t) + 2\sin t(2\cos t)\,dt}{\sqrt{4\cos^2 t + 4\sin^2 t}} = \int_0^{\pi} \frac{\cancelto{0}{4\sin t\cos t - 4\cos t\sin t}}{2}\,dt = \boxed{0}$$ ✓

find f for $\frac{xi + yj}{\sqrt{x^2+y^2}}$

$f_x = \frac{x}{\sqrt{x^2+y^2}} \Rightarrow f = \sqrt{x^2+y^2} + g(y) + C$

thus $f = \sqrt{x^2+y^2} + C$

$f_y = \frac{y}{\sqrt{x^2+y^2}} \qquad f = \sqrt{x^2+y^2} + h(x) + C$

by Thm 3.3 $\int_C F \cdot ds = f(B) - f(A) = \sqrt{4\cos^2\pi + 4\sin^2\pi} + C - \sqrt{4\cos^2 0 + 4\sin^2 0} - C$

$$= \sqrt{4} + C - \sqrt{4} - C = \boxed{0}$$ ✓

(19) Let $\vec{F}$ be the grav force field of a mass M on a part of mass m:

$$\vec{F} = -\frac{GMm}{(x^2+y^2+z^2)^{3/2}}(xi+yj+zk)$$

Given that G, M, m are all constants, show that the work done by $\vec{F}$ as a particle of mass m moves from $\vec{x}_0=(x_0,y_0,z_0)$ to $\vec{x}_1=(x_1,y_1,z_1)$ depends only on $\|\vec{x}_0\|$ and $\|\vec{x}_1\|$.

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$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-GMmx}{(x^2+y^2+z^2)^{3/2}} & \frac{-GMmy}{(x^2+y^2+z^2)^{3/2}} & \frac{-GMmz}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix} = i\left(\frac{GMm3yz}{(y^2+x^2+z^2)^{5/2}} - \frac{GMm3yz}{(y^2+x^2+z^2)^{5/2}}\right)$$
$$+j\left(\frac{GMm3xz}{(y^2+x^2+z^2)^{5/2}} - \frac{GMm3xz}{(y^2+x^2+z^2)^{5/2}}\right)$$
$$+k\left(\frac{GMm3xy}{(y^2+x^2+z^2)^{5/2}} - \frac{GMm3xy}{(y^2+x^2+z^2)^{5/2}}\right)$$
$$= 0 \checkmark$$

thus $\vec{F}$ is cons and thm 3.3 applies

find f: $f_x = \frac{-GMmx}{(x^2+y^2+z^2)^{3/2}} \Rightarrow f = \frac{GMm}{\sqrt{x^2+y^2+z^2}}$ ✓

this will appear for all partial derivatives

$\therefore f = \frac{GMm}{\sqrt{x^2+y^2+z^2}} + C$

thus by Thm 3.3

$$\text{work} = \int_C \vec{F}\cdot ds = f(x_1,y_1,z_1) - f(x_0,y_0,z_0) = \frac{GMm}{\sqrt{x_1^2+y_1^2+z_1^2}} + C - \frac{GMm}{\sqrt{x_0^2+y_0^2+z_0^2}} - C$$

$$= \frac{GMm}{\|\vec{x}_1\|} - \frac{GMm}{\|\vec{x}_0\|} = GMm\left(\frac{1}{\|\vec{x}_1\|} - \frac{1}{\|\vec{x}_0\|}\right) \checkmark$$

(11) Consider the vect field $\vec{F} = -\frac{y}{x^2+y^2} i + \frac{x}{x^2+y^2} j$

a) calc $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = i(0-0) + j(0-0) + k\left(\frac{y^2-x^2}{(y^2+x^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right)$

$= 0$

$\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$

$\frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)}$

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b) Evaluate $\oint_C F \cdot ds$ where C is the unit circle $x^2+y^2=1$

$\vec{x}(t) = (\cos t, \sin t)$

$\int_0^{2\pi} \frac{-\sin t\, dx}{\sin^2 t + \cos^2 t} + \frac{\cos t\, dy}{\sin^2 t + \cos^2 t} = \int_0^{2\pi} \frac{\sin^2 t}{1} + \frac{\cos^2 t}{1} dt = \int_0^{2\pi} 1\, dt = \boxed{2\pi}$ ✓

c) Is $\vec{F}$ conservative?

No because $\oint_C \vec{F} \cdot d\vec{s}$ does not equal zero.

d) Since the domain of $\vec{F}$ isn't simply connected region (undefined at $(0,0)$) it does not necessarily follow ~~as~~ as stated in thm 3.5 that $\nabla \times F = 0$ implies $\vec{F} = \nabla f$ so $\vec{F}$ might not be ~~a~~ conservative even if $\nabla \times F = 0$

Answer to 6.4.22

Show $\int_{\vec{x}} \vec{F}\cdot d\vec{s} = \frac{1}{2}m(v(b))^2 - \frac{1}{2}m(v(a))^2$ where $v(t) = \|\vec{v}(t)\|$

$\vec{F} = m\vec{a}$ $\quad \int_{\vec{x}} m\vec{a}\cdot d\vec{s} = \int_a^b m\vec{a}(t)\cdot\vec{x}'(t)\,dt = m\int_a^b \vec{x}''(t)\cdot\vec{x}'(t)\,dt$

$= m\int_a^b \vec{v}'(t)\cdot\vec{v}(t)\,dt$

Two possibilities:

a) The straightforward way

$\vec{v}(t) = (v_1(t), v_2(t), v_3(t))$ $\quad \int_{\vec{x}} \vec{F}\cdot d\vec{s} = m\int_a^b (v_1'(t), v_2'(t), v_3'(t))$
$\cdot (v_1(t), v_2(t), v_3(t))\,dt$

$= m\left(\int_a^b v_1' v_1\,dt + \int_a^b v_2' v_2\,dt + \int_a^b v_3' v_3\,dt\right)$ now substitution

$u_1(t) = v_1(t)$ $\quad u_2(t) = v_2(t)$ $\quad u_3(t) = v_3(t)$
$du_1 = v_1'(t)\,dt$ $\quad du_2 = v_2'(t)\,dt$ $\quad du_3 = v_3'(t)$

$= m\left(\int u_1\,du_1 + \int u_2\,du_2 + \int u_3\,du_3\right) = \frac{m}{2}\left(u_1^2\Big|_{u_1(a)}^{u_1(b)} + u_2^2\Big|_{u_2(a)}^{u_2(b)} + u_3^2\Big|_{u_3(a)}^{u_3(b)}\right)$

$= \frac{m}{2}\left(\underbrace{v_1(t)^2 + v_2(t)^2 + v_3(t)^2}_{\text{That is } \vec{v}\cdot\vec{v} \text{ or } \|\vec{v}\|^2 = v(t)^2}\Big|_a^b\right) = \frac{m}{2}\left(v(t)^2\Big|_a^b\right) = \underline{\frac{m}{2}v(b)^2 - \frac{m}{2}v(a)^2}$

b) The trick way

Notice: $\frac{d}{dt}(\vec{v}\cdot\vec{v}) \overset{\text{product rule}}{=} \vec{v}'\cdot\vec{v} + \vec{v}\cdot\vec{v}' = 2(\vec{v}\cdot\vec{v}')$

So $\int_{\vec{x}} \vec{F}\cdot d\vec{s} = m\int_a^b \vec{v}'(t)\cdot\vec{v}(t)\,dt = m\int_a^b \frac{1}{2}\frac{d}{dt}(\vec{v}(t)\cdot\vec{v}(t))\,dt$

$= \frac{m}{2}\int_a^b \frac{d}{dt}(v(t)^2)\,dt$ by FTOC $= \underline{\frac{m}{2}v(b)^2 + \frac{m}{2}v(a)^2}$