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Math61 Sec2
HW # 7
Due: 5/27/2004
TA: Po

Sec 7.1 # 1, 7, 11, 19, 22
Sec 7.2 # 1, 6, 7, 17, 20

1) Let $X: \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrized surface given by

$$X(s,t) = (s^2 - t^2,\ s+t,\ s^2 + 3t)$$

a) Determine a normal vector to this surface at the point $(3,1,1) = X(2,-1)$

$X_s(2,-1) = (2s,\ 1,\ 2s) = (2 \cdot 2,\ 1,\ 2 \cdot 2) = (4,1,4)$

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$X_t(2,-1) = (-2t,\ 1,\ 3) = (-2 \cdot -1, 1, 3) = (2,1,3)$

$$\vec{N}(s,t) = \|X_s \times X_t\| = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 1 & 4 \\ 2 & 1 & 3 \end{vmatrix}$$

$= (3-4)\hat{\imath} - (12-8)\hat{\jmath} + (4-2)\hat{k}$

$= \boxed{-\hat{\imath} - 4\hat{\jmath} + 2\hat{k}} = (-1,-4,2)$

b) Find an equation for the plane tangent to this surface at the point $(3,1,1)$

Tangent Plane: $N(s_0,t_0) \cdot (\vec{x} - X(s_0,t_0)) = 0$

$\Rightarrow N(2,-1) \cdot (\vec{x} - (3,1,1)) = 0$

$\Rightarrow (-1,-4,2) \cdot ((x-3),(y-1),(z-1)) = 0$

from part (a)

$\Rightarrow -(x-3) - 4(y-1) + 2(z-1) = 0$

$\Rightarrow -x + 3 - 4y + 4 + 2z - 2 = 0$

$\Rightarrow -x - 4y + 2z + 5 = 0$

$\Rightarrow \boxed{x + 4y - 2z = 5}$

7) Let S be the surface parametrized by $x = s\cos t$
$y = s\sin t$, $z = s^2$ where $s \geq 0$, $0 \leq t \leq 2\pi$

a) At what points is S smooth? Find an equation for the tangent plane at the point $(1, \sqrt{3}, 4)$

S is smooth where $N(s_0, t_0) = X_s \times X_T \neq 0$

$X_s = (\cos t, \sin t, 2s)$
$X_T = (-s\sin t, s\cos t, 0)$

$$X_s \times X_T = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos t & \sin t & 2s \\ -s\cdot\sin t & s\cos t & 0 \end{vmatrix} =$$

$= (-2s^2\cos t)\,\hat{\imath} - (+2s^2\sin t)\,\hat{\jmath} + (s\cos^2 t + s\sin^2 t)\,\hat{k}$

$= (-2s^2\cos t)\,\hat{\imath} - (2s^2\sin t)\,\hat{\jmath} + (s(\overset{=1}{\cancel{\cos^2 t + \sin^2 t}}))\,\hat{k}$

$N(s_0 t_0) = (-2s_0^2\cos t_0, -2s_0^2\sin t_0, s_0)$

$N(s_0 t_0) = \vec{0}$ only when $s_0 = 0$

$X(0, t_0) = (0\cdot\cos t_0, 0\cdot\sin t_0, 0^2) = (0,0,0)$

$\boxed{\text{so } S \text{ is smooth except point } (0,0,0)}$ ✓

7a)

Tangent plane at point $(1, \sqrt{3}, 4)$

$X(s,t) = (s\cos t,\ s\sin t,\ s^2) = (1, \sqrt{3}, 4)$

$$\begin{cases} s\cos t = 1 \\ s\sin t = \sqrt{3} \\ s^2 = 4 \end{cases}$$

$s = \pm 2$ but since $s \geq 0$, $s = 2$

$$\begin{cases} -2\cos t = 1 \\ -2\sin t = \sqrt{3} \end{cases} \Rightarrow \begin{matrix} \cos t = -\frac{1}{2} \\ \sin t = -\frac{\sqrt{3}}{2} \end{matrix} \Rightarrow t = \frac{4\pi}{3} \text{ for } 0 \leq t \leq 2\pi$$

so $N(2, \frac{4\pi}{3}) = (2 \cdot 2^2 \cos\frac{4\pi}{3},\ 2 \cdot 2^2 \sin\frac{4\pi}{3},\ 2)$

$= (8 \cdot -\frac{1}{2},\ 8 \cdot -\frac{\sqrt{3}}{2},\ 2)$

$= (-4, -4\sqrt{3}, 2)$

Tangent plane: $N(2, \frac{\pi}{3}) \cdot (\vec{x} - (1, \sqrt{3}, 4)) = 0$

$\Rightarrow (-4, -4\sqrt{3}, 2) \cdot ((x-1), (y-\sqrt{3}), (z-4)) = 0$

$\Rightarrow -4(x-1) - 4\sqrt{3}(y - \sqrt{3}) + 2(z-4) = 0$

$\Rightarrow -4x + 4 - 4\sqrt{3}y + 12 + 2z - 8 = 0$

$\Rightarrow -4x - 4\sqrt{3}y + 2z = -8$

$\Rightarrow \boxed{2x + 2\sqrt{3}y - z = 4}$ ✓

7 b) Sketch the graph of S. Can you recognize S as a familiar surface

$$\begin{cases} x = s\cos t \\ y = s\sin t \end{cases}$$

In the xy-plane, the graph looks like a circle of radius s. (varies)

$z = s^2$ suggests a parabolic shape (in the yz and xz plane)

As s increases, z increases, as well as the radius of the circle in the xy plane

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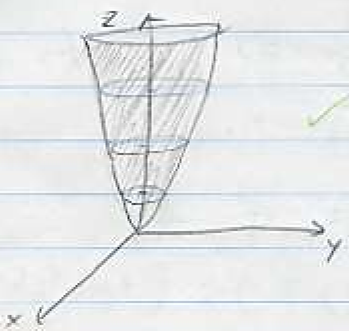

Thus, S is a paraboloid

c) Describe S by an equation of the form $z = f(x,y)$

$$x^2 + y^2 = s^2\cos^2 t + s^2\sin^2 t = s^2(\sin^2 t + \cos^2 t) = s^2 = z.$$

$$\therefore \boxed{z = x^2 + y^2}$$

d) Using your answer in part (c), discuss whether S has a tangent plane at every point

Take part (c) .. $z = x^2 + y^2 \Rightarrow x^2 + y^2 - z = 0$

$\Rightarrow \nabla(x^2 + y^2 - z) = (2x, 2y, -1)$ in which is the normal vector

Since the normal vector is defined everywhere, there is tangent plane at every point

Especially at the problem point $(0,0,0)$, it has a tangent plane with the equation $z = 0$

11.) Given the sphere of radius 2 centered at $(2,-1,0)$, find an equation for the plane tangent to it at the point $(1,0,\sqrt{2})$ in three ways

a) by considering the sphere as the graph of the function

$$f(x,y) = \sqrt{4-(x-2)^2-(y+1)^2}$$

Tangent plane is defined by the equation

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$$z = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x}-\vec{a})$$

$$\Rightarrow z = f(1,0) + \nabla f(1,0) \cdot ((x,y)-(1,0))$$
$$= \sqrt{4-(1-2)^2-(0+1)^2} + \nabla f(1,0) \cdot (x-1,y)$$
$$= \sqrt{4-(-1)^2-1^2} + \nabla f(1,0) \cdot (x-1,y)$$
$$= \sqrt{2} + \nabla f(1,0) \cdot (x-1,y)$$

$$f = (4-(x-2)^2-(y+1)^2)^{\frac{1}{2}}$$
$$f_x = \tfrac{1}{2}(4-(x-2)^2-(y+1)^2)^{-\frac{1}{2}} \cdot 2(x-2) \cdot -1$$
$$= \frac{-(x-2)}{\sqrt{4-(x-2)^2-(y+1)^2}}$$
$$f_y = \tfrac{1}{2}(4-(x-2)^2-(y+1)^2)^{-\frac{1}{2}} \cdot 2(y+1) \cdot -1$$
$$= \frac{-(y+1)}{\sqrt{4-(x-2)^2-(y+1)^2}}$$

$$\nabla f(1,0) = \left(\frac{-(1-2)}{\sqrt{4-(1-2)^2-(0+1)^2}}, \frac{-(0+1)}{\sqrt{4-(1-2)^2-(0+1)^2}}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow z = \sqrt{2} + \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right) \cdot (x-1,y)$$
$$\Rightarrow z = \sqrt{2} + \tfrac{1}{\sqrt{2}}x - \tfrac{1}{\sqrt{2}} + \left(-\tfrac{y}{\sqrt{2}}\right)$$

$$\Rightarrow \sqrt{2}z = 2 + x - 1 - y$$
$$\Rightarrow \boxed{-x+y+\sqrt{2}z = 1}$$

b) by considering the sphere as a level surface as the function

$$\vec{F}(x,y,z) = (x-2)^2 + (y+1)^2 + z^2$$

<u>Find the normal vector</u>

$$\nabla \vec{F} = (2(x-2),\ 2(y+1),\ 2z)$$
$$= (2x-4,\ 2y+2,\ 2z)$$

so

$$\nabla \vec{F}(1,0,\sqrt{2}) = (2\cdot 1-4,\ 2\cdot 0+2,\ 2\cdot\sqrt{2})$$
$$= (-2,\ 2,\ 2\sqrt{2})$$ ← which is the normal vector

Tangent plane is expressed by the equation

Normal vector $\cdot (\vec{x} - \vec{a}) = 0$ at point $\vec{a}$.

$$\Rightarrow \nabla \vec{F}(1,0,\sqrt{2}) \cdot (\vec{x} - (1,0,\sqrt{2})) = 0$$

$$\Rightarrow (-2,2,2\sqrt{2}) \cdot (x-1,\ y,\ z-\sqrt{2}) = 0$$

$$-2x + 2 + 2y + 2\sqrt{2}z - 4 = 0$$
$$-x + 1 + y + \sqrt{2}z - 2 = 0$$
$$\boxed{-x + y + \sqrt{2}z = 1}$$

1(c) by considering the sphere as the surface parametrized by

$$X(s,t) = (2\sin s\cos t + 2,\ 2\sin s\sin t - 1,\ 2\cos s)$$

$N = X_s \times X_t$

$X_s = (2\cos s\cos t,\ 2\cos s\sin t,\ -2\sin s)$
$X_t = (-2\sin s\sin t,\ 2\sin s\cos t,\ 0)$

Meanwhile,

$$\begin{cases} x = 2\sin s\cos t + 2 = 1 \\ y = 2\sin s\sin t - 1 = 0 \\ z = 2\cos s = \sqrt{2} \end{cases}$$ at point $(1, 0, \sqrt{2})$

$\Rightarrow \begin{cases} \cos s = \frac{\sqrt{2}}{2} \\ \sin s\sin t = \frac{1}{2} \\ \sin s\cos t = -\frac{1}{2} \end{cases}$ From $\cos s = \frac{\sqrt{2}}{2}$, $s = \frac{\pi}{4}$ or $\frac{7\pi}{4}$

Check for $\begin{cases} \sin s\sin t = \frac{1}{2} \\ \sin s\cos t = -\frac{1}{2} \end{cases}$ ✓

If $s = \frac{\pi}{4}$

$\Rightarrow \begin{array}{l} \frac{\sqrt{2}}{2}\sin t = \frac{1}{2} \\ \frac{\sqrt{2}}{2}\cos t = -\frac{1}{2} \end{array} \Rightarrow t = \frac{3\pi}{4}$

so when $(x,y,z) = (1, 0, \sqrt{2})$, $(s,t) = (\frac{\pi}{4}, \frac{3\pi}{4})$

$\Rightarrow X_s = (2\cos\frac{\pi}{4}\cos\frac{3\pi}{4},\ 2\cos\frac{\pi}{4}\sin\frac{3\pi}{4},\ -2\sin\frac{\pi}{4})$
$= (2\cdot\frac{\sqrt{2}}{2}\cdot -\frac{\sqrt{2}}{2},\ 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2},\ -2\frac{\sqrt{2}}{2})$

$= (-1,\ 1,\ -\sqrt{2})$

$X_t = (-2\cdot\sin\frac{\pi}{4}\sin\frac{3\pi}{4},\ 2\sin\frac{\pi}{4}\cos\frac{3\pi}{4},\ 0) = (-2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2},\ 2\cdot\frac{\sqrt{2}}{2}\cdot -\frac{\sqrt{2}}{2},\ 0)$
$= (-1, -1, 0)$

$$N = X_s \times X_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -\sqrt{2} \\ -1 & -1 & 0 \end{vmatrix} = (-\sqrt{2})\hat{i} - (-\sqrt{2})\hat{j} + (1+1)\hat{k}$$

$$= (-\sqrt{2}, \sqrt{2}, 2)$$

Tangent plane is expressed by the equation

✓ $N \cdot (\vec{x} - \vec{a}) = 0$ at point $\vec{a}$

$$\Rightarrow (-\sqrt{2}, \sqrt{2}, 2) \cdot ((x,y,z) - (1, 0, \sqrt{2})) = 0$$

$$\Rightarrow (-\sqrt{2}, \sqrt{2}, 2) \cdot (x-1, y, z-\sqrt{2}) = 0$$

$$\Rightarrow -\sqrt{2}x + \sqrt{2} + \sqrt{2}y + 2z - 2\sqrt{2} = 0$$

$$\Big\} \div \sqrt{2}$$

$$\Rightarrow -x + 1 + y + \sqrt{2}z - 2 = 0$$

$$\boxed{-x + y + \sqrt{2}z = 1}$$

19) A cylindrical hole of radius b is bored through a ball or radius a $(a > b)$ to form a ring. Find the outer surface area of the ring.

Use cylindrical coordinates

$$\begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\varphi \\ z = a\cos\varphi \end{cases}$$

The crux of this question is the limit of φ

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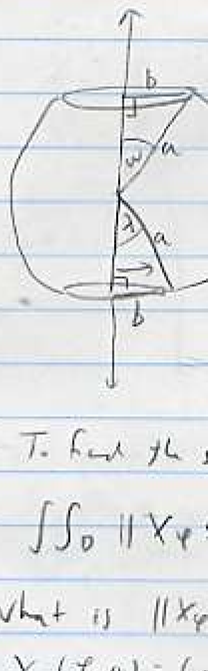


As seen from the diagram to the left

$w = \lambda = \sin^{-1}\frac{b}{a}$, so the limit of φ is

$$\varphi : \sin^{-1}\frac{b}{a} \longrightarrow \pi - \sin^{-1}\frac{b}{a}$$

$$\theta : 0 \to 2\pi \text{ as always}$$

To find the surface area, I use the formula

$$\iint_D \|X_\varphi \times X_\theta\| \, d\varphi \, d\theta$$

What is $\|X_\varphi \times X_\theta\|$?

$$X(\varphi, \theta) = (a\sin\varphi\cos\theta, \ a\sin\varphi\sin\theta, \ a\cos\varphi)$$

$$X_\varphi \times X_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= (a^2\sin^2\varphi\cos\theta)\hat{i} - (-a^2\sin^2\varphi\sin\theta)\hat{j} + (a^2\sin\varphi\cos\varphi\cos^2\theta + a^2\sin\varphi\cos\varphi\sin^2\theta)\hat{k}$$

$$= (a^2\sin^2\varphi\cos\theta, \ a^2\sin^2\varphi\sin\theta, \ a^2\sin\varphi\cos\varphi)$$

$$\|X_\varphi \times X_\theta\| = \sqrt{a^4\sin^4\varphi\cos^2\theta + a^4\sin^4\varphi\sin^2\theta + a^4\sin^2\varphi\cos^2\varphi}$$

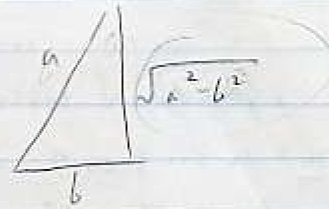
$$= a^2\sqrt{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta + \sin^2\varphi\cos^2\varphi}$$

$$= a^2\sqrt{\sin^2\varphi}$$

$$= a^2\sin\varphi$$

$$\iint \rho \, \| x_\varphi \times x_\theta \| \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_{\sin^{-1}\frac{b}{a}}^{\pi - \sin^{-1}\frac{b}{a}} a^2 \sin\varphi \, d\varphi \, d\theta$$



$$= a^2 \int_0^{2\pi} -\cos\varphi \Big|_{\sin^{-1}\frac{b}{a}}^{\pi - \sin^{-1}\frac{b}{a}} d\theta$$

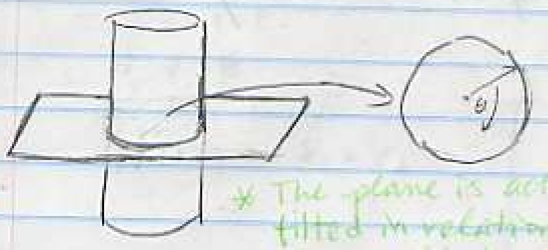
$$= a^2 \int_0^{2\pi} \left(\frac{1}{a}\sqrt{a^2-b^2} + \frac{1}{a}\sqrt{a^2-b^2} \right) d\theta$$

$$= 2a \int_0^{2\pi} \sqrt{a^2-b^2} \, d\theta$$

$$= \boxed{4\pi a \sqrt{a^2-b^2}}$$

22) Calculate the surface area of the portion of the plane $x+y+z=a$ cut out by the cylinder $x^2+y^2=a^2$ in two ways

(a) by using formula (6)



$\rho : 0 \to a \qquad \theta : 0 \to 2\pi$

Use polar coordinates

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = a - \rho\cos\theta + \rho\sin\theta \end{cases}$$

* The plane is actually tilted in relation to the cylinder.

$x+y+z=a \Rightarrow z = a-x-y$

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$X_\rho = (\cos\theta, \sin\theta, \sin\theta - \cos\theta)$

$X_\theta = (-a\sin\theta, a\cos\theta, \rho\sin\theta + \rho\cos\theta)$

$$X_\rho \times X_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & \sin\theta - \cos\theta \\ -\rho\sin\theta & \rho\cos\theta & \rho\sin\theta + \rho\cos\theta \end{vmatrix}$$

$= (\rho\sin^2\theta + \rho\cos\theta\sin\theta - \rho\cos\theta\sin\theta + \rho\cos^2\theta)\hat{i} - (\rho\sin\theta\cos\theta + \rho\cos^2\theta + \rho\sin^2\theta - \rho\sin\theta\cos\theta)\hat{j} + (\rho\cos^2\theta + \rho\sin^2\theta)\hat{k}$

$= \rho\hat{i} - \rho\hat{j} + \rho\hat{k} \Rightarrow \| X_\rho \times X_\theta \| = \sqrt{3\rho^2} = \sqrt{3}\rho$

Thus $\int_0^a \int_0^{2\pi} \sqrt{3}\rho \, d\theta \, d\rho$

$= 2\pi \int_0^a \sqrt{3}\rho \, d\rho = \boxed{\sqrt{3}\pi a^2}$

b) by using formula (9).

Surface area of the graph of $f(x,y)$ over D

$$= \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dx\,dy$$

Let $f(x,y) = a - x - y$

$\Rightarrow \quad f_x = -1$ and $f_y = -1$

and D is the region made by $x^2+y^2=a^2$, so it is a circle of radius a.

$$\Rightarrow \iint_D \sqrt{f_x^2+f_y^2+1}\,dx\,dy = \iint_D \sqrt{(-1)^2+(-1)^2+1}\,dx\,dy$$

$$= \iint_D \sqrt{3}\,dx\,dy = \sqrt{3}\iint_D dx\,dy$$

and observe that

$$\iint_D dx\,dy = \text{area of } D = \text{area of circle of radius } a = \pi a^2$$

$$\Rightarrow \sqrt{3}\iint_D dx\,dy = \boxed{\sqrt{3}\,\pi a^2}$$

Sec 7.2

1) Let $X(s,t) = (s, s+t, t)$, $0 \le s \le 1$, $0 \le t \le 2$. Find
$\iint_X (x^2+y^2+z^2)\, dS$

$$\begin{cases} x = s \\ y = s+t \\ z = t \end{cases}$$

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$$N(s,t) = X_s \times X_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (1-0)\hat{i} - (1-0)\hat{j} + (1-0)\hat{k}$$

$$= \hat{i} - \hat{j} + \hat{k} = (1,-1,1)$$

$$\|N(s,t)\| = \sqrt{1^2+(-1)^2+(1)^2} = \sqrt{3}$$

$$\iint_X (x^2+y^2+z^2)\, dS = \int_0^2\int_0^1 (s^2+(s+t)^2+t^2)\sqrt{3}\, ds\, dt$$

$$= \sqrt{3}\int_0^2\int_0^1 (s^2+(s^2+2st+t^2)+t^2)\, ds\, dt$$

$$= \sqrt{3}\int_0^2\int_0^1 (2s^2+2st+2t^2)\, ds\, dt$$

$$= \sqrt{3}\int_0^2 \frac{2s^3}{3} + \frac{\cancel{2}s^2t}{\cancel{2}} + 2t^2s \Big|_0^1 dt$$

$$= \sqrt{3}\int_0^2 \frac{2}{3} + t + 2t^2\, dt$$

$$= \sqrt{3}\left(\frac{2}{3}t + \frac{t^2}{2} + \frac{2t^3}{3}\right)\Big|_0^2 = \sqrt{3}\left(\frac{2}{3}\cdot 2 + \frac{2^2}{2} + \frac{2\cdot 2^3}{3}\right)$$

$$= \boxed{\frac{26\sqrt{3}}{3}}$$ ✓

6) Find $\iint_S (x^2+y^2)\,dS$, where S is the lateral surface of the cylinder of radius a and height h whose axis is the z-axis

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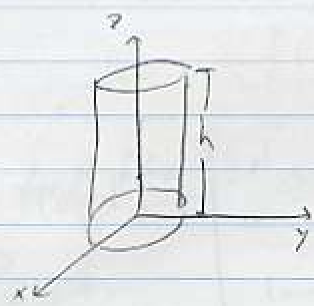

Parametrize cylinder: S by

$$\begin{cases} x = a\cos s & 0 \le s \le 2\pi \\ y = a\sin s & \\ z = t & 0 \le t \le h \end{cases}$$

$$X(s,t) = (a\cos s,\ a\sin s,\ t)$$

$$N(s,t) = X_s \times X_t = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -a\sin s & a\cos s & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= a\cos s\,\hat{\imath} - (-a\sin s)\hat{\jmath} + 0\,\hat{k} = (a\cos s,\ a\sin s,\ 0)$$

$$\|N(s,t)\| = \sqrt{a^2\cos^2 s + a^2\sin^2 s} = \sqrt{a^2(\sin^2 s + \cos^2 s)} = a$$

$$\iint_S (x^2+y^2)\,dS = \int_0^h \int_0^{2\pi} (a^2\cos^2 s + a^2\sin^2 s)\cdot a\,ds\,dt$$

$$= \int_0^h \int_0^{2\pi} (a^2(\cos^2 s + \sin^2 s))\cdot a\,ds\,dt$$

$$= \int_0^h \int_0^{2\pi} a^3\,ds\,dt$$

$$= a^3 \int_0^h \int_0^{2\pi} ds\,dt = a^3 \int_0^h 2\pi\,dt = \boxed{2\pi a^3 h}$$ ✓

7) Let S be a sphere of radius a

a) Find $\iint_S (x^2+y^2+z^2)\,dS$

Parametrize S.

$$\begin{cases} x = a\sin\varphi\cos\theta & 0 \le \varphi \le \pi \\ y = a\sin\varphi\sin\theta & 0 \le \theta \le 2\pi \\ z = a\cos\varphi \end{cases}$$

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$$X(\varphi,\theta) = (a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi)$$

$$N(\varphi,\theta) = X_\varphi \times X_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= (a^2\sin^2\varphi\cos\theta)\hat{i} - (-a^2\sin^2\varphi\sin\theta)\hat{j} + (a^2\sin\varphi\cos\varphi\cos^2\theta + a^2\sin\varphi\cos\varphi\sin^2\theta)\hat{k}$$

$$= (a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\sin\varphi\cos\varphi)$$

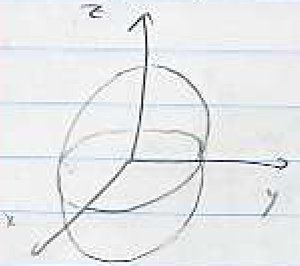
$$\|N(\varphi,\theta)\| = \sqrt{a^4\sin^4\varphi\cos^2\theta + a^4\sin^4\varphi\sin^2\theta + a^4\sin^2\varphi\cos^2\varphi}$$
$$= a^2\sqrt{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta + \sin^2\varphi\cos^2\varphi}$$
$$= a^2\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$
$$= a^2\sqrt{\sin^2\varphi(\sin^2\varphi + \cos^2\varphi)}$$
$$= \underline{a^2\sin\varphi}\ \checkmark$$

$$\iint_S (x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^{\pi} (a^2\sin^2\varphi\cos^2\theta + a^2\sin^2\varphi\sin^2\theta + a^2\cos^2\varphi)\cdot(a^2\sin\varphi)\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} (a^2(\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi))(a^2\sin\varphi)\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} (a^2)(a^2\sin\varphi)\,d\varphi\,d\theta$$
$$= a^4\int_0^{2\pi}\int_0^{\pi} \sin\varphi\,d\varphi\,d\theta$$
$$= a^4\int_0^{2\pi} -\cos\varphi\Big|_0^{\pi} d\theta = a^4\int_0^{2\pi}(-\cos\pi + \cos 0)\,d\theta = a^4\int_0^{2\pi} 2\,d\theta$$

$$= \boxed{4\pi a^4}\ \checkmark$$

b) Use symmetry and part (a) to easily find

$\iint_S y^2\,dS$.



Observe that

$$\iint_S y^2\,dS = \iint_S x^2\,dS = \iint_S z^2\,dS$$

because of the symmetry of sphere

Thus,

$$\frac{1}{3}\iint_S (x^2+y^2+z^2)\,dS = \iint_S y^2\,dS = \boxed{\frac{4}{3}\pi a^4}$$

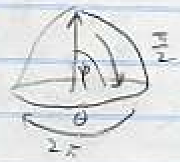
17-20) Find the flux of the given vector field $\vec{F}$ across the upper hemisphere $x^2+y^2+z^2=a^2$, $z \geq 0$. Orient the hemisphere with an upward-pointing normal

17) $\vec{F} = y\hat{j}$

Parametrize the upper hemisphere

$\frac{10}{10}$

$$\begin{cases} x = a\cos\theta\sin\varphi & 0 \leq \theta \leq 2\pi \\ y = a\sin\theta\sin\varphi & 0 \leq \varphi \leq \frac{\pi}{2} \\ z = a\cos\varphi & \end{cases}$$



As found in problem 16,

$$N(\varphi,\theta) = (a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\sin\varphi\cos\varphi)$$

If we normalize N, we find that

$$\hat{n}(\varphi,\theta) = \frac{N(\varphi,\theta)}{\|N(\varphi,\theta)\|} = \frac{\cancel{a^2\sin\varphi}(\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)}{\cancel{a^2\sin\varphi}}$$ ← found in question 16

$$= (\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)$$

which clearly shows that $\hat{n}$, and thus N, is pointing upwards.

Hence, we have

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \vec{F}(x(\varphi,\theta))\cdot N(\varphi,\theta)\,d\theta\,d\varphi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} (0,\ a\sin\theta\sin\varphi,\ 0)\cdot(a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\sin\varphi\cos\varphi)\,d\theta\,d\varphi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} a^3\sin^3\varphi\sin^2\theta\,d\theta\,d\varphi \overset{TI-89}{=} \boxed{\frac{2\pi a^3}{3}}$$ ✓

20) $\vec{F} = x^2\hat{\imath} + xy\hat{\jmath} + xz\hat{k}$

We found in question 17 that

$$N(\varphi,\theta) = (a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\sin\varphi\cos\varphi)$$

$\frac{10}{10}$

and since $\begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\theta \\ z = a\cos\varphi \end{cases}$ $\quad 0 \le \theta \le 2\pi$, $0 \le \varphi \le \frac{\pi}{2}$

$$\vec{F}(x(\varphi,\theta)) = (a^2\sin^2\varphi\cos^2\theta,\ a^2\sin^2\varphi\sin\theta\cos\theta,\ a^2\sin\varphi\cos\varphi\cos\theta)$$

$$\iint_S \vec{F}\cdot d\vec{s} = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \vec{F}(x(\varphi,\theta))\cdot N(\varphi,\theta)\,d\theta\,d\varphi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} (a^2\sin^2\varphi\cos^2\theta,\ a^2\sin^2\varphi\sin\theta\cos\theta,\ a^2\sin\varphi\cos\varphi\cos\theta)\cdot (a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\sin\varphi\cos\varphi)\,d\theta\,d\varphi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} a^4\sin^4\varphi\cos^3\theta + a^4\sin^4\varphi\sin^2\theta\cos\theta + a^4\sin^2\varphi\cos^2\varphi\cos\theta)\,d\theta\,d\varphi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} a^4\sin^2\varphi\cos\theta\,(\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi)\,d\theta\,d\varphi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} a^4\sin^2\varphi\cos\theta\,d\theta\,d\varphi$$ ✓

$\overset{TI-89}{=} \boxed{0}$