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HW # 9

Math 61 Sec 2

Due: 5/31/2004

Section 7.3 # 1, 2, 5, 10 (Stoke's Theorem)

Section 7.3 # 9, 11, 12, 14 (Gauss's Divergence Theorem)

1-2) Verify Stokes' Theorem for the given surface and vector field.

1) S is defined by $x^2 + y^2 + 5z = 1$, $z \geq 0$, oriented by upward normal;

$$\vec{F} = xz\vec{i} + yz\vec{j} + (x^2 + y^2)\vec{k}$$

$$S: x^2 + y^2 + 5z = 1$$

If $z \geq 0$, it is clear that S is defined over the x - y plane when $z = 0$.So let $z = 0$

$$x^2 + y^2 + 5 \cdot 0 = 1 \Rightarrow x^2 + y^2 = 1$$

So S consists of the circle $C = \{(x, y, z) \mid x^2 + y^2 = 1, z = 0\}$

By Stokes' Theorem, then

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

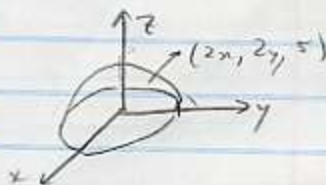
$$\text{L.H.S: } \vec{F} = xz\vec{i} + yz\vec{j} + (x^2 + y^2)\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & x^2 + y^2 \end{vmatrix} = (2y - y)\vec{i} - (2x - x)\vec{j} + 0\vec{k}$$

$$= (y, -x, 0)$$

Find the gradient of $x^2 + y^2 + 5z$ to find its upward normal

$$\nabla S = (2x, 2y, 5), \text{ which is oriented upward}$$



By using formula (5) of § 7.2, we have, where $D = \{(x, y) | x^2 + y^2 \leq 1\}$

$$\begin{aligned}\iint_S \nabla \times \vec{F} \cdot d\vec{S} &= \iint_D (y, -x, 0) \cdot (2x, 2y, 5) dx dy \\&= \iint_D (2xy - 2xy + 0) dx dy \\&= \iint_D 0 dx dy = \boxed{0}\end{aligned}$$

RHS: We may parametrize ∂S as

$$\begin{cases} x = \cos \theta \\ y = \sin \theta \\ z = 0 \end{cases} \quad 0 \leq \theta \leq 2\pi$$



(which yields the orientation desired for ∂S)

$$\begin{aligned}\oint_{\partial S} \vec{F} \cdot d\vec{s} &= \int_0^{2\pi} \vec{F}(x(t)) \cdot x'(t) dt \\&= \int_0^{2\pi} (0, 0, \cos^2 \theta + \sin^2 \theta) \cdot (-\sin \theta, \cos \theta, 0) dt \\&= \int_0^{2\pi} 0 dt = \boxed{0}\end{aligned}$$

Since $LHS = RHS$, Stoke's theorem thus holds.

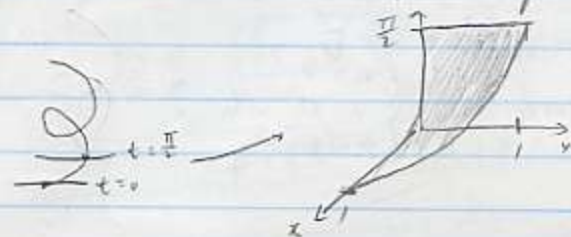
2) S is parametrized by $\vec{X}(s,t) = (s \cos t, s \sin t, t)$,

$$0 \leq s \leq 1, 0 \leq t \leq \frac{\pi}{2}$$

$$\begin{cases} x = s \cos t \\ y = s \sin t \\ z = t \end{cases} \quad \begin{matrix} 0 \leq s \leq 1 \\ 0 \leq t \leq \frac{\pi}{2} \end{matrix}$$

$$\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$$

S is a helix which stops at a quarter circle



By Stoke's Theorem

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$$

LHS

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = (1-0)\hat{i} - (0-1)\hat{j} + (1-0)\hat{k} = (1, 1, 1)$$

$$\begin{aligned} \vec{X}_s \times \vec{X}_t &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & \sin t & 0 \\ -s \sin t & s \cos t & 1 \end{vmatrix} = (s \sin t)\hat{i} - (s \cos t)\hat{j} + (s \cos^2 t + s \sin^2 t)\hat{k} \\ &= (s \sin t, -s \cos t, s) \end{aligned}$$

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_S \nabla \times \vec{F} \cdot (\vec{X}_s + \vec{X}_t) ds dt$$

$$= \int_0^{\pi/2} \int_0^1 (1, 1, 1) \cdot (s \sin t, -s \cos t, s) ds dt$$

$$= \int_0^{\pi/2} \int_0^1 (s \sin t - s \cos t + s) ds dt$$

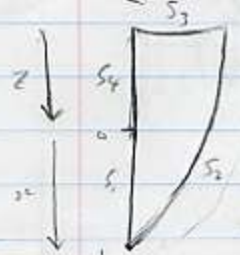
$$= \int_0^{\pi/2} \left(s \sin t - s \cos t + \frac{s^2}{2} \right) \Big|_0^1 dt$$

$$= \int_0^{\pi/2} \sin t - \cos t + \frac{1}{2} dt = -\cos t - \sin t + \frac{1}{2} t \Big|_0^{\pi/2}$$

$$= \left(-\cos \frac{\pi}{2} - \sin \frac{\pi}{2} + \frac{1}{2} \cdot \frac{\pi}{2} \right) - \left(-\cos 0 - \sin 0 + \frac{1}{2} \cdot 0 \right)$$

$$= \left(-1 + \frac{\pi}{4} \right) - (-1) = \frac{\pi}{4}$$

RHS



$$\oint_{\partial S} \vec{F} \cdot d\vec{s} = \int_{S_1} \vec{F} \cdot d\vec{s} + \int_{S_2} \vec{F} \cdot d\vec{s} + \int_{S_3} \vec{F} \cdot d\vec{s} + \int_{S_4} \vec{F} \cdot d\vec{s}$$

Parameter S_1, S_2, S_3, S_4

$$S_1 \begin{cases} x = t \\ y = 0 \\ z = 0 \end{cases} \quad t: 0 \rightarrow 1$$

$$S_2 \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases} \quad t: 0 \rightarrow \frac{\pi}{2}$$

$$S_3 \begin{cases} x = 0 \\ y = t \\ z = \frac{\pi}{2} \end{cases} \quad t: 1 \rightarrow 0$$

$$S_4 \begin{cases} x = 0 \\ y = 0 \\ z = t \end{cases} \quad t: \frac{\pi}{2} \rightarrow 0$$

$$\int_{S_1} \vec{F} \cdot d\vec{s} = \int_{S_1} \vec{F} \cdot \vec{x}'(t) dt = \int_{S_1} (z, x, y) \cdot (1, 0, 0) dt$$

$$\vec{x}(t) = (t, 0, 0)$$

$$= \int_{S_1} z dt = \int_{S_1} 0 dt = 0$$

$$\int_{S_2} \vec{F} \cdot d\vec{s} = \int_{S_2} (z, x, y) \cdot (-\sin t, \cos t, 1) dt = \int_{S_2} (-z \sin t + x \cos t + y) dt$$

$$\vec{x}(t) = (\cos t, \sin t, t)$$

$$= \int_{S_2} (-t \sin t + \cos^2 t + \sin t) dt$$

$$= \int_0^{\frac{\pi}{2}} -t \sin t dt + \int_0^{\frac{\pi}{2}} \cos^2 t dt + \int_0^{\frac{\pi}{2}} \sin t dt = \frac{\pi}{4}$$

$$\int_{S_3} \vec{F} \cdot d\vec{s} = \int_{S_3} (z, x, y) \cdot (0, 1, 0) dt = \int_{S_3} x dt = \int_{S_3} 0 dt = 0$$

$$\vec{x}(t) = (0, t, \frac{\pi}{2})$$

$$\int_{S_4} \vec{F} \cdot d\vec{s} = \int_{S_4} (z, x, y) \cdot (0, 0, 1) dt = \int_{S_4} y dt = \int_{S_4} 0 dt = 0$$

$$\vec{x}(t) = (0, 0, t)$$

$$\oint_{\partial S} \vec{F} \cdot d\vec{s} = 0 + \frac{\pi}{4} + 0 + 0 = \boxed{\frac{\pi}{4}}$$

LHS = RHS, verifying Stokes' theorem.

5) Let S be the "silo surface," that is, S is the union of two smooth surfaces S_1 and S_2 .

$$S_1: x^2 + y^2 = 9 \quad 0 \leq z \leq 8$$

$$S_2: x^2 + y^2 + (z - 8)^2 = 9 \quad z \geq 8$$

Find $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$, where

$$\vec{F} = (x^3 + xz + yz^3)\hat{i} + (xyz^3 + y^3)\hat{j} + x^2z^5\hat{k}$$

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S}$$

S_1 : It is a cylinder with a radius of 3 with height from $z=0 \rightarrow 8$

S_2 : It is the upper half of a sphere with radius of 3, centered at $(0, 0, 8)$



By Stokes's theorem, $\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$ where ∂S consists of the circle (like a projection of S onto the xy -plane)
 $C = \{(x, y, z) \mid x^2 + y^2 = 9, z = 0\}$

$$\Rightarrow \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r} = \oint_{\partial S'} \vec{F} \cdot d\vec{r} = \iint_{S'} \nabla \times \vec{F} \cdot d\vec{S}$$

where we let S' be just the circle C .

$$\int_S \nabla \times \vec{F} \cdot d\vec{S} = \int_{S'} \nabla \times \vec{F} \cdot d\vec{S}$$

$$= \int_{S'} (\nabla \times \vec{F} \cdot \hat{n}) dS$$

We orient S' by $\hat{n} = (0, 0, 1)$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 + xz + yz^2 & xy^2 + 1 & x^2z^2 \end{vmatrix} = (0 - 3xyz^2)\hat{i} - (2xz^5 - (x + 2yz))\hat{j} + (yz^3 - z^2)\hat{k}$$

$$= (-3xyz^2, x + 2yz - 2xz^5, yz^3 - z^2)$$

$$\int_{S'} (\nabla \times \vec{F} \cdot \hat{n}) dS = \int_{S'} (yz^3 - z^2) dS$$

and since C is $x^2 + y^2 = 9, z = 0$

$$\Rightarrow \int_{S'} (y(0)^3 - 0^2) dS = \boxed{0}$$

10.) Verify that Stokes's theorem implies Green's theorem

Assume $\vec{F}$ is independent of z and its k -component is identically zero.

$$\Rightarrow \vec{F}(x, y, z) = M(x, y)\hat{i} + N(x, y)\hat{j} + 0\hat{k}$$

Stokes's theorem states that

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$$

where ∂S is the boundary of the surface S .

$\Rightarrow$ LHS:

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} \quad \vec{F} = M(x, y)\hat{i} + N(x, y)\hat{j}, z = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M(x, y) & N(x, y) & 0 \end{vmatrix} = (0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$$



For this $\mathbb{R}^2$ region S

to be oriented properly,

the normal has to be pointed upwards,

$$\text{so } \hat{n} = (0, 0, 1).$$

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_S \nabla \times \vec{F} \cdot \hat{n} dS = \iint_S (0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) \cdot (0, 0, 1) dS$$

$$= \iint_S \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} dx dy$$

$\Rightarrow$ RHS:

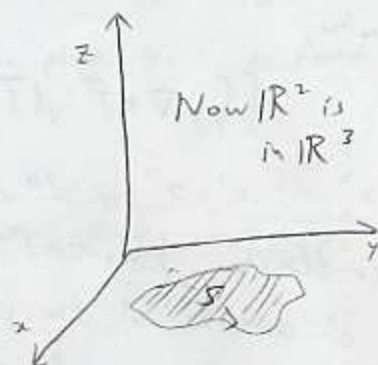
$$\oint_{\partial S} \vec{F} \cdot d\vec{s} = \oint_{\partial S} (M, N, 0) \cdot (dx, dy, dz) = \oint_{\partial S} M dx + N dy$$

$\Rightarrow$ RHS = LHS

$$\oint_{\partial S} M dx + N dy = \iint_S \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} dx dy, \text{ which verifies that}$$

Green's Theorem is a special case of Stokes's Theorem,

where $\vec{F}$ is planar field (independent of z).



9) Verify Gauss's theorem for the given three-dimensional region P and vector field $\vec{F}$.

$$\vec{F} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \quad P = \{(x, y, z) \mid a^2 \leq x^2 + y^2 + z^2 \leq b^2\}$$

D is the region between sphere with radius a and sphere with radius b .



$$\Rightarrow a < b$$

Then, by Gauss's theorem

$$\oiint_{\partial P} \vec{F} \cdot d\vec{S} = \iiint_P \nabla \cdot \vec{F} dV$$

LHS

$$\oiint_{\partial P} \vec{F} \cdot d\vec{S} = \oiint_{\partial D_1} \vec{F} \cdot d\vec{S} + \oiint_{\partial D_2} \vec{F} \cdot d\vec{S}$$

where D_1 is the boundary of the outer sphere and

∂D_2 is the boundary of the inner sphere

The unit normal vector has to be directed such that it is pointing away from the region P . Thus n_1 has to be pointing outwards and n_2 has to be inwards.

$$\partial D_1 \Rightarrow x^2 + y^2 + z^2 = b^2$$

$$\partial D_2 \Rightarrow x^2 + y^2 + z^2 = a^2$$

$$\oiint_{\partial P} \vec{F} \cdot d\vec{S} = \oiint_{\partial D_1} \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \right) d\vec{S} = \oiint_{\partial D_1} \left(\frac{x}{b}, \frac{y}{b}, \frac{z}{b} \right) d\vec{S}$$

$$S: \begin{cases} x = b \cos \theta \sin \varphi \\ y = b \sin \theta \sin \varphi \\ z = b \cos \varphi \end{cases} \quad \begin{matrix} \varphi: 0 \rightarrow \pi \\ \theta: 0 \rightarrow 2\pi \end{matrix} \quad X = (b \cos \theta \sin \varphi, b \sin \theta \sin \varphi, b \cos \varphi)$$

$$X_\theta \times X_\varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -b \sin \theta \sin \varphi & b \cos \theta \sin \varphi & 0 \\ b \cos \theta \cos \varphi & b \sin \theta \cos \varphi & -b \sin \varphi \end{vmatrix} = \begin{pmatrix} (-b^2 \sin^2 \varphi \cos \theta) \hat{i} - (b^2 \sin^2 \varphi \sin \theta) \hat{j} \\ + (-b^2 \sin^2 \theta \sin \varphi \cos \varphi - b^2 \cos^2 \theta \sin \varphi \cos \varphi) \hat{k} \end{pmatrix}$$

$$= (-b^2 \sin^2 \varphi \cos \theta, -b^2 \sin^2 \varphi \sin \theta, -b^2 \sin \varphi \cos \varphi)$$

$$\text{But since } n_1 \text{ is pointed outwards, } n_1 = (b^2 \sin^2 \varphi \cos \theta, b^2 \sin^2 \varphi \sin \theta, b^2 \sin \varphi \cos \varphi)$$

$$\Rightarrow \oiint_{\partial D_1} \left(\frac{b \cos \theta \sin \varphi}{b}, \frac{b \sin \theta \sin \varphi}{b}, \frac{b \cos \varphi}{b} \right) \cdot (b^2 \sin^2 \varphi \cos \theta, b^2 \sin^2 \varphi \sin \theta, b^2 \sin \varphi \cos \varphi) d\varphi d\theta$$

$$= \oiint_{\partial D_1} b^3 \sin^3 \varphi \cos^2 \theta + b^3 \sin^3 \varphi \sin^2 \theta + b^3 \sin^2 \varphi \cos^2 \varphi d\varphi d\theta$$

$$= \iint_{D_0} b^2 \sin \varphi (\sin^2 \varphi \cos^2 \theta + \sin^2 \varphi \sin^2 \theta + \cos^2 \varphi) d\varphi d\theta$$

$$= \iint_{D_0} b^2 \sin \varphi d\varphi d\theta = \int_0^{2\pi} \int_0^\pi b^2 \sin \varphi d\varphi d\theta$$

$$= b^2 \int_0^{2\pi} [-\cos \varphi]_0^\pi d\theta = b^2 \int_0^{2\pi} 2 d\theta = \underline{4\pi b^2}$$

$$S_2 \begin{cases} x = a \cos \theta \sin \varphi & \varphi: 0 \rightarrow \pi \\ y = a \sin \theta \sin \varphi & \theta: 0 \rightarrow 2\pi \\ z = a \cos \varphi \end{cases}$$

Since n_z is pointed inwards, $\vec{n}_z = (-a^2 \sin^2 \varphi \cos \theta, -a^2 \sin^2 \varphi \sin \theta, -a^2 \sin \varphi \cos \varphi)$

$$\iint_{D_2} \vec{F} \cdot d\vec{S} = \iint_{D_2} \left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a} \right) \cdot (-a^2 \sin^2 \varphi \cos \theta, -a^2 \sin^2 \varphi \sin \theta, -a^2 \sin \varphi \cos \varphi) d\varphi d\theta$$

$$= - \iint_{D_2} a^2 \sin \varphi d\varphi d\theta = - \int_0^{2\pi} \int_0^\pi a^2 \sin \varphi d\varphi d\theta = \underline{-4\pi a^2}$$

$$\iint_{D_0} \vec{F} \cdot d\vec{S} = \iint_{D_0} \vec{F} \cdot d\vec{S} + \iint_{D_2} \vec{F} \cdot d\vec{S} = 4\pi b^2 - 4\pi a^2 = \underline{4\pi(b^2 - a^2)}$$

$$\text{RHS: } \iiint_V \nabla \cdot \vec{F} dV$$

$$\begin{aligned} \nabla \cdot \vec{F} &= \left(\frac{\partial}{\partial x} \frac{x^2}{\sqrt{x^2+y^2+z^2}} + \frac{\partial}{\partial y} \frac{y^2}{\sqrt{x^2+y^2+z^2}} + \frac{\partial}{\partial z} \frac{z^2}{\sqrt{x^2+y^2+z^2}} \right) \\ &= \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}} \\ &= \frac{2x^2+2y^2+2z^2}{(x^2+y^2+z^2)^{3/2}} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{(x^2+y^2+z^2)^{1/2}} \end{aligned}$$

Change to spherical coordinates

$$\begin{cases} x = \rho \cos \theta \sin \varphi \\ y = \rho \sin \theta \sin \varphi \\ z = \rho \cos \varphi \end{cases} \quad \begin{matrix} \rho: a \rightarrow b \\ \theta: 0 \rightarrow 2\pi \\ \varphi: 0 \rightarrow \pi \end{matrix} \Rightarrow x^2+y^2+z^2 = \rho^2$$

$$\iiint_V \nabla \cdot \vec{F} dV = \iiint_V \nabla \cdot \vec{F} dx dy dz = \iiint_{\rho, \varphi, \theta} \nabla \cdot \vec{F} \left| \frac{\partial(x,y,z)}{\partial(\rho, \varphi, \theta)} \right| d\rho d\varphi d\theta$$

$$\frac{\partial(x,y,z)}{\partial(\rho, \varphi, \theta)} = \det \begin{bmatrix} x_\rho & x_\varphi & x_\theta \\ y_\rho & y_\varphi & y_\theta \\ z_\rho & z_\varphi & z_\theta \end{bmatrix} = \det \begin{bmatrix} \sin \varphi \cos \theta & \rho \cos \varphi \cos \theta & -\rho \sin \varphi \sin \theta \\ \sin \varphi \sin \theta & \rho \cos \varphi \sin \theta & \rho \sin \varphi \cos \theta \\ \cos \varphi & -\rho \sin \varphi & 0 \end{bmatrix}$$

$$= \cos \varphi (\rho^2 \cos^2 \theta \sin \varphi \cos \varphi + \rho^2 \sin^2 \theta \sin \varphi \cos \varphi) + \rho \sin \varphi (\rho \cos^2 \theta \sin^2 \varphi + \rho \sin^2 \theta \sin^2 \varphi)$$

$$= \rho^2 \cos \varphi (\sin \varphi \cos \varphi) + \rho^2 \sin^3 \varphi = \rho^2 \sin \varphi (\cos^2 \varphi + \sin^2 \varphi) = \rho^2 \sin \varphi$$

and $\sin \varphi$ is always non-negative from $0 \rightarrow \pi$, so

$$\Rightarrow \iiint_V \nabla \cdot \vec{F} dV = \int_0^{2\pi} \int_0^\pi \int_a^b \frac{2}{\rho} \rho^2 \sin \varphi d\rho d\varphi d\theta \quad \left(\left| \frac{\partial(x,y,z)}{\partial(\rho, \varphi, \theta)} \right| = \rho^2 \sin \varphi \right)$$

$$= \int_0^{2\pi} \int_0^\pi \frac{\rho^2}{2} \sin \varphi \Big|_a^b d\varphi d\theta = \int_0^{2\pi} \int_0^\pi (b^2 - a^2) \sin \varphi d\varphi d\theta = (b^2 - a^2) \int_0^{2\pi} -\cos \varphi \Big|_0^\pi d\theta$$

$$= (b^2 - a^2) \int_0^{2\pi} 2 d\theta = \boxed{4\pi(b^2 - a^2)}$$

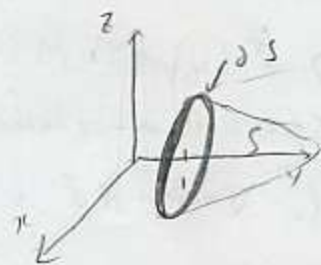
∴ LHS = RHS, verifying Gauss's theorem

11) let S be defined by $y = 10 - x^2 - z^2$ with $y \geq 1$, oriented with right-hand pointing normal. Let

$$\vec{F} = (2xyz + 5z)\vec{i} + e^x \cos yz \vec{j} + x^2 y \vec{k}$$

Determine

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S}$$



$$y = 10 - x^2 - z^2 \Rightarrow x^2 + z^2 + y = 10, \text{ so when } y = 1, x^2 + z^2 = 9$$

By Stokes' theorem, $\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$, where ∂S consists of the circle

$$C = \{(x, y, z) \mid x^2 + z^2 = 9, y = 1\}$$

$$\Rightarrow \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s} = \oint_{\partial S} \vec{F} \cdot d\vec{t} = \iint_{S'} \nabla \times \vec{F} \cdot d\vec{S}$$

where we let S' be the circle C .

$$\begin{aligned} \Rightarrow \iint_S \nabla \times \vec{F} \cdot d\vec{S} &= \iint_{S'} \nabla \times \vec{F} \cdot d\vec{S} \\ &= \iint_{S'} (\nabla \times \vec{F} \cdot \hat{n}) dS \end{aligned}$$

We orient S' by $\hat{n} = (0, 1, 0)$

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz + 5z & e^x \cos yz & x^2 y \end{vmatrix} = (6z + ye^x \sin yz)\vec{i} - (2xz - (2xz + 5))\vec{j} \\ &\quad + (e^x \cos yz - 2xz)\vec{k} \\ &= (x^2 + ye^x \sin yz, 5, e^x \cos yz - 2xz) \end{aligned}$$

$$\Rightarrow \iint_{S'} (\nabla \times \vec{F} \cdot \hat{n}) dS = \iint_{S'} (x^2 + ye^x \sin yz, 5, e^x \cos yz - 2xz) \cdot (0, 1, 0) dS$$

$$= \iint_{S'} 5 dS = 5 \cdot \text{area of } S' = 5(9\pi) = \boxed{45\pi}$$

↑
area of circle with radius 3

12) Let S be defined by $z = e^{1-x^2-y^2}$, $z \geq 1$, oriented by upward normal.
and let $\vec{F} = x\hat{i} + y\hat{j} + (2-2z)\hat{k}$. Use Gauss's theorem to calculate

$$\iint_S \vec{F} \cdot d\vec{S}.$$

Consider the piecewise smooth, closed surface created by taking the union of S and S' , where S' is the portion of the plane $z=1$ enclosed by ∂S .
 $1 = e^{1-(x^2+y^2)} \Rightarrow$ the disk $x^2+y^2 \leq 1, z=1$.

S is oriented by upward, or outward normal, so S' must also be oriented outward, or downward normal $\hat{n} = (0, 0, -1)$.

Then, by Gauss's theorem,

$$\iint_S \vec{F} \cdot d\vec{S} + \iint_{S'} \vec{F} \cdot d\vec{S} = \iiint_{\mathcal{V}} \nabla \cdot \vec{F} dV$$

$$\nabla \cdot \vec{F} = (1+1-2) = 0$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} + \iint_{S'} \vec{F} \cdot d\vec{S} = 0,$$

$$\text{so } \iint_S \vec{F} \cdot d\vec{S} = -\iint_{S'} \vec{F} \cdot (0, 0, -1) dS = \iint_R (2-2z) dx dy$$

where R is the unit disk $x^2+y^2 \leq 1$ in the plane $z=1$.

$$\Rightarrow \iint_R (2-2z) dx dy = \iint_R (2-2 \cdot 1) dx dy = \iint_R 0 dx dy = \boxed{0}$$

14) Use Gauss's theorem to evaluate

$\iint_S \vec{F} \cdot d\vec{S}$,
where $\vec{F} = ze^x \hat{i} + 3y \hat{j} + (2 - yz^7) \hat{k}$ and S is the union of the five
"lateral" faces of the unit cube $[0, 1] \times [0, 1] \times [0, 1]$.

That is, the $z=0$ face is not part of S .

Take the union of S and S' , where S' is the portion of the plane $z=0$
enclosed by ∂S .

Then by Gauss's theorem

$$\iint_S \vec{F} \cdot d\vec{S} + \iint_{S'} \vec{F} \cdot d\vec{S} = \iiint_{\partial S \cup S'} \vec{F} \cdot d\vec{S} = \iiint_D \nabla \cdot \vec{F} dV$$

$$\nabla \cdot \vec{F} = (2xe^x + 3 - 7yz^6)$$

(Note that D is the unit cube $[0, 1] \times [0, 1] \times [0, 1]$)

$$S_0: x: 0 \rightarrow 1$$

$$y: 0 \rightarrow 1$$

$$z: 0 \rightarrow 1$$

$$\iiint_D \nabla \cdot \vec{F} dV = \int_0^1 \int_0^1 \int_0^1 (2xe^x + 3 - 7yz^6) dx dy dz$$

$$\int 2xe^x dx$$

$$\text{Let } u = x^2$$

$$du = 2x dx$$

$$dx = \frac{du}{2x}$$

$$\int 2xe^u \frac{du}{2x} = \int e^u du$$

$$= \int_0^1 \int_0^1 (ze^x|_0^1 + 3x|_0^1 - 7yz^6 x|_0^1) dy dz$$

$$= \int_0^1 \int_0^1 (ze^{-z} + 3 - 7yz^6) dy dz$$

$$= \int_0^1 (ze^{-z} + 3 - \frac{7y^2 z^6}{2})|_0^1 dz$$

$$= \int_0^1 (ze^{-z} + 3 - \frac{7z^6}{2}) dz$$

$$= \frac{z^2 e^{-z}}{2}|_0^1 - \frac{z^2}{2}|_0^1 + 3z - \frac{7z^7}{14}|_0^1$$

$$= \frac{e}{2} - \frac{1}{2} + 3 - \frac{1}{2} = \frac{e}{2} + 2$$

$$\iint_{S'} \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 (ze^x, 3y, 2 - yz^7) \cdot (0, 0, -1) ds dt$$

$$= \int_0^1 \int_0^1 yz^7 - 2 ds dt$$

$$= \int_0^1 \int_0^1 ((0)^7 - 2) ds dt = \int_0^1 \int_0^1 -2 ds dt = -2$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \iiint_D \nabla \cdot \vec{F} dV - \iint_{S'} \vec{F} \cdot d\vec{S}$$

$$= \frac{e}{2} + 2 - (-2) = \frac{e}{2} + 4$$

Parametrize S'

$$\begin{cases} x=s & s: 0 \rightarrow 1 \\ y=t & t: 0 \rightarrow 1 \\ z=0 \end{cases}$$

$$X_s \times X_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0, 0, 1)$$

but the normal vector must be oriented
away from the region so it must be $(0, 0, -1)$