

#1 sln

Key to Quiz I (Summer Math 61)

Find Critical pts:

$$\nabla f = 0 \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 18x^2 - 6y = 0 \\ \frac{\partial f}{\partial y} = -6x - 2y = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 - y = 0 \quad \dots ① \\ 3x + y = 0 \quad \dots ② \end{cases}$$

$$\begin{aligned} ① + ② &\Rightarrow 3x^2 + 3x = 0 \Rightarrow x^2 + x = 0 \Rightarrow x(x+1) = 0 \\ &\Rightarrow x = 0 \text{ or } x = -1 \end{aligned}$$

$$\text{If } x = 0, \quad ② \Rightarrow y = 0$$

$$\text{If } x = -1, \quad ② \Rightarrow y = 3$$

$\Rightarrow$ Critical pts $(0, 0)$, $(-1, 3)$.

$$\text{Now } Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 36x & -6 \\ -6 & -2 \end{bmatrix}$$

$$Hf(0, 0) = \begin{bmatrix} 0 & -6 \\ -6 & -2 \end{bmatrix}$$

$$\Rightarrow \det Hf(0, 0) = -36 < 0$$

$\Rightarrow (0, 0)$ is a saddle pt.

$$Hf(-1, 3) = \begin{bmatrix} -36 & -6 \\ -6 & -2 \end{bmatrix}$$

$$\det Hf(-1, 3) = 36 \times 2 - 36 = 36 > 0$$

$$f_{xx}(-1, 3) = -36 < 0$$

$\Rightarrow (-1, 3)$ local max pt.

#2 SKN

Want to max/min $f(x, y, z) = x^2 + y^2$
under the constraint $g(x, y) = 5x^2 + 6xy + 5y^2 - 4 = 0$

Step 1 $\begin{cases} \nabla f = \lambda \nabla g \\ g(x, y) = 0 \end{cases}$

$$\Rightarrow \begin{cases} \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} 10x + 6y \\ 6x + 10y \end{pmatrix} \\ g(x, y) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2x = 2\lambda(5x + 3y) \\ 2y = 2\lambda(3x + 5y) \\ g(x, y) = 0 \end{cases}$$

Note the set $\{(x, y) | 5x^2 + 6xy + 5y^2 = 4\}$ is compact since it is an ellipse. To see this, you can complete the square as following: $x^2 + \frac{6}{5}xy + y^2 = \frac{5}{4}$

$$\Rightarrow x^2 + \frac{6}{5}xy + \left(\frac{3}{5}\right)^2 y^2 - \left(\frac{3}{5}\right)^2 y^2 + y^2 = \frac{5}{4}$$

$$\Rightarrow \left(x + \frac{3}{5}y\right)^2 + \left(\frac{4}{5}\right)^2 y^2 = \frac{5}{4}$$

So there $\exists$ global max^m and global min

$$\Rightarrow \begin{cases} x = \lambda(5x + 3y) \\ y = \lambda(3x + 5y) \\ g(x, y) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (5\lambda - 1)x + 3\lambda y = 0 \\ 3\lambda x + (5\lambda - 1)y = 0 \end{cases} \Rightarrow \begin{bmatrix} 5\lambda - 1 & 3\lambda \\ 3\lambda & 5\lambda - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note $\begin{cases} x=0 \\ y=0 \end{cases}$ is not a solution since

$$g(0, 0) \neq 0$$

thus the homogeneous system has nonzero solution. This happens iff

$$\det \begin{bmatrix} 5\lambda - 1 & 3\lambda \\ 3\lambda & 5\lambda - 1 \end{bmatrix} = 0 \Rightarrow (5\lambda - 1)^2 - 9\lambda^2 = 0$$

$$\Rightarrow 16\lambda^2 - 10\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = \frac{10 \pm \sqrt{100 - 4 \times 16}}{2 \times 16}$$

$$\Rightarrow \lambda_{1,2} = \frac{10 \pm 6}{2 \times 16}$$

$$\Rightarrow \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{8}$$

When $y = x$,

$$5x^2 + 6xy + 5y^2 = 4 \Rightarrow 5x^2 + 6x^2 + 5x^2 = 4$$

$$\Rightarrow 16x^2 = 4 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm \frac{1}{2}$$

$$\Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{1}{2}\right) \text{ are critical pts.}$$

When $y = -x$

$$5x^2 - 6x^2 + 5x^2 = 4 \Rightarrow 4x^2 = 4$$

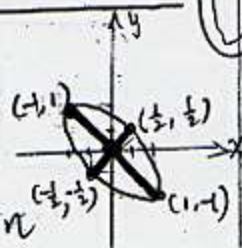
$$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

$$\Rightarrow (1, -1), (-1, 1) \text{ are critical pts.}$$

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = f\left(-\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}$$

$$f(1, -1) = f(-1, 1) = 2$$

$$\Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{1}{2}\right) \text{ global min}$$



For $\lambda_1 = \frac{1}{2}$

$$\Rightarrow \frac{3}{2}x + \frac{3}{2}y = 0$$

$$\Rightarrow x + y = 0$$

$$\Rightarrow y = -x$$

For $\lambda_2 = \frac{1}{8}$

$$\Rightarrow -\frac{3}{8}x + \frac{3}{8}y = 0$$

$$\Rightarrow -x + y = 0$$

$$\Rightarrow y = x$$

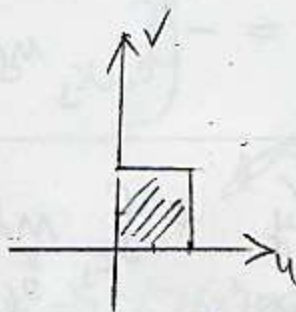
#3 soln

$$\text{let } \begin{cases} u=x \\ v=2y-x \end{cases} \Rightarrow \begin{cases} x=u \\ y=\frac{u+v}{2} \end{cases} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

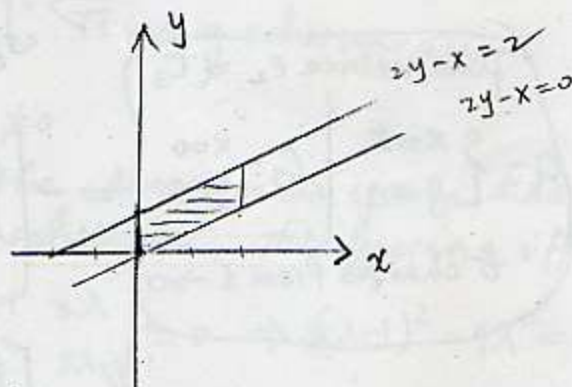
The original region that the integral taking place is

$$\begin{cases} \text{from } y=\frac{x}{2} \text{ to } y=\frac{x}{2}+1 \\ \text{from } x=0 \text{ to } x=2 \end{cases} \quad \text{i.e.,} \quad \begin{cases} \text{from } 2y-x=0 \text{ to } 2y-x=2 \\ \text{from } x=0 \text{ to } x=2 \end{cases}$$

$$\text{thus } \begin{cases} 0 \leq u \leq 2 \\ 0 \leq v \leq 2 \end{cases}$$



$$T: \begin{cases} x=u \\ y=\frac{u+v}{2} \end{cases}$$



$$\text{thus } I = \int_{v=0}^2 \int_{u=0}^2 u^5 v e^{v^2} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

$$= \int_{u=0}^2 u^5 du \int_{v=0}^2 v e^{v^2} \frac{1}{2} dv$$

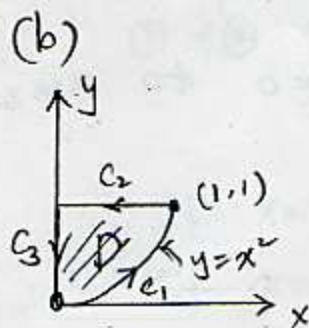
$$= \left(\frac{u^6}{6} \Big|_{u=0}^2 \right) \left(\frac{1}{4} e^{v^2} \Big|_{v=0}^2 \right)$$

$$= \boxed{\frac{8}{3}(e^4-1)}$$

#4 sln

$$(a) \oint_C \underbrace{(e^y \cos x + e^x)}_M dx + \underbrace{(e^y \sin x + y^2)}_N dy$$

by Green's thm $\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_D (e^y \cos x - e^y \cos x) dx dy = 0$



Let $C = C_1 \cup C_2 \cup C_3$, then $\oint_C m dx + n dy = 0$

Now $\oint_C m dx + n dy = \int_{C_1} m dx + n dy + \int_{C_2 \cup C_3} m dx + n dy$

$$\Rightarrow \int_{C_1} m dx + n dy = - \left(\int_{C_2 \cup C_3} m dx + n dy \right)$$

parametrize C_2 & C_3

$C_2: \begin{cases} x=t \\ y=1 \end{cases} \quad C_3: \begin{cases} x=0 \\ y=t \end{cases}$

t changes from $1 \rightarrow 0$

$$= - \left[\int_{C_2} m dx + n dy + \int_{C_3} m dx + n dy \right]$$

$\swarrow$ $y=1$ $\swarrow$ $x=0$

$$= - \left[\int_{t=1}^0 (e^1 \cos t + e^t) dt + \int_{t=1}^0 (e^t \sin 0 + t^2) dt \right]$$

$$= - \left[e \sin t + e^t \Big|_{t=1}^0 + \frac{t^3}{3} \Big|_{t=1}^0 \right]$$

$$= - \left[\{e \sin 0 + e^0 - (e \sin 1 + e)\} + \{0 - \frac{1}{3}\} \right]$$

$$= \boxed{e(1 + \sin 1) - \frac{2}{3}}$$