Math 12: Handout DS3
(but write the HW number from the webpage on your assignment)

1. As an environmental engineer, you have been asked to analyze the dumping of a pollutant in a nearby lake, and find that the amount of pollution entering the lake from all sources is about 20 pounds per year. Fortunately, sunlight gradually breaks down the pollutant into harmless by-products; each year, 10% of the pollutant is lost by this process.

Currently in the lake, there are 0.2 pounds of pollutant per cubic mile of water. Safe levels have been defined by the EPA to be 0.1 pounds per cubic mile. The lake contains 1500 cubic miles of water.

In order to simplify the situation, we assume that the 20 pounds of pollutant all gets added at the end of the year. The dynamical system which models the above process is given by

$$x_{n+1} = 0.9x_n + 20.$$

(a) Will the lake ever reach safe levels? Determine your answer by using a fixed point and stability analysis.

(b) Now suppose you had a method to remove more pollution so that 20% of the pollutant was removed each year. Will the lake ever reach safe levels? Justify. If so, use Discrete Tool (or a programmable calculator) to determine how many years it takes.

(c) Suppose that you were wrong about your model and in fact the amount of pollutant that sunlight breaks down is not 10% or 20% but is $1.64\sqrt{x_n}$ pounds per year. Use Discrete Tool to determine whether the lake will ever reach safe levels. If so, how many years will it take?

2. Suppose you would like to find the roots of $f(x)$. In Newton's method, one iterates the function

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

(a) Show that if p is a root of f, then p is a fixed point for g.

(b) We know that iteration of g seems to converge as long as the initial seed x_0 is "close enough" to a root of f. In other words, p appears to be an *attracting* fixed point of g when p is a root of f. Use the Multiplier Theorem on g to show that this is true.

3. Consider the dynamical system given by iteration of the map

$$f(x) = -\frac{1}{2}(x^3 + x).$$

(a) Both 1 and -1 are periodic points of period 2, and part of the same orbit. Use Discrete Tool to guess whether this is a stable periodic orbit, and then prove your guess using the Multiplier Theorem for periodic orbits.

(b) Plot $f(f(x))$ in Maple to determine if there are any *other* periodic points of *prime* period 2. The prime period of a fixed point p is the smallest number n such that $f^n(p) = p$. (To define the function f in Maple, use the command `f := x -> -(x^3+x)/2;` . Then you can plot `f(f(x))` .)

4. Consider the logistic map $f(x) = \lambda x(1-x)$ when $\lambda = 3.8319$. A period three orbit for this map is (approximately) $\{0.5, 0.948, 0.1543\}$. Use the Multiplier Theorem for periodic orbits to determine whether this periodic orbit is stable or not.

5. Does the function $f(x) = x - x^2$ have any periodic points besides the fixed point at $x = 0$? If so find all periodic points. If not, justify why not.